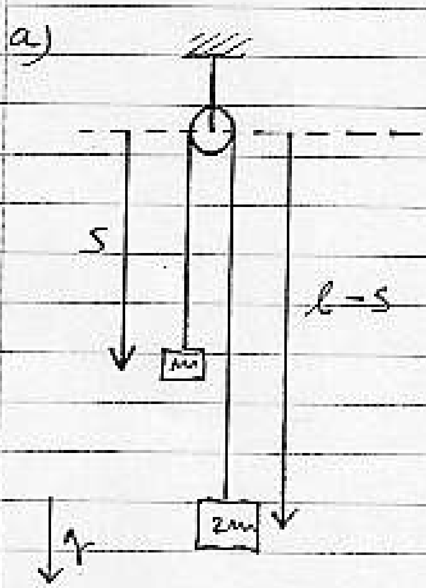
opgave no.		naam	
		studienummer	
		vak AE3-914	code
		datum 27-3-08	fac.

Gebruik voor elke opgave een afzonderlijk vel papier !

1 a)



Generalised coordinate: s

$$T = \frac{1}{2} m \dot{s}^2 + \frac{1}{2} 2m (\dot{l} - \dot{s})^2$$

$$= \frac{1}{2} m \dot{s}^2 + m \dot{s}^2$$

$$= \frac{3}{2} m \dot{s}^2$$

$$V = -mgs - 2mg(l - s)$$

$$= mgs \cancel{- 2mgl}$$

$$L = T - V = \frac{3}{2} m \dot{s}^2 - mgs$$

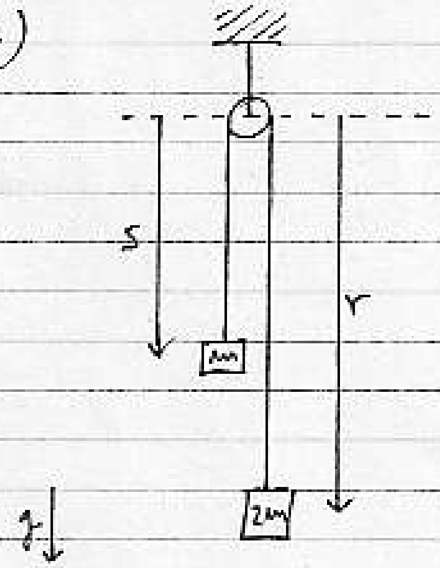
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = 0$$

$$\frac{\partial L}{\partial \dot{s}} = 3m\dot{s} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = 3m\ddot{s} \qquad \frac{\partial L}{\partial s} = -mg$$

$$3m\ddot{s} + mg = 0$$

$$\boxed{\ddot{s} + g/3 = 0}$$

b)



Generalised coordinates: s, r

Constraint:

$$f(s,r) = s + r - \ell = 0$$

$$T = \frac{1}{2} m \dot{s}^2 + \frac{1}{2} 2m \dot{r}^2$$
$$= \frac{1}{2} m \dot{s}^2 + m \dot{r}^2$$

$$V = -mgs - 2mgr$$

$$L = T - V = \frac{1}{2} m \dot{s}^2 + m\dot{r}^2 + mgs + 2mgr$$

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} &= \lambda \frac{\partial f}{\partial s} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} &= \lambda \frac{\partial f}{\partial r} \\ f(s, r) &= 0 \end{aligned}\right\}$$

$$\frac{\partial L}{\partial \dot{s}} = m\dot{s} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = m\ddot{s} \quad \frac{\partial L}{\partial s} = mg \quad \frac{\partial f}{\partial s} = 1$$

$$\frac{\partial L}{\partial \dot{r}} = 2m\dot{r} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 2m\ddot{r} \quad \frac{\partial L}{\partial r} = 2mg \quad \frac{\partial f}{\partial r} = 1$$

$$\left.\begin{aligned} m\ddot{s} - mg &= \partial \\ 2m\ddot{r} - 2mg &= \partial \\ s + r &= \ell \end{aligned}\right\}$$

Problem 2

General expression for velocity.

$$\underline{v} = \underline{v}_{xyz} + \underline{\omega} \times \underline{r}_{rel} + \underline{v}_{rel}$$

Rigid Body: $\Rightarrow$ $\underline{v}_{rel} = \underline{0}$

Origin fixed: $\underline{v}_{xyz} = \underline{0}$ $\underline{r}_{rel} = \underline{r}$

$$\underline{v} = \underline{\omega} \times \underline{r}$$

$$T = \frac{1}{2}\int_V \underline{v} \cdot \underline{v}\, dm = \frac{1}{2}\int_V (\underline{\omega} \times \underline{r}) \cdot (\underline{\omega} \times \underline{r})\, dm$$

$$= \frac{1}{2}\int_V \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix}^2 dm$$

$$= \frac{1}{2}\int_V \left[(\omega_y z - \omega_z y)\,\underline{i} + (\omega_z x - \omega_x z)\,\underline{j} + (\omega_x y - \omega_y x)\,\underline{k}\right]^2 dm$$

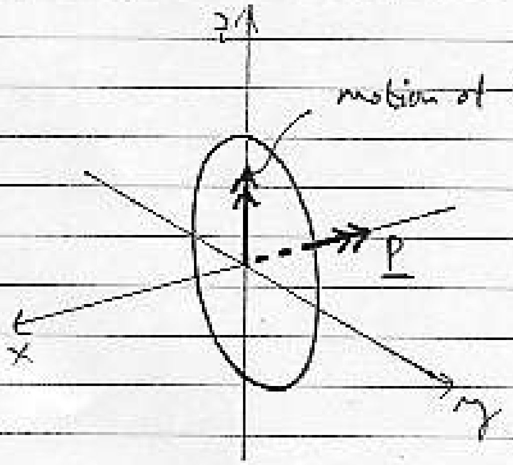
$$= \frac{1}{2}\int_V \Big[(y^2+z^2)\omega_x^2 + (x^2+z^2)\omega_y^2 + (x^2+y^2)\omega_z^2$$
$$-2xy\,\omega_x\omega_y - 2xz\,\omega_x\omega_z - 2yz\,\omega_y\omega_z\Big]\, dm$$

$$= \frac{1}{2}\begin{pmatrix}\omega_x & \omega_y & \omega_z\end{pmatrix}\int_V \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} dm \begin{pmatrix}\omega_x \\ \omega_y \\ \omega_z\end{pmatrix}$$

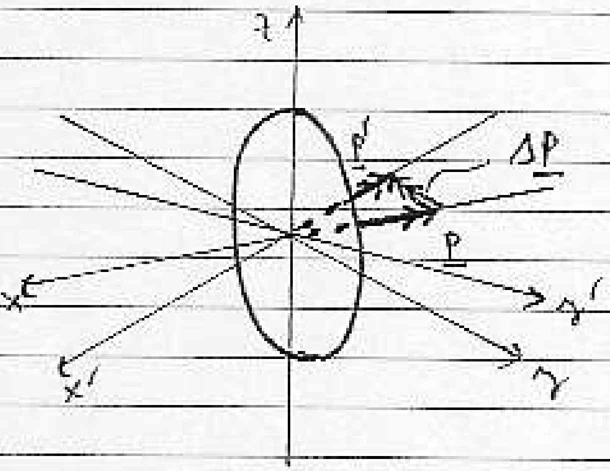
$$= \frac{1}{2}\,\underline{\omega}^T \underline{I}_0\, \underline{\omega}$$

3

Angular momentum diagram:

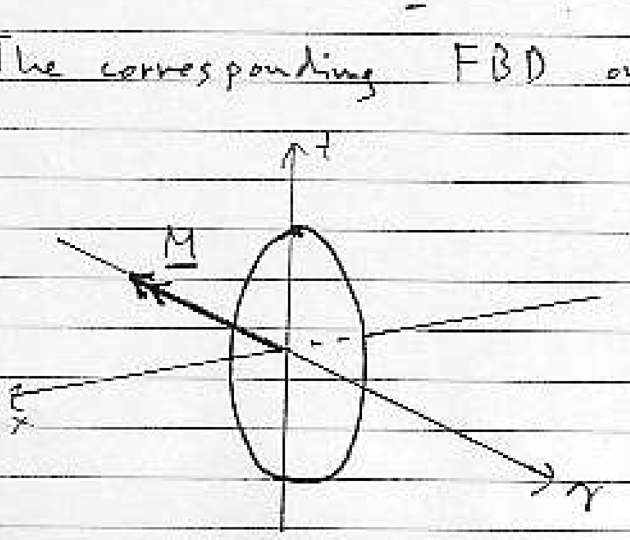


Change in angular momentum due to motion of the shaft:

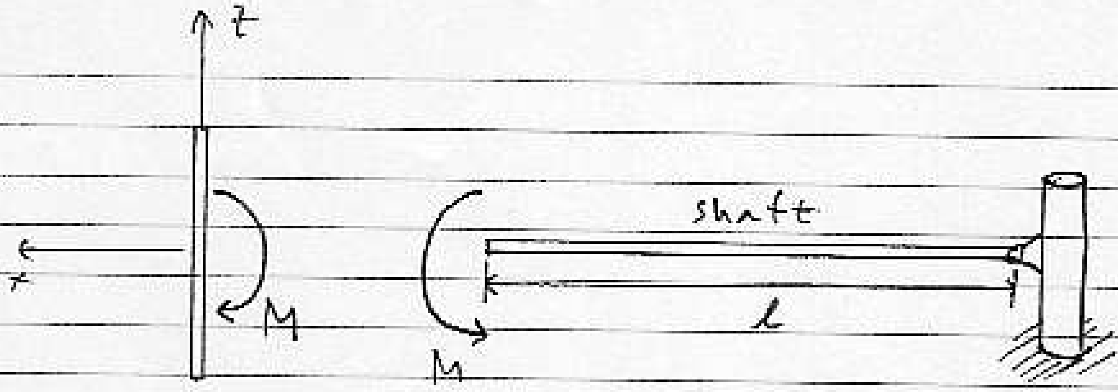


When $\Delta t \to 0$ $\Delta \underline{p}$ is parallel to the y-axis.

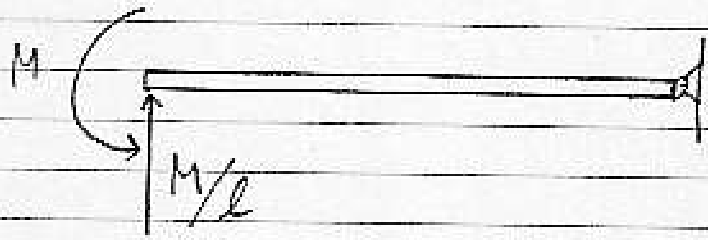
The corresponding FBD on the disk is:



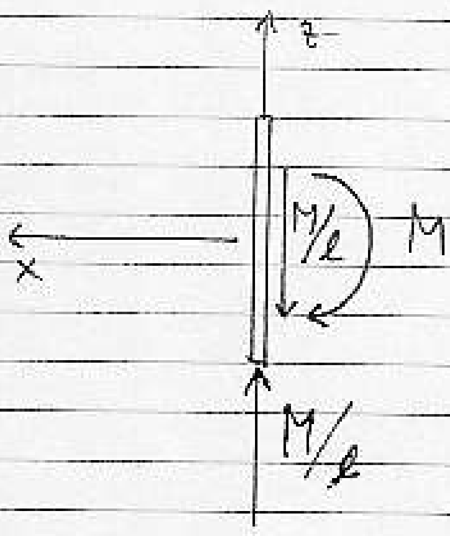
see overleaf



This corresponds to a CCW moment applied to the shaft. This needs to be equilibrated by an upward reaction force amounting to M/l, with l being the length of the shaft.

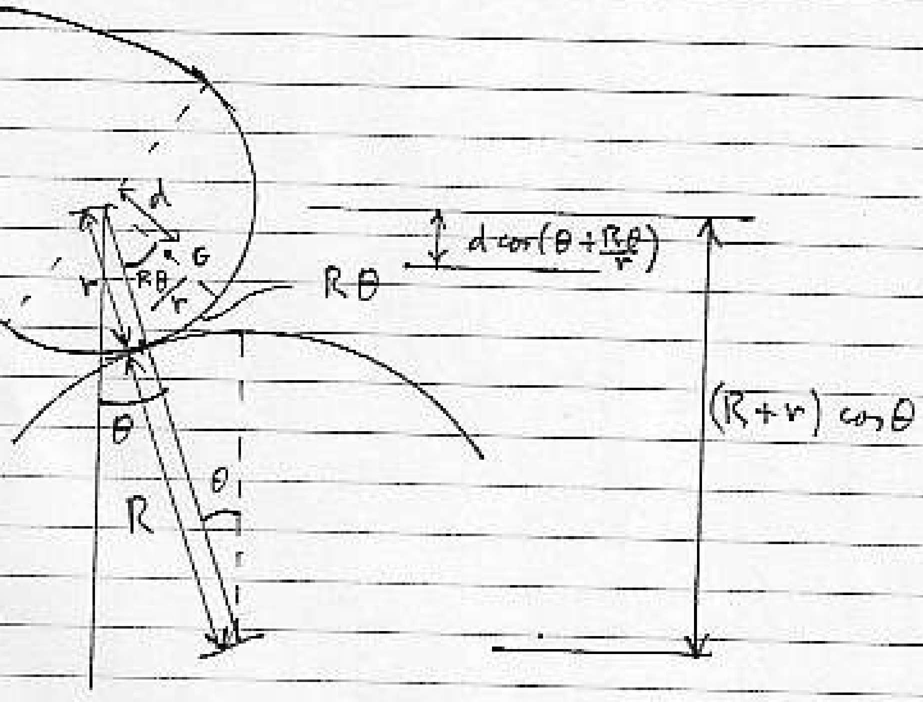


Consequently, the static reaction at the contact point is increased by M/l



4

a)



$$V = mg\left[(R+r)\cos\theta - d\cos\left(\frac{R+r}{r}\theta\right)\right]$$

b) $R = 0{,}012\,m$, $r = 0{,}021\,m$, $m = 0{,}05\,kg$, $g = 10\,m/s^2$

$$V = 0{,}5\,(0{,}063\cos\theta - d\cos 3\theta)$$

$$V' = 0{,}5\,(-0{,}063\sin\theta + 3d\sin 3\theta)$$

$V'(0) = 0 \Rightarrow \theta$ is equilibrium point

$$V'' = 0{,}5\,(-0{,}063\cos\theta + 9d\cos 3\theta)$$

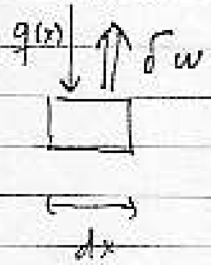
$V''(0) = 0{,}5\,(-0{,}063 + 9d) > 0$ for stability

$$\boxed{d > 0{,}007\,m}$$

5 a)

The distributed load provides a virtual work density $\delta \mathcal{W}^e$ as

$$\delta \mathcal{W}^e = -q(x)\,\delta w = -c\,w_{xx}\,\delta w$$



It is not immediate to devise a generalised potential density that would lead to that virtual work density upon taking a variation.

An experienced analyst would propose the expression

$$\mathcal{V}^{\gamma} = -\frac{1}{2}\,c\,w_x^2$$

after stating an analogy between the second spatial derivative vs. the squared first derivative, and the acceleration vs. the kinetic energy on a single-degree-of-freedom system.

When one is unable to find $\mathcal{V}^{\gamma}$ the virtual work density is incorporated to the variation of the Lagrangian when applying Hamilton's principle, see answer to question c.

b)

Kinetic energy density:

$$\mathcal{T} = \frac{1}{2}\rho A w_t^2$$

Elastic potential energy density:

$$\mathcal{V}^e = \frac{1}{2} EI w_{xx}^2$$

Lagrangian density:

$$\mathcal{L} = \mathcal{T} - \mathcal{V}^e = \frac{1}{2}\rho A w_t^2 - \frac{1}{2} EI w_{xx}^2$$

Remark:

If the analyst has been capable of finding a generalised potential density as shown in question a) then this should be added to the Lagrangian density as

$$\mathcal{L} = \mathcal{T} - \mathcal{V}^e - \mathcal{V}^g = \frac{1}{2}\rho A w_t^2 - \frac{1}{2} EI w_{xx}^2 + \frac{1}{2} c w_x^2$$

c) Consider a time interval $(t_1, t_2]$, the motion is such that the action is stationary:

$$\delta I(w) = \int_{t_1}^{t_2} (\delta L + \delta W)\, dt = 0$$

Remark: if a generalised potential has been found this is incorporated into L and then $\delta W \equiv 0$

$$\delta I(w) = \int_{t_1}^{t_2}\int_0^{l} (\delta \mathcal{L} + \delta \mathcal{W})\, dx\, dt = 0$$

$$= \int_{t_1}^{t_2}\int_0^{l} \left\{ \delta\left[\frac{1}{2}\rho A w_t^2 - \frac{1}{2} EJ w_{xx}^2\right] + \left[-c w_{xx}\, \delta w\right]\right\} dx\,dt$$

$$= \int_{t_1}^{t_2}\int_0^{l} \left(\delta \mathcal{L}(w_t, w_{xx}) - c w_{xx}\, \delta w\right) dx\, dt$$

$$= \int_{t_1}^{t_2}\int_0^{l} \left(\frac{\partial \mathcal{L}}{\partial w_t}\delta w_t + \frac{\partial \mathcal{L}}{\partial w_{xx}}\delta w_{xx} - c w_{xx}\, \delta w\right) dx\,dt$$

$$= \int_0^{l} \left[\frac{\partial \mathcal{L}}{\partial w_t}\delta w\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial w_t}\right)\delta w\, dt\right] dx \; +$$

$$+ \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial w_{xx}}\delta w_x\Big|_0^{l} - \int_0^{l} \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial w_{xx}}\right)\delta w_x\, dx\right] dt + \int_{t_1}^{t_2}\int_0^{l} -c w_{xx}\, \delta w\, dx\, dt$$

$$= \int_0^l \frac{\partial \mathcal{L}}{\partial w_t} \delta w \Big|_{t_1}^{t_2} dx - \int_0^l \int_{t_1}^{t_2} \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial w_t}\right) \delta w \, dt \, dx$$

$$+ \int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial w_{xx}} \delta w_x \Big|_0^l dt - \int_{t_1}^{t_2} \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial w_{xx}}\right) \delta w \Big|_0^l dt$$

$$+ \int_{t_1}^{t_2}\int_0^l \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial w_{xx}}\right) \delta w \, dx \, dt - \int_{t_1}^{t_2}\int_0^l c\, w_{xx} \delta w \, dx \, dt$$

$$= \int_{t_1}^{t_2}\int_0^l \left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial w_{xx}}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial w_t}\right) - c\, w_{xx}\right] \delta w \, dx \, dt$$

+ BOUNDARY TERMS (NOT CONSIDERED IN THIS PROBLEM)

$\delta I = 0$ for any δw $\Rightarrow$

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial w_{xx}}\right) - c\, w_{xx} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial w_t}\right) = 0$$

$$\boxed{\rho A w_{tt} + c\, w_{xx} + EJ\, w_{xxxx} = 0}$$

EQUATION OF MOTION