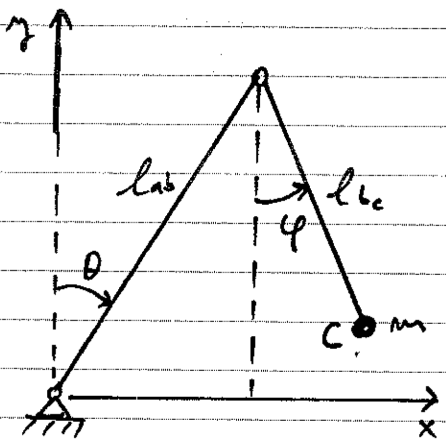
| opgave no. | | naam | |
|---|---|---|---|
| | | studienummer | |
| | | vak D&S | code |
| | | datum 24-6-08 | fac. |

**Gebruik voor elke opgave een afzonderlijk vel papier !**

1

a)




$$\left.\begin{aligned} x &= l_{ab}\sin\theta + l_{bc}\sin\varphi \\ y &= l_{ab}\cos\theta - l_{bc}\cos\varphi \end{aligned}\right\}$$

$$\left.\begin{aligned} \dot{x} &= l_{ab}\dot{\theta}\cos\theta + l_{bc}\dot{\varphi}\cos\varphi \\ \dot{y} &= -l_{ab}\dot{\theta}\sin\theta + l_{bc}\dot{\varphi}\sin\varphi \end{aligned}\right\}$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) =$$

$$= \frac{1}{2}m\left[(l_{ab}\dot{\theta}\cos\theta + l_{bc}\dot{\varphi}\cos\varphi)^2 + (-l_{ab}\dot{\theta}\sin\theta + l_{bc}\dot{\varphi}\sin\varphi)^2\right]$$

5

$$= \frac{1}{2}m\left[l_{ab}^2\dot{\theta}^2\cos^2\theta + 2l_{ab}l_{bc}\dot{\theta}\dot{\varphi}\cos\theta\cos\varphi + l_{bc}^2\dot{\varphi}^2\cos^2\varphi + l_{ab}^2\dot{\theta}^2\sin^2\theta - 2l_{ab}l_{bc}\dot{\theta}\dot{\varphi}\sin\theta\sin\varphi + l_{bc}^2\dot{\varphi}^2\sin^2\varphi\right]$$

$$= \frac{1}{2}m\left(l_{ab}^2\dot{\theta}^2 + l_{bc}^2\dot{\varphi}^2 + 2l_{ab}l_{bc}\dot{\theta}\dot{\varphi}\cos(\theta+\varphi)\right)$$

3

$$V = mg(l_{ab}\cos\theta - l_{bc}\cos\varphi)$$

$$L = T - V =$$

$$= \frac{1}{2}m\left(l_{ab}^2\dot{\theta}^2 + l_{bc}^2\dot{\varphi}^2 + 2l_{ab}l_{bc}\dot{\theta}\dot{\varphi}\cos(\theta+\varphi)\right) - mg\left(l_{ab}\cos\theta - l_{bc}\cos\varphi\right)$$

b)

$l_{ab} = 50\,m \quad l_{bc} = 25\,m \quad m = 4000\,kg \quad g = 10\,m/s$

$$L = \frac{1}{2}\,4000\left(50^2\dot\theta^2 + 25^2\dot\varphi^2 + 2\cdot 50\cdot 25\,\dot\theta\dot\varphi\cos(\theta+\varphi)\right)$$

$$- 4000\cdot 10\cdot(50\cos\theta - 25\cos\varphi)$$

5 $$= \left(5\dot\theta^2 + 1{,}25\dot\varphi^2 + 5\dot\theta\dot\varphi\cos(\theta+\varphi) - 2\cos\theta + \cos\varphi\right)\times 10^6$$

$$\left.\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0\\ \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\varphi}\right) - \frac{\partial L}{\partial\varphi} = 0\end{aligned}\right\}$$

$$\left.\begin{aligned}\frac{\partial L}{\partial\dot\theta} &= \left(10\dot\theta + 5\dot\varphi\cos(\theta+\varphi)\right)\times 10^6\\ \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) &= \left(10\ddot\theta + 5\left(\ddot\varphi\cos(\theta+\varphi) - \dot\varphi\sin(\theta+\varphi)(\dot\theta+\dot\varphi)\right)\right)\times 10^6\\ \frac{\partial L}{\partial\theta} &= \left(-5\dot\theta\dot\varphi\sin(\theta+\varphi) + 2\sin\theta\right)\times 10^6\end{aligned}\right\}$$

$$\left.\begin{aligned}\frac{\partial L}{\partial\dot\varphi} &= \left(2{,}5\dot\varphi + 5\dot\theta\cos(\theta+\varphi)\right)\times 10^6\\ \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\varphi}\right) &= \left(2{,}5\ddot\varphi + 5\left(\ddot\theta\cos(\theta+\varphi) - \dot\theta\sin(\theta+\varphi)(\dot\theta+\dot\varphi)\right)\right)\times 10^6\\ \frac{\partial L}{\partial\varphi} &= \left(-5\dot\theta\dot\varphi\sin(\theta+\varphi) - \sin\varphi\right)\times 10^6\end{aligned}\right\}$$

$$10\ddot{\theta} + 5\left(\ddot{\varphi}\cos(\theta+\varphi) - \dot{\varphi}\sin(\theta+\varphi)(\dot{\theta}+\dot{\varphi})\right) + 5\dot{\theta}\dot{\varphi}\sin(\theta+\varphi) - 2\sin\theta = 0$$

$$2{,}5\ddot{\varphi} + 5\left(\ddot{\theta}\cos(\theta+\varphi) - \dot{\theta}\sin(\theta+\varphi)(\dot{\theta}+\dot{\varphi})\right) + 5\dot{\theta}\dot{\varphi}\sin(\theta+\varphi) + \sin\varphi = 0$$

c)

$$\theta(t) = \frac{\pi}{6} + \frac{\pi}{18}\sin\frac{\pi t}{4}$$

$$\dot{\theta}(t) = \frac{\pi^2}{72}\cos\frac{\pi t}{4}$$

5

$$L = \left[5\frac{\pi^4}{72^2}\cos^2\frac{\pi t}{4} + 1{,}25\dot{\varphi}^2 + 5\frac{\pi^2}{72}\dot{\varphi}\cos\left(\frac{\pi}{6}+\frac{\pi}{18}\sin\frac{\pi t}{4}+\varphi\right) - 2\cos\left(\frac{\pi}{6}+\frac{\pi}{18}\sin\frac{\pi t}{4}\right) + \cos\varphi\right] \times 10^6$$

$$L = L(\dot{\varphi}, \varphi, t)$$

$$\frac{\partial L}{\partial \dot{\varphi}} = \left[2.5\dot{\varphi} + 5\frac{\pi^2}{72}\cos\left(\frac{\pi}{6}+\frac{\pi}{18}\sin\frac{\pi t}{4}+\varphi\right)\right] \times 10^6$$

2

$$h = \frac{\partial L}{\partial \dot{\varphi}}\dot{\varphi} - L =$$

$$= \left[2.5\dot{\varphi}^2 + 5\frac{\pi^2}{72}\dot{\varphi}\cos\left(\frac{\pi}{6}+\frac{\pi}{18}\sin\frac{\pi t}{4}+\varphi\right) - \frac{5\pi^4}{72^2}\cos^2\frac{\pi t}{4} - 1{,}25\dot{\varphi}^2 - \frac{5\pi^2}{72}\dot{\varphi}\cos\left(\frac{\pi}{6}+\frac{\pi}{18}\sin\frac{\pi t}{4}+\varphi\right) + 2\cos\left(\frac{\pi}{6}+\frac{\pi}{18}\sin\frac{\pi t}{4}\right) - \cos\varphi\right] \times 10^6$$

$$= \left[1{,}25\dot{\varphi}^2 - \frac{5\pi^4}{72^2}\cos^2\frac{\pi t}{4} + 2\cos\left(\frac{\pi}{6}+\frac{\pi}{18}\sin\frac{\pi t}{4}\right) - \cos\varphi\right] \times 10^6$$

d) No, because $L = L(t)$ and consequently

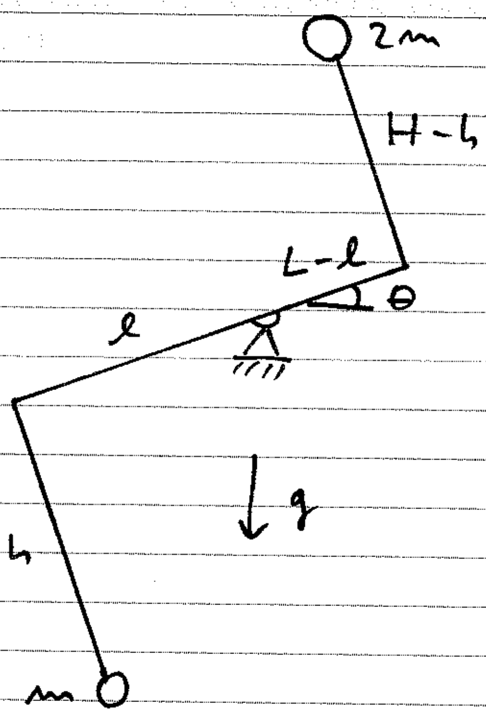
5

$$\frac{dh}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\dot{\varphi} - L\right) = -\frac{\partial L}{\partial t} \neq 0$$

so $h \neq$ Constant

(2)

a)




5 $$V = 2mg\left[(L-\ell)\sin\theta + (H-h)\cos\theta\right] - mg\,(\ell\sin\theta + h\cos\theta)$$

b)

$$V' = 2mg\left[(L-\ell)\cos\theta - (H-h)\sin\theta\right] - mg\,(\ell\cos\theta - h\sin\theta)$$

Equilibrium at $\theta = 0 \Rightarrow V'(0) = 0$

$$V'(0) = 2mg\left[(L-\ell)\cos 0 - (H-h)\sin 0\right] - mg\,(\ell\cos 0 - h\sin 0) =$$

$$= 2mg(L-\ell) - mg\ell = 0$$

5 $$\ell = \frac{2}{3}L$$

c)

$$V'' = 2mg\left[-(L-\ell)\sin\theta - (H-h)\cos\theta\right] - mg\left(-\ell\sin\theta - h\cos\theta\right)$$

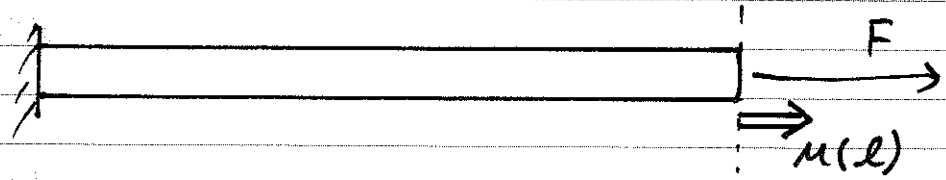
$$V''(0) = 2mg\left(-(H-h)\right) - mg(-h) > 0$$

$$-2H + 2h + h > 0$$

5 $$h > \frac{2}{3}H$$

**3**

a)




$$\delta W = F\,\delta u(l)$$

$$\delta W = -\delta V^{gen} = F\,\delta u(l)$$

$$\delta V^{gen} = -F\,\delta u(l)$$

5 $$V^{gen} = -F\,u(l)$$

b)

$$V = V^e + V^{gen} = \int_0^l \frac{1}{2} EA\, u'^2\, dx - F\,u(l)$$

Ritz's method:

$$u(x) = \alpha \cdot 1 + \beta x + \gamma x^2 \qquad u'(x) = \beta + 2\gamma x$$

$$u(0) = 0 \Rightarrow \alpha = 0$$

$$V = \int_0^l \frac{1}{2} EA(\beta + 2\gamma x)^2\, dx - F\cdot(\beta l + \gamma l^2)$$

$$= \int_0^l \frac{1}{2} EA(\beta^2 + 4\beta\gamma x + 4\gamma^2 x^2)\, dx - F(\beta l + \gamma l^2)$$

5

$$= \left[\frac{1}{2} EA\left(\beta^2 x + 4\beta\gamma \frac{x^2}{2} + 4\gamma^2 \frac{x^3}{3}\right)\right]_0^l - F(\beta l + \gamma l^2) =$$

$$= \frac{1}{2} EA\left(\beta^2 \ell + 2\beta\gamma \ell^2 + \frac{4}{3}\gamma^2 \ell^3\right) - F(\beta\ell + \gamma\ell^2)$$

$$\left.\begin{aligned} \frac{\partial V}{\partial \beta} &= EA(\beta\ell + \gamma\ell^2) - F\ell = 0 \\ \frac{\partial V}{\partial \gamma} &= EA\left(\beta\ell^2 + \frac{4}{3}\gamma\ell^3\right) - F\ell^2 = 0 \end{aligned}\right\}$$

$$\beta = \frac{\begin{vmatrix} F\ell & EA\ell^2 \\ F\ell^2 & \frac{4EA}{3}\ell^3 \end{vmatrix}}{\begin{vmatrix} EA\ell & EA\ell^2 \\ EA\ell^2 & \frac{4EA}{3}\ell^3 \end{vmatrix}} = \frac{\frac{1}{3} F\, EA\,\ell^4}{\frac{1}{3}(EA)^2\ell^4} = \frac{F}{EA}$$

$$\gamma = \frac{\begin{vmatrix} EA\ell & F\ell \\ EA\ell^2 & F\ell^2 \end{vmatrix}}{\begin{vmatrix} EA\ell & EA\ell^2 \\ EA\ell^2 & \frac{4EA}{3}\ell^3 \end{vmatrix}} = \frac{0}{\frac{1}{3}(EA)^2\ell^4} = 0$$

10

$$u(x) = \frac{Fx}{EA}$$

c) The exact solution for this bar in tension is anyway $u(x) = \frac{Fx}{EA}$, which can be viewed as a polynomial (AE1-914 matter)

The approximated solution obtained through Ritz's method coincides with the exact one for this reason, that is why $\gamma = 0$ 
5 anyway. If higher order terms had been included, their corresponding coefficient would have been 0 as well, as the exact solution would have been found though.

(4)

$$\ddot{x} + \dot{x} + \sqrt{2}(\cos x - \sin x) = 0$$

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -y - \sqrt{2}(\cos x - \sin x) \end{aligned}\right\}$$

Equilibrium points:

$$\dot{x} = \dot{y} = 0$$

5 $$\left.\begin{aligned} y &= 0 \\ \cos x - \sin x &= 0 \end{aligned}\right\} \quad \tan x = 1 \Rightarrow x = \begin{cases} \pi/4 \\ 5\pi/4 \end{cases}$$

Stability:

$x = \frac{\pi}{4}$

Linearisation:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \sqrt{2}(\sin x + \cos x) & -1 \end{bmatrix}_{x=\frac{\pi}{4}} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

5 $$\begin{vmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{vmatrix} = 0 = \lambda^2 + \lambda - 2 \qquad \lambda = \frac{-1 \pm \sqrt{1+8}}{2}$$

$$\lambda = \begin{cases} 1 > 0 \Rightarrow \underline{\text{unstable}} \\ -2 < 0 \end{cases}$$

$$x = \frac{5\pi}{4}$$

Linearisation:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \sqrt{2}\,(\sin x + \cos x) & -1 \end{bmatrix}\Bigg|_{x=\frac{5\pi}{4}} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

5 $$\begin{vmatrix} -\lambda & 1 \\ -2 & -1-\lambda \end{vmatrix} = 0 = \lambda^2 + \lambda + 2$$

$$\lambda = \frac{-1 \pm \sqrt{1-8}}{2} = -\frac{1}{2} \pm \sqrt{\frac{-7}{4}} \in \mathbb{C}$$

with $\mathrm{Re}(\lambda) < 0$

$\Rightarrow$ <u>stable</u>

5

a) $$\frac{\partial T}{\partial \dot{\theta}} = I\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = I\ddot{\theta}$$

$$\frac{\partial T}{\partial \theta} = I\dot{\phi}^2 \sin\theta\cos\theta + I_s(\dot{\phi}\cos\theta + \dot{\psi})\dot{\phi}(-\sin\theta)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = 0$$

$$I\ddot{\theta} - I\dot{\phi}^2\sin\theta\cos\theta + I_s\dot{\phi}(\dot{\phi}\cos\theta + \dot{\psi})\sin\theta = 0$$

Steady motion:

$$\ddot{\theta} = \dot{\theta} = 0$$

$$I\dot{\phi}\cos\theta = I_s(\dot{\phi}\cos\theta + \dot{\psi})$$

$$(I - I_s)\dot{\phi}\cos\theta = I_s\dot{\psi}$$

10 $$\boxed{\dot{\phi}\cos\theta = \frac{I_s}{I - I_s}\dot{\psi}}$$

b)

$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

5 $$\omega_3 = \frac{I_s}{I - I_s}\dot{\psi} + \dot{\psi} = \frac{I}{I - I_s}\dot{\psi}$$

$$\dot{\psi} = \frac{I - I_s}{I}\,\omega_3$$

c)

$$I = I_1 = I_2 = \frac{1}{5}\,5{,}976\times10^{24}\cdot\left(\left(6378\cdot10^3\right)^2 + \left(6356\cdot10^3\right)^2\right)$$

$$= 9{,}6904\times10^{37}\ \text{kg m}^2$$

$$I_s = I_3 = \frac{1}{5}\,5{,}976\times10^{24}\left(2\cdot\left(6378\cdot10^3\right)^2\right)$$

$$= 9{,}7239\times10^{37}\ \text{kg m}^2$$

$$\omega_3 = 2\pi\ \text{rad/day}\cdot\frac{1\ \text{day}}{24\text{h}}\cdot\frac{1\text{h}}{3600\text{s}} = 7{,}2722\times10^{-5}\ \text{rad/s}$$

5 $$\dot{\psi} = \frac{I - I_s}{I}\,\omega_3 = -2{,}5140\times10^{-7}\ \text{rad/s}$$

$$\dot{\phi} = \frac{\omega_3 - \dot{\psi}}{\cos 0} = 7{,}2973\times10^{-5}\ \text{rad/s}$$