opgave no.	D & S

naam	
studienummer	
vak	code ae3-914
datum 20-1-2005	fac.

Gebruik voor elke opgave een afzonderlijk vel papier !

1

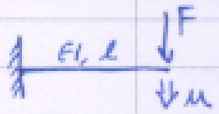
a.

$$\theta = \frac{F\ell^2}{2EI} \Rightarrow F = \frac{2EI}{\ell^2}\theta$$

$$u = \frac{F\ell^3}{3EI} \Rightarrow F = \frac{3EI}{\ell^3}u$$

$$\frac{2EI}{\ell^2}\theta = \frac{3EI}{\ell^3}u \Rightarrow \boxed{\theta = \frac{3u}{2\ell}}$$

b. $T = \frac{1}{2} m \dot{u}^2$ Kinetic energy



(Hint)

$F = \frac{3EI}{\ell^3}u \rightarrow \frac{3EI}{\ell^3}$ analogous to spring constant

$V = \frac{1}{2}\frac{3EI}{\ell^3}u^2$ Potential energy

$$L = T - V = \frac{1}{2} m \dot{u}^2 - \frac{1}{2}\frac{3EI}{\ell^3}u^2$$

$$\delta W = F\,\delta u = c(V_\infty \theta - \dot{u})\,\delta u = \underbrace{c\left(V_\infty \frac{3u}{2\ell} - \dot{u}\right)}_{Q}\delta u$$

Thus, we have the generalised force

$$Q = c\left(V_\infty \frac{3u}{2\ell} - \dot{u}\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}}\right) - \frac{\partial L}{\partial u} = Q$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}}\right) = m\ddot{u} \qquad \frac{\partial L}{\partial u} = -\frac{3EI}{\ell^3}u$$

$$m\ddot{u} + \frac{3EI}{\ell^3}u = c\left(V_\infty \frac{3u}{2\ell} - \dot{u}\right)$$

$$\boxed{m\ddot{u} + c\dot{u} + \left(\frac{3EI}{\ell^3} - \frac{3cV_\infty}{2\ell}\right)u = 0}$$

c.

$$\left.\begin{aligned} \dot{u} &= v \\ \dot{v} &= \frac{1}{m}\left[-cv - \left(\frac{3EI}{\ell^3} - \frac{3cV_\infty}{2\ell}\right)u\right] \end{aligned}\right\}$$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{m}\left(\frac{3cV_\infty}{2\ell} - \frac{3EI}{\ell^3}\right) & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ \frac{1}{m}\left(\frac{3cV_\infty}{2\ell} - \frac{3EI}{\ell^3}\right) & -\frac{c}{m} - \lambda \end{vmatrix} = \lambda\left(\frac{c}{m} + \lambda\right) - \frac{1}{m}\left(\frac{3cV_\infty}{2\ell} - \frac{3EI}{\ell^3}\right)$$

$$= 0 \qquad \Longrightarrow$$

$$\lambda^2 + \frac{c}{m}\lambda - \frac{1}{m}\left(\frac{3cV_\infty}{2l} - \frac{3EI}{l^3}\right) = 0$$

$$\lambda = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 + 4\cdot\frac{1}{m}\left(\frac{3cV_\infty}{2l} - \frac{3EI}{l^3}\right)}}{2}$$

Unstable if $\lambda > 0$

$$-\frac{c}{m} + \sqrt{\left(\frac{c}{m}\right)^2 + \frac{4}{m}\left(\frac{3cV_\infty}{2l} - \frac{3EI}{l^3}\right)} > 0$$

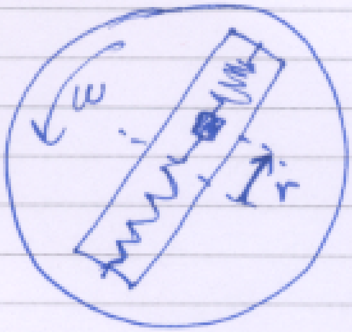
$$\left(\frac{c}{m}\right)^2 + \frac{4}{m}\left(\frac{3cV_\infty}{2l} - \frac{3EI}{l^3}\right) > \left(\frac{c}{m}\right)^2$$

$$\frac{3cV_\infty}{2l} - \frac{3EI}{l^3} > 0$$

$$\boxed{V_\infty > \frac{2EI}{cl^2}}$$

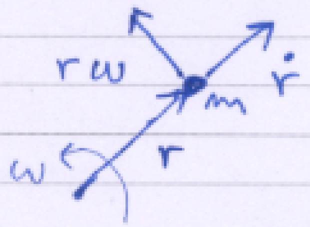
2

a.



Displacement of mass w.r.t. centre is chosen as generalised coordinate

r



$$T = \frac{1}{2} m(\dot{r}^2 + r^2\omega^2)$$

$$V = \underbrace{2}_{\text{two springs}} \cdot \frac{1}{2} k r^2 = k r^2$$

$$\boxed{L = T - V = \frac{1}{2} m(\dot{r}^2 + r^2\omega^2) - k r^2}$$

b. $\frac{\partial L}{\partial t} = 0 \Rightarrow$ Jacobi integral h <u>is</u> an integral of motion.

$$h = \frac{\partial L}{\partial \dot{r}}\dot{r} - L = m\dot{r}\,\dot{r} - \frac{1}{2}m(\dot{r}^2 + r^2\omega^2) + kr^2$$

$$= \boxed{\frac{1}{2}m\dot{r}^2 - \frac{1}{2}m r^2\omega^2 + kr^2}$$

c. Effective potential is recognised in h as

$$V_{eff} = kr^2 - \frac{1}{2}m r^2\omega^2$$

d. Since an effective potential is available, stability can best be studied on the nature of its extrema.

$r = 0$ is a maximum of V_{eff} $\Rightarrow$ unstable point

$$V'_{eff}(r) = 2kr - mr\omega^2$$

$V'_{eff}(0) = 0 \Rightarrow$ indeed equilibrium point

$$V''_{eff}(r) = 2k - m\omega^2$$

$V''_{eff}(0) < 0 \Rightarrow r = 0$ is a maximum

$$V''_{eff}(0) = 2k - m\omega^2 < 0$$

$$\boxed{\omega^2 > \frac{2k}{m}}$$ for instability

3

a. $$T = \frac{1}{2}\left[I(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + I_s(\dot{\phi}\cos\theta + \dot{\psi})^2 \right]$$

$$V = mgl\cos\theta$$

$$L = T - V$$

ϕ and ψ are ignorable coordinates

θ is an explicit coordinate

Steady motion must be sought for the equation corresponding to the explicit coordinate:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = I\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = I\dot{\phi}^2 \sin\theta\cos\theta + I_s(\dot{\phi}\cos\theta + \dot{\psi})\dot{\phi}(-\sin\theta) + mgl\sin\theta$$

$$I\ddot{\theta} - \sin\theta\left(I\dot{\phi}^2\cos\theta - I_s(\dot{\phi}\cos\theta + \dot{\psi})\dot{\phi} + mgl\right) = 0$$

Steady motion $\Rightarrow$ $\dot{\theta} = \ddot{\theta} = 0$

For $\theta = 90°$ we have:

$$0 = \underbrace{\sin 90°}_{1}\left(I\dot{\phi}^2 \underbrace{\cos 90°}_{0} - I_s(\dot{\phi}\underbrace{\cos 90°}_{0} + \dot{\psi})\dot{\phi} + mgl\right)$$

$$I_s \dot{\psi}\dot{\phi} = mgl$$

Since $I_s = \frac{1}{2} mr^2$.

$$\boxed{\dot{\psi}\dot{\phi} = \frac{2gl}{r^2}}$$

b.

For $\theta = 60°$ we have:

$$0 = \sin 60° \underbrace{\left(I\dot{\phi}^2 \overbrace{\cos 60°}^{1/2} - I_s(\dot{\phi}\overbrace{\cos 60°}^{1/2} + \dot{\psi})\dot{\phi} + mgl\right)}_{0}$$

$$I\dot{\phi}^2 \cdot \frac{1}{2} - I_s\left(\dot{\phi}\cdot\frac{1}{2} + \dot{\psi}\right)\dot{\phi} + mgl = 0$$

$$\frac{1}{2}(I - I_s)\dot{\phi}^2 - I_s\dot{\psi}\dot{\phi} + mgl = 0$$

$$\dot{\phi} = \frac{I_s \dot{\psi} \pm \sqrt{I_s^2 \dot{\psi}^2 - 2(I - I_s) mgl}}{I - I_s}$$

Substituting $I_s = \frac{1}{2} m r^2$ and

$I = \frac{1}{4} m r^2 + m l^2$ (steiner) this results into

$$\boxed{\dot{\phi} = \frac{2 r^2 \dot{\psi} \pm 2 \sqrt{r^4 \dot{\psi}^2 - (4 l^2 - r^2) 2 g l}}{4 l^2 - r^2}}$$

Furthermore, the quantity under the root must be non-negative:

$$r^4 \dot{\psi}^2 - (4 l^2 - r^2) 2 g l \geqslant 0$$

$$\boxed{\dot{\psi}^2 \geqslant \frac{4 l^2 - r^2}{r^4} 2 g l}$$

which is a condition on the minimum required spin in the case $\theta = 60^\circ$

4

a. $dL = dT - dV$

$$dT = \frac{1}{2}\dot{\phi}^2 dI = \frac{1}{2}\phi_t^2 \frac{1}{2}\rho\pi r^4 dx$$

$$= \frac{1}{4}\rho\pi r^4 \phi_t^2 dx$$

$$dV = \frac{1}{2} G I_p \phi_x^2 dx = \frac{1}{2} G \frac{1}{2}\pi r^4 \phi_x^2 dx$$

$$= \frac{1}{4} G\pi r^4 \phi_x^2 dx$$

$$\boxed{dL = \frac{1}{4}\pi r^4 (\rho\phi_t^2 - G\phi_x^2)\, dx}$$

b. Rotation is fixed at $x = 0 \Rightarrow$

$$\delta\phi(0,t) = 0 \Rightarrow$$

$\boxed{\phi(0,t) = 0 \text{ is an essential B.C.}}$

c.

$$I = \int_{t_a}^{t_b}\int_0^{l} dL\,dt = \int_{t_a}^{t_b}\int_0^{l} \frac{1}{4}\pi r^4\left(\rho\phi_t^2 - G\phi_x^2\right) dx\,dt$$

$$= \int_{t_a}^{t_b}\int_0^{l} F(\phi_t, \phi_x)\,dx\,dt$$

$$\delta I = \int_{t_a}^{t_b}\int_0^{l}\left(\frac{\partial F}{\partial \phi_t}\delta\phi_t + \frac{\partial F}{\partial \phi_x}\delta\phi_x\right) dx\,dt$$

$$= \int_0^{l} \frac{\partial F}{\partial \phi_t}\delta\phi\Big|_{t_a}^{t_b} dx - \int_{t_a}^{t_b}\int_0^{l}\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \phi_t}\right)\delta\phi\,dx\,dt$$

$$+ \int_{t_a}^{t_b} \frac{\partial F}{\partial \phi_x}\delta\phi\Big|_0^{l} dt - \int_{t_a}^{t_b}\int_0^{l}\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \phi_x}\right)\delta\phi\,dx\,dt =$$

$$= \left\{ \int_0^{\ell} \frac{\partial F}{\partial \phi_t} \delta\phi \Big|_{t_a}^{t_b} dx \right.$$

$= 0$ (essential B.C.)

$$+ \int_{t_a}^{t_b} \left[\frac{\partial F}{\partial \phi_x}\Big|_{x=\ell} \delta\phi(\ell,t) - \frac{\partial F}{\partial \phi_x}\Big|_{x=0} \delta\phi(0,t) \right] dt$$

$$\left. - \int_{t_a}^{t_b} \int_0^{\ell} \left[\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \phi_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \phi_x}\right) \right] \delta\phi \, dx\, dt \right\} = 0$$

$\delta I = 0 \Rightarrow$ each term in the sum $= 0$ for any variation $\delta\phi$

Consequently:

$$\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \phi_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \phi_x}\right) = 0$$

(Euler-Lagrange equation)

reworked for $F = \frac{1}{4}\pi r^4 (\rho \phi_t^2 - G \phi_x^2)$:

$$\boxed{\rho \phi_{tt} = G \phi_{xx}}$$ Equation of motion

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Natural boundary condition:

$$\left.\frac{\partial F}{\partial \phi_x}\right|_{x=\ell} \underbrace{\delta\phi(\ell,t)}_{\neq 0} = 0 \Rightarrow \left.\frac{\partial F}{\partial \phi_x}\right|_{x=\ell} = 0$$

which is reworked as

$$\boxed{\phi_x(\ell,t) = 0}$$ is a natural boundary condition

Boundary conditions on $\phi(x,t_a)$ and $\phi(x,t_b)$ are not asked for.

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$$\underline{\underline{I}} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

$$I_{xx} = \int (y^2+z^2)\,dm = \sum m_i (y_i^2 + z_i^2)$$

$$= 2m\,2a^2 + 4m\,a^2 + 3m\,2a^2 = 14\,ma^2$$

$$I_{yy} = \int (x^2+z^2)\,dm = \sum m_i (x_i^2 + z_i^2)$$

$$= 2m\,2a^2 + 4m\,a^2 + 3m\,2a^2 = 14\,ma^2$$

$$I_{zz} = \int (x^2+y^2)\,dm = \sum m_i (x_i^2 + y_i^2)$$

$$= 2m\,2a^2 + 4m\,2a^2 + 3m\,2a^2 = 18\,ma^2$$

$$I_{xy} = \int xy\,dm = \sum m_i x_i y_i = -2ma^2 - 4ma^2 + 3ma^2 = -3ma^2$$

$$I_{xz} = \int xz\,dm = \sum m_i x_i z_i = 2ma^2 - 3ma^2 = -ma^2$$

$$I_{yz} = \int yz\,dm = \sum m_i y_i z_i = -2ma^2 - 3ma^2 = -5ma^2$$

$\longrightarrow$

$$\underline{\underline{I}} = \begin{bmatrix} 14 & 3 & 1 \\ 3 & 14 & 5 \\ 1 & 5 & 18 \end{bmatrix} m a^2$$