| opgave no. | | naam | |
|---|---|---|---|
| | | studienummer | |
| | | vak D&S | code AE3-914 |
| | | datum 22-6-04 | fac. |

Gebruik voor elke opgave een afzonderlijk vel papier !

# Problem 1

$\underline{F}$ is conservative $\Rightarrow$ $\underline{F} = -\nabla V$

consequently $\nabla \times \underline{F} = -\nabla \times \nabla V =$

$$= -(\nabla \times \nabla) V = \underline{0}$$

Alternative (analitically):

$$\underline{F} = F_x \underline{i} + F_y \underline{j} + F_z \underline{k} =$$

$$= -\frac{\partial V}{\partial x}\underline{i} - \frac{\partial V}{\partial y}\underline{j} - \frac{\partial V}{\partial z}\underline{k}$$

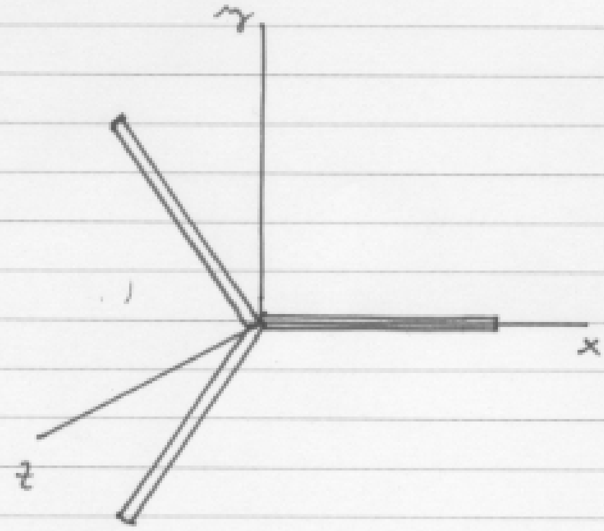
$$\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{\partial V}{\partial x} & -\frac{\partial V}{\partial y} & -\frac{\partial V}{\partial z} \end{vmatrix}$$

$$= \frac{\partial^2}{\partial y \partial z}(V-V)\underline{i} + \frac{\partial^2}{\partial x \partial z}(V-V)\underline{j} + \frac{\partial^2}{\partial x \partial y}(V-V)\underline{k}$$

$$= 0\underline{i} + 0\underline{j} + 0\underline{k} = \underline{0}$$

Problem 2

a)




z ≡ rotor shaft

$$I_3 = I_{zz} = 3 I_b$$

In xy-plane there are three symmetry axes, consequently all directions are principal in that plane and thus:

$$I_{xx} = I_{yy} = I_1 = I_2 \qquad (1)$$

Moreover, the object is contained in the xy-plane, consequently:

$$I_{zz} = \int (x^2 + y^2)\,dm = \int x^2\,dm + \int y^2\,dm$$

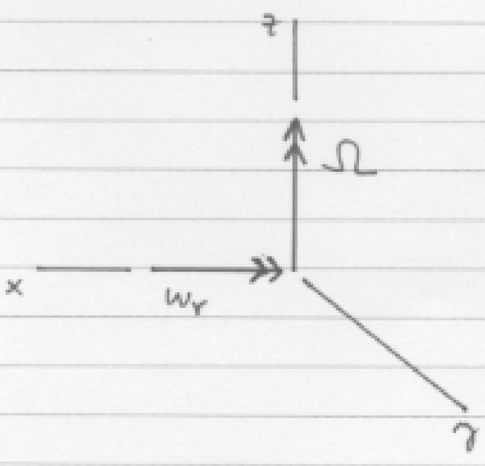
$$= I_{yy} + I_{xx} \qquad (2)$$

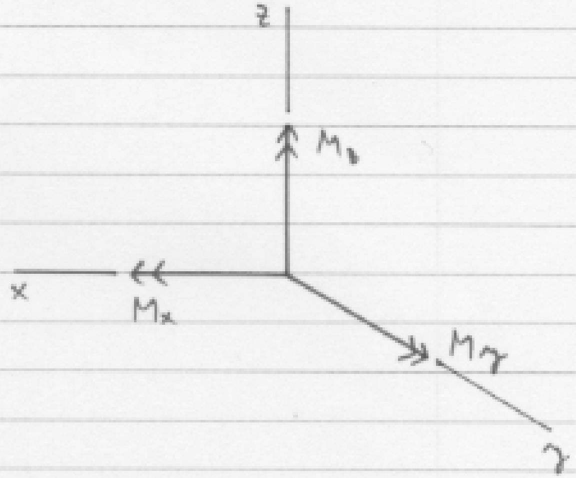
(1) and (2) together provide

$$I_{xx} = I_{yy} = \frac{I_{zz}}{2}; \quad I_1 = I_2 = \frac{I_3}{2}$$

$$I_1 = \frac{3}{2} I_b; \quad I_2 = \frac{3}{2} I_b; \quad I_3 = 3 I_b$$

(b) Forward rotor:




Euler's equations:

$$I_1 \dot{\omega}_x - (I_2 - I_3)\omega_y \omega_z = M_x$$

$$I_2 \dot{\omega}_y - (I_3 - I_1)\omega_x \omega_z = M_y$$

$$I_3 \dot{\omega}_z - (I_1 - I_2)\omega_x \omega_y = M_z$$

Data: $\omega_x = -\omega_r$; $\omega_y = 0$; $\omega_z = \Omega$

$\dot{\omega}_x = ?$; $\dot{\omega}_y = ?$; $\dot{\omega}_z = ?$

$\underline{\omega}_{xy} = -\omega_r \underline{i}$; $\dfrac{d\underline{\omega}_{xy}}{dt} = \underline{0}$

$$\frac{d\underline{\omega}_{xy}}{dt} = \dot{\omega}_x \underline{i} + \dot{\omega}_y \underline{j} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega_x & \omega_y & \omega_z \\ \omega_x & \omega_y & 0 \end{vmatrix} = \underline{0}$$

$\Rightarrow \dot{\omega}_x = \omega_y \omega_z$; $\dot{\omega}_y = -\omega_x \omega_z$

$$M_x = I_1 \cdot 0 - (I_2 - I_3)\cdot 0 \cdot \Omega = 0$$

$$M_y = I_2(\omega_r \Omega) - (I_3 - I_1)(-\omega_r)\Omega = I_3 \omega_r \Omega$$

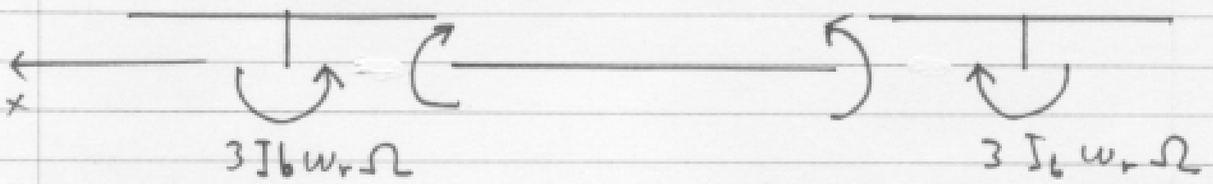
$$= 3 I_b \omega_r \Omega$$

Aft rotor:

Exactly the same as for the forward rotor, but $\Omega$ is now replaced by $-\Omega$

$$M_y = -3 I_b w_r \Omega$$

Helicopter:




The bending moment amounts to

$$+3 I_b w_r \Omega$$

Problem 3

(a)

$$I_f = 2\int_0^b f^2\, dA = 2\int_0^b f^2\, t\sqrt{1+f'^2}\, dy$$

$$= 2t\int_0^b f^2\sqrt{1+f'^2}\, dy$$

Euler-Lagrange equation:

For $\quad I = \int_0^b F(y, f, f')\, dy \quad$ we have

$$\frac{\partial F}{\partial f} - \frac{d}{dy}\left(\frac{\partial F}{\partial f'}\right) = 0 \qquad (1)$$

$$\frac{\partial F}{\partial f} = 2f\sqrt{1+f'^2}$$

$$\frac{\partial F}{\partial f'} = f^2\frac{f'}{\sqrt{1+f'^2}}$$

$$\frac{d}{dy}\left(\frac{\partial F}{\partial f'}\right) = \frac{2ff'^2 + f^2f'' + 2ff'^4}{(1+f'^2)^{3/2}}$$

Substituting in (1) and rearranging yields

$$2 + 2f'^2 - ff'' = 0$$

(b)

$$\forall \delta f \quad 0 = \delta I = \int_0^b \delta F(y, f, f')\, dy$$

$$= \int_0^b \left[\frac{\partial F}{\partial f}\delta f + \frac{\partial F}{\partial f'}\delta f'\right] dy$$

$$= \int_0^b \frac{\partial F}{\partial f}\delta f\, dy + \frac{\partial F}{\partial f'}\delta f\Big|_0^b$$

$$- \int_0^b \frac{d}{dy}\left(\frac{\partial F}{\partial f'}\right)\delta f \quad \Rightarrow$$

$$\forall \delta f \quad \frac{\partial F}{\partial f'}\delta f\Big|_0^b = 0 \Rightarrow \quad \frac{\partial F}{\partial f'}\Big|_0 \delta f(0) = 0 \qquad (1)$$

$$\frac{\partial F}{\partial f'}\Big|_b \delta f(b) = 0 \qquad (2)$$

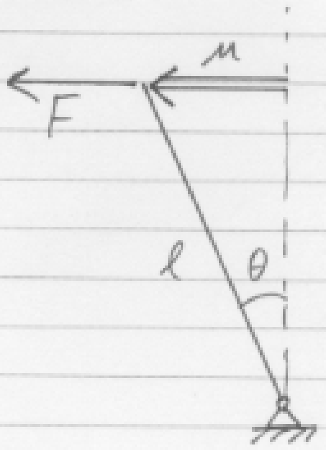
(1) is always 0 because $f(0) = \frac{h}{2} \Rightarrow \delta f(0) = 0$

(2): $f(b)$ is not given, thus $\frac{\partial F}{\partial f'}\Big|_b = 0 \rightarrow$ natural boundary condition

$$\frac{\partial F}{\partial f'} = \frac{f^2 f'}{\sqrt{1+f'^2}} = 0 \Rightarrow \left.\begin{cases} f = 0 \\ f' = 0 \end{cases}\right\} \Rightarrow \boxed{\begin{matrix} f(b) = 0 \\ \text{or} \\ f'(b) = 0 \end{matrix}}$$

Problem 4

(a)




$$x = l\sin\theta; \quad \delta x = l\cos\theta\,\delta\theta$$

$$\delta W = F\,\delta x = \underbrace{F l\cos\theta}_{Q_\theta}\,\delta\theta$$

$$Q_\theta = F l\cos\theta$$

which is a moment

(b) $T = \frac{1}{2}I\dot\theta^2 = \frac{1}{6}ml^2\dot\theta^2; \qquad V = \frac{1}{2}mgl\cos\theta;$

$$L = T - V = \frac{1}{6}ml^2\dot\theta^2 - \frac{1}{2}mgl\cos\theta$$

Equation of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = Q_\theta$$

$$\frac{1}{3}ml\ddot\theta - \frac{1}{2}mg\sin\theta - F\cos\theta = 0$$

(c) $\theta = 0; \quad \ddot\theta = 0$

$$\frac{1}{3}ml\cdot 0 - \frac{1}{2}mg\sin 0 - F(0)\cos 0 = 0$$

$$\boxed{F(0) = 0}$$

(d) Convert to first order:

$$\left.\begin{aligned}\dot\theta &= \varphi \\ \dot\varphi &= \frac{3g\sin\theta}{2l} + \frac{3F\cos\theta}{ml}\end{aligned}\right\}$$

Linearization for $\theta = 0$

$$\begin{bmatrix}\dot\theta \\ \dot\varphi\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{3g\cos 0}{2l} + \frac{3F'(0)\cos 0 - F(0)\sin 0}{ml} & 0 \end{bmatrix}\begin{bmatrix}\theta \\ \varphi\end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ \frac{3g}{2l} + \frac{3F'(0)}{ml} & 0 \end{bmatrix}\begin{bmatrix}\theta \\ \varphi\end{bmatrix}$$

Eigenvalues:

$$\begin{vmatrix} -\lambda & 1 \\ \frac{3}{l}\left(\frac{g}{2} + \frac{F'(0)}{m}\right) & -\lambda \end{vmatrix} = \lambda^2 - \frac{3}{l}\left(\frac{g}{2} + \frac{F'(0)}{m}\right) = 0$$

Stable if $\lambda \in \mathrm{Im} \Rightarrow \frac{g}{2} + \frac{F'(0)}{m} < 0$

$$\boxed{F'(0) < -\frac{mg}{2}}$$

Problem 5

(a)

$$T = \frac{1}{2} m \left( \dot{s}^2 + s^2 \cos^2\alpha\, \dot{\theta}^2 \right) + \frac{1}{2} m \dot{s}^2$$

$$= m \left( \dot{s}^2 + \frac{1}{2} s^2 \cos^2\alpha\, \dot{\theta}^2 \right)$$

$$V = mg\, s \sin\alpha + mgs = (1+\sin\alpha) mgs$$

$$L = m \left( \dot{s}^2 + \frac{\cos^2\alpha}{2} s^2 \dot{\theta}^2 \right) - (1+\sin\alpha) mgs$$

(b)

$\theta$ is ignorable

$$\frac{\partial L}{\partial \dot{\theta}} = C_\theta; \quad m \cos^2\alpha\, s^2 \dot{\theta} = C_\theta$$

(conservation of angular momentum)

(c)

$$\frac{d}{dt}\left( \frac{\partial L}{\partial \dot{s}} \right) = 2m\ddot{s}$$

$$\frac{\partial L}{\partial s} = m \cos^2\alpha\, s\, \dot{\theta}^2 - (1+\sin\alpha) mg$$

$$\dot{\theta} = \frac{C_\theta}{m \cos^2\alpha\, s^2}$$

Equation of motion:

$$2m\ddot{s} - m\cos^2\alpha\, s\,\frac{C_\theta^2}{m^2\cos^4\alpha\, s^4} + (1+\sin\alpha)mg = 0$$

$$\ddot{s} - \frac{C_\theta^2}{2m^2\cos^2\alpha\, s^3} + \frac{(1+\sin\alpha)g}{2} = 0$$

Steady motion:

$$\ddot{s} = \dot{s} = 0$$

$$C_\theta = m\cos^2\alpha\, s^2\dot{\theta}$$

$$\frac{C_\theta^2}{2m^2\cos^2\alpha\, s^3} = \frac{(1+\sin\alpha)g}{2}$$

$$\frac{m^2\cos^4\alpha\, s^4\dot{\theta}^2}{2m^2\cos^2\alpha\, s^3} = \frac{(1+\sin\alpha)g}{2}$$

$$\cos^2\alpha\, s\,\dot{\theta}^2 = (1+\sin\alpha)g$$

(d) Convert to first order:

$$\dot{s} = r$$

$$\dot{r} = \frac{C_\theta^2}{2m^2\cos^2\alpha\, s^3} - \frac{(1+\sin\alpha)g}{2}$$

Linearise:

$$\begin{bmatrix} \dot{s} \\ \dot{r} \end{bmatrix} = \left.\begin{bmatrix} 0 & 1 \\ \dfrac{-3C_\theta^2}{2m^2\cos^2\alpha\, s^4} & 0 \end{bmatrix}\right|_{s\dot{\theta}^2 = \frac{1+\sin\alpha}{\cos^2\alpha} g} \begin{bmatrix} s \\ r \end{bmatrix}$$

Eigenvalues:

$$\begin{vmatrix} -\lambda & 1 \\ \dfrac{-3C_\theta^2}{2m^2\cos^2\alpha\, s^4} & -\lambda \end{vmatrix} = \left. \lambda^2 + \frac{3C_\theta^2}{2m^2\cos^2\alpha\, s^4} \right|_{s\dot{\theta}^2 = \frac{1+\sin\alpha}{\cos^2\alpha} g} = 0$$

Since $\dfrac{3C_\theta^2}{2m^2\cos^2\alpha\, s^4} > 0$ always,

$\lambda \in \mathrm{Im} \Rightarrow$ Stable