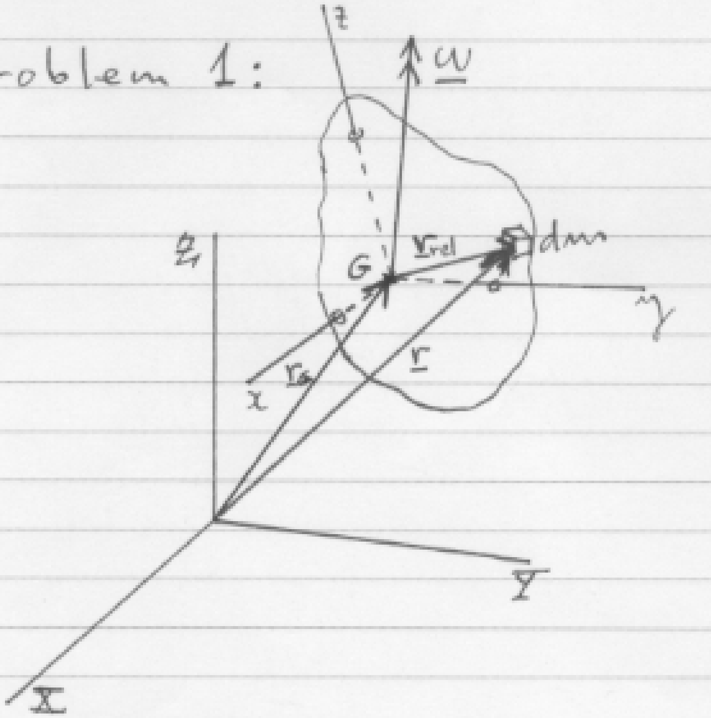
opgave no.		naam	
		studienummer	
		vak D & S	code AE3-914
		datum 9-1-4	fac.

Gebruik voor elke opgave een afzonderlijk vel papier !

Problem 1:



$$\underline{r} = \underline{r}_G + \underline{r}_{rel}$$

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{d\underline{r}_G}{dt} + \frac{d\underline{r}_{rel}}{dt} = \underline{v}_G + \underline{\omega} \times \underline{r}_{rel}$$

$$T = \frac{1}{2}\int_v \underline{v}\cdot\underline{v}\, dm = \frac{1}{2}\int_v (\underline{v}_G + \underline{\omega}\times\underline{r}_{rel})\cdot(\underline{v}_G + \underline{\omega}\times\underline{r}_{rel})\, dm$$

$$= \frac{1}{2}\underline{v}_G\cdot\underline{v}_G\int_v dm + \underline{v}_G\cdot\left[\underline{\omega}\times\int_v \underline{r}_{rel}\, dm\right]$$

$$+ \frac{1}{2}\int_v (\underline{\omega}\times\underline{r}_{rel})\cdot(\underline{\omega}\times\underline{r}_{rel})\, dm$$

$$= \frac{1}{2} m v_G^2 + \underline{v}_G\cdot(\underline{\omega}\times\underline{Q}_G) + \frac{1}{2}\int_v (\underline{\omega}\times\underline{r}_{rel})\cdot(\underline{\omega}\times\underline{r}_{rel})\, dm$$

$\underline{Q}_G = \underline{0}$ (First moment of mass about G)

$$\frac{1}{2}\int_v (\underline{\omega}\times\underline{r}_{rel})\cdot(\underline{\omega}\times\underline{r}_{rel})\, dm = \longrightarrow$$

$$= \frac{1}{2} \int_v \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ w_x & w_y & w_z \\ x & y & z \end{vmatrix}^2 dm =$$

$$= \frac{1}{2} \int_v \left[(w_y z - w_z y)\underline{i} + (w_z x - w_x z)\underline{j} + (w_x y - w_y x)\underline{k} \right]^2 dm$$

$$= \frac{1}{2} \int_v \left[(w_y z - w_z y)^2 + (w_z x - w_x z)^2 + (w_x y - w_y x)^2 \right] dm$$

$$= \frac{1}{2} \int_v \left[(y^2+z^2) w_x^2 + (x^2+z^2) w_y^2 + (x^2+y^2) w_z^2 - 2xy\, w_x w_y - 2xz\, w_x w_z - 2yz\, w_y w_z \right] dm$$

$$= \frac{1}{2} (w_x \quad w_y \quad w_z) \int_v \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} dm \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}$$

$$= \frac{1}{2} \underline{w}^T \underline{\underline{I}}_G \underline{w}$$

$$T = \frac{1}{2} m v_G^2 + \underline{v}_G \cdot (\underline{w} \times \underline{0}) + \frac{1}{2} \underline{w}^T \underline{\underline{I}}_G \underline{w}$$

$$= \frac{1}{2} m v_G^2 + \frac{1}{2} \underline{w}^T \underline{\underline{I}}_G \underline{w}$$

Problem 2:

a. $L = T - V =$

$$= \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) + m \frac{k}{r^3}$$

b. θ is not found in $L \Rightarrow \theta$ is ignorable

$\frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} = C_\theta$ is the integral of motion

$m r^2 \dot{\theta} = C_\theta$ represents conservation of angular momentum

c.
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$ $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$

$$\frac{\partial L}{\partial r} = m r \dot{\theta}^2 - m \frac{3k}{r^4}$$

$$m\ddot{r} - m r\dot{\theta}^2 + m\frac{3k}{r^4} = 0$$

$$\ddot{r} - r\dot{\theta}^2 + \frac{3k}{r^4} = 0$$

$$\dot{\theta}^2 = \frac{C_\theta^2}{m^2 r^4}$$

$$\boxed{\ddot{r} - \frac{C_\theta^2}{m^2 r^3} + \frac{3k}{r^4} = 0}$$ Equation of motion for r.

Steady motion for $\ddot{r} = \dot{r} = 0$

$$\frac{3k}{r^4} = \frac{C_\theta^2}{m^2 r^3}$$

$$\frac{3k}{r^4} = \frac{m^2 r^4 \dot{\theta}^2}{m^2 r^3}$$

$$\boxed{3k = r^5 \dot{\theta}^2}$$ Condition for steady motion

d.

$$\left.\begin{aligned} \dot{r} &= s \\ \dot{s} &= \frac{C_\theta^2}{m^2 r^3} - \frac{3k}{r^4} \end{aligned}\right\}$$

Linearisation:

$$\begin{bmatrix} \dot{r} \\ \dot{s} \end{bmatrix} = \left.\begin{bmatrix} 0 & 1 \\ -\frac{3C_\theta^2}{m^2 r^4} + \frac{12k}{r^5} & 0 \end{bmatrix}\right|_{3k = r^5\dot{\theta}^2} \begin{pmatrix} r \\ s \end{pmatrix} \longrightarrow$$

$$= \begin{bmatrix} 0 & 1 \\ -3\dfrac{m^2 r^4 \dot{\theta}^2}{m^2 r^4} + \dfrac{12k}{r^5} & 0 \end{bmatrix} \begin{pmatrix} r \\ s \end{pmatrix} \Bigg|_{3k = r^5 \dot{\theta}^2}$$

$$= \begin{bmatrix} 0 & 1 \\ -\dfrac{9k}{r^5} + \dfrac{12k}{r^5} & 0 \end{bmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{3k}{r^5} & 0 \end{bmatrix} \begin{pmatrix} r \\ s \end{pmatrix}$$

Eigenvalues:

$$\begin{vmatrix} -\lambda & 1 \\ \dfrac{3k}{r^5} & -\lambda \end{vmatrix} = \lambda^2 - \frac{3k}{r^5} = 0$$

$$\lambda = \pm\sqrt{3k/r^5} \in \mathbb{R}$$

The steady motion is unstable

Problem 3:

a.

$$dL = dT - dV$$

$$dT = \frac{1}{2} v^2 dm = \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 \rho A(x)\, dx$$

$$= \frac{1}{2} u_t^2 \, \rho \pi (r_0 + \frac{x}{l} r_1)^2 dx$$

$$dV = \frac{1}{2} E A \varepsilon^2 dx = \frac{1}{2} E A(x) \left(\frac{\partial u}{\partial x}\right)^2 dx$$

$$= \frac{1}{2} E u_x^2 \, \pi (r_0 + \frac{x}{l} r_1)^2 dx$$

$$dL = \frac{1}{2} \pi (r_0 + \frac{x}{l} r_1)^2 (\rho u_t^2 - E u_x^2)\, dx$$

b.

$$I[u(x,t)] = \int_{t_a}^{t_b} \int_0^l dL\, dt$$

$$= \int_{t_a}^{t_b} \int_0^l \frac{1}{2} \pi (r_0 + \frac{x}{l} r_1)^2 (\rho u_t^2 - E u_x^2)\, dx\, dt$$

c.

$$I[u(x,t)] = \int_{t_a}^{t_b} \int_0^l F(x, u_x, u_t)\, dx\, dt$$

$$\delta I = 0 \Rightarrow \quad \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial u_t}\right) = 0$$

(Euler-Lagrange equation)

$\longrightarrow$

$$\frac{\partial F}{\partial u_x} = \cancel{\frac{1}{2}}\pi\left(r_0 + \frac{x}{\ell} r_1\right)^2 \left(-\cancel{2} E u_x\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) = -2\pi\left(r_0 + \frac{x}{\ell} r_1\right)\frac{r_1}{\ell} E u_x - \pi\left(r_0 + \frac{x}{\ell} r_1\right)^2 E u_{xx}$$

$$\frac{\partial F}{\partial u_t} = \cancel{\frac{1}{2}}\pi\left(r_0 + \frac{x}{\ell} r_1\right)^2 \cdot \cancel{2}\rho u_t$$

$$\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial u_t}\right) = \pi\left(r_0 + \frac{x}{\ell} r_1\right)^2 \cdot \rho u_{tt}$$

$$\frac{\partial F}{\partial u} = 0$$

Substituting in E-L equation:

$$0 = 2\cancel{\pi}\cancel{\left(r_0 + \frac{x}{\ell} r_1\right)}\frac{r_1}{\ell} E u_x + \cancel{\pi}\left(r_0 + \frac{x}{\ell} r_1\right)^{\cancel{2}} E u_{xx} - \cancel{\pi}\left(r_0 + \frac{x}{\ell} r_1\right)^{\cancel{2}} \cdot \rho u_{tt}$$

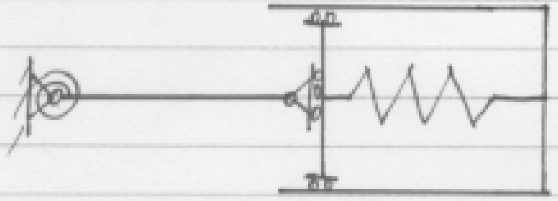
$$E\left[\left(r_0 + \frac{x}{\ell} r_1\right) u_{xx} + 2\frac{r_1}{\ell} u_x\right] = \rho\left(r_0 + \frac{x}{\ell} r_1\right) u_{tt}$$

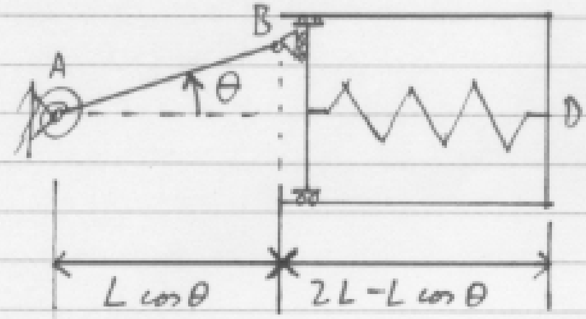
or, rearranging terms

$$E\left[u_{xx} + \frac{2 r_1}{\ell\left(r_0 + \frac{x}{\ell} r_1\right)} u_x\right] = \rho u_{tt}$$

Problem 4:

a.





θ: generalised coordinate

$$\mathcal{L} = T - V$$

$$T = \frac{1}{2} I_A \dot{\theta}^2$$

$$V = \frac{1}{2} k_A \theta^2 + \frac{1}{2} k_{BD} \left(2L - L\cos\theta - (1 + \alpha \Delta Z)L\right)^2$$

$$= \frac{1}{2} k_A \theta^2 + \frac{1}{2} k_{BD} L^2 (1 - \cos\theta - \alpha \Delta Z)^2$$

$$\mathcal{L} = \frac{1}{2} I_A \dot{\theta}^2 - \frac{1}{2} k_A \theta^2 - \frac{1}{2} k_{BD} L^2 (1 - \cos\theta - \alpha \Delta Z)^2$$

b.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = I_A \ddot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -k_A \theta - k_{BD} L^2 (1 - \cos\theta - \alpha \Delta Z) \sin\theta$$

$$I_A \ddot{\theta} + k_A \theta + k_{BD} L^2 (1 - \cos\theta - \alpha \Delta Z) \sin\theta = 0$$

c.

$$\dot{\theta} = \omega$$

$$\dot{\omega} = -\frac{1}{I_A}\left[k_A\theta + k_{BD}L^2(1-\cos\theta - \alpha\Delta z)\sin\theta\right]$$

Linearisation for $\theta = 0$:

$$\begin{pmatrix}\dot{\theta}\\ \dot{\omega}\end{pmatrix} = \left.\begin{bmatrix} 0 & 1 \\ -\frac{1}{I_A}\left(k_A + k_{BD}L^2\left(\sin^2\theta + (1-\cos\theta-\alpha\Delta z)\cos\theta\right)\right) & 0\end{bmatrix}\right|_{\theta=0}\begin{pmatrix}\theta\\ \omega\end{pmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -\frac{1}{I_A}(k_A - k_{BD}L^2\alpha\Delta z) & 0\end{bmatrix}\begin{pmatrix}\theta\\ \omega\end{pmatrix}$$

Eigenvalues:

$$\begin{vmatrix} -\lambda & 1 \\ -\frac{1}{I_A}(k_A - k_{BD}L^2\alpha\Delta z) & -\lambda\end{vmatrix} =$$

$$= \lambda^2 + \frac{1}{I_A}(k_A - k_{BD}L^2\alpha\Delta z) = 0$$

$\theta = 0$ is unstable if $\lambda \in \mathbb{R}$ and $\exists\, \lambda > 0$

$$\lambda = \pm\sqrt{(k_{BD}L^2\alpha\Delta z - k_A)/I_A} \Rightarrow \boxed{\Delta z > \frac{k_A}{k_{BD}L^2\alpha}}$$

Problem 5:

a. Non-ignorable coordinate $\rightarrow$ θ (Nutation angle)

$$T = \frac{1}{2}\left[I(\dot\phi^2 \sin^2\theta + \dot\theta^2) + I_s(\dot\phi\cos\theta + \dot\psi)^2 \right]$$

$V \neq V(\phi, \theta, \psi)$ in space, so $V = 0$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) = I\ddot\theta$$

$$\frac{\partial T}{\partial \theta} = I\dot\phi^2 \sin\theta\cos\theta - I_s(\dot\phi\cos\theta + \dot\psi)\dot\phi\sin\theta$$

$$I\ddot\theta - I\dot\phi^2\sin\theta\cos\theta + I_s(\dot\phi\cos\theta + \dot\psi)\dot\phi\sin\theta = 0 \quad (*)$$

Formally this is not the equation asked for yet, because the ignorable coordinates are still there. We find now the integrals of motion for ϕ and ψ:

$$\left.\begin{aligned} \frac{\partial L}{\partial \dot\phi} &= I\dot\phi\sin^2\theta + I_s(\dot\phi\cos\theta + \dot\psi)\cos\theta = C_\phi \\ \frac{\partial L}{\partial \dot\psi} &= I_s(\dot\phi\cos\theta + \dot\psi) = C_\psi \end{aligned}\right\}$$

This is reworked as:

$$\dot\phi\cos\theta + \dot\psi = \frac{C_\psi}{I_s}$$

$$\dot\phi = \frac{C_\phi - C_\psi\cos\theta}{I\sin^2\theta}$$

Substituting into (*) we get:

$$I\ddot{\theta} - I\,\frac{(C_\phi - C_\psi \cos\theta)^2}{I^2 \sin^4\theta}\sin\theta\cos\theta$$

$$+ I_s\,\frac{C_\psi}{I_s}\cdot\frac{C_\phi - C_\psi\cos\theta}{I\sin^2\theta}\sin\theta = 0$$

Rearranging terms:

$$\boxed{I\ddot{\theta} - \frac{(C_\phi - C_\psi\cos\theta)^2}{I\sin^3\theta}\cos\theta + \frac{C_\phi - C_\psi\cos\theta}{I\sin\theta}\cdot C_\psi = 0}$$

with $I = Mk^2$

b. The conditions for steady motion can be found directly from equation (*):

$$\ddot{\theta} = \dot{\theta} = 0$$

$$I\cdot 0 - I\dot{\phi}^{\cancel{2}}\,\cancel{\sin\theta}\cos\theta + I_s(\dot{\phi}\cos\theta + \dot{\psi})\cancel{\dot{\phi}}\,\cancel{\sin\theta} = 0$$

$$-I\dot{\phi}\cos\theta + I_s(\dot{\phi}\cos\theta + \dot{\psi}) = 0$$

$$\cos\theta = \frac{I_s\dot{\psi}}{(I - I_s)\dot{\phi}} = \frac{k_s^2\dot{\psi}}{(k^2 - k_s^2)\dot{\phi}}$$

c.

$$\frac{\dot{\psi}}{\dot{\phi}} = 3, \quad \cos\theta = \cos 10^\circ = 0{,}9848$$

$$\frac{k^2 - k_s^2}{k_s^2} = \frac{\dot{\psi}}{\dot{\phi}}\cdot\frac{1}{\cos\theta}; \quad \frac{k}{k_s} = \sqrt{1 + 3/\cos 10} = 2{,}0115$$

$$\boxed{\frac{k_s}{k} \simeq \frac{1}{2}}$$