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1

$$\frac{dh}{dt} = \frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right) = \ddot{q}\frac{\partial L}{\partial \dot{q}} + \dot{q}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{dL}{dt} =$$

$$= \ddot{q}\frac{\partial L}{\partial \dot{q}} + \dot{q}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial \dot{q}}\ddot{q} - \frac{\partial L}{\partial q}\dot{q} =$$

$$= \ddot{q}\underbrace{\left(\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial \dot{q}}\right)}_{=0} + \dot{q}\underbrace{\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\right)}_{=0} =$$

$$= 0 \qquad \text{q.e.d.}$$

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2 a. The potential energy of a system must attain a strict relative minimum on a stable equilibrium configuration.

b.

$$V = -mg\,2l\sin\theta + mgl\cos 2\theta$$

$$= mgl(\cos 2\theta - 2\sin\theta)$$

Equilibrium:

$$\frac{dV}{d\theta} = 0 \Rightarrow mgl(-2\sin 2\theta - 2\cos\theta) = 0$$

$$2\sin\theta\cos\theta + \cos\theta = 0$$

$$\cos\theta(2\sin\theta + 1) = 0$$

$$\cos\theta = 0 \Rightarrow \theta = \pm 90^\circ$$

$$\sin\theta = -\frac{1}{2} \Rightarrow \theta = \begin{cases} 210^\circ \\ 330^\circ \end{cases}$$

Stability:

$$\frac{d^2V}{d\theta^2} = (-4\cos 2\theta + 2\sin\theta)mgl$$

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$\theta = +90°$: $\dfrac{d^2V}{d\theta^2} = (-4\cos(180) + 2\sin(90))mgl =$

$= 6mgl > 0 \Rightarrow$ minimum

STABLE

$\theta = -90°$: $\dfrac{d^2V}{d\theta^2} = (-4\cos(-180) + 2\sin(-90))mgl =$

$= 2mgl > 0 \Rightarrow$ minimum

STABLE

$\theta = 210°$: $\dfrac{d^2V}{d\theta^2} = (-4\cos(420) + 2\sin(210))mgl =$

$= -3mgl < 0 \Rightarrow$ maximum

UNSTABLE

$\theta = 330°$: $\dfrac{d^2V}{d\theta^2} = (-4\cos(660) + 2\sin(330))mgl =$

$= -3mgl < 0 \Rightarrow$ maximum

UNSTABLE

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3 a.

$I_{xy} = I_{xz} = I_{yz} = 0$ because of symmetry

$I_{xx} = \frac{1}{3} m l^2 + 4 m l^2 = \frac{13}{3} m l^2$

$I_{yy} = 2 \cdot \frac{1}{3} (2m)(2l)^2 = \frac{16}{3} m l^2$

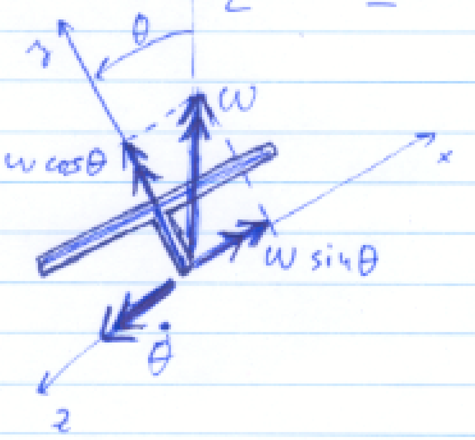
$I_{zz} = \underbrace{I_{xx} + I_{yy}}_{} = \frac{29}{3} m l^2$

↳ because it is a flat body in xy

$$\underline{\underline{I}}_o = \begin{bmatrix} \frac{13}{3} & 0 & 0 \\ 0 & \frac{16}{3} & 0 \\ 0 & 0 & \frac{29}{3} \end{bmatrix} m l^2$$

b.

$T = \frac{1}{2} \underline{\omega}^T \underline{\underline{I}}_o \underline{\omega}$



$$\underline{\omega} = \begin{pmatrix} \omega \sin\theta \\ \omega \cos\theta \\ \dot{\theta} \end{pmatrix}$$

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$$T = \frac{1}{2} m l^2 \begin{bmatrix} \omega \sin\theta & \omega\cos\theta & \dot\theta \end{bmatrix} \begin{bmatrix} \frac{13}{3} & 0 & 0 \\ 0 & \frac{16}{3} & 0 \\ 0 & 0 & \frac{29}{3} \end{bmatrix} \begin{bmatrix} \omega\sin\theta \\ \omega\cos\theta \\ \dot\theta \end{bmatrix}$$

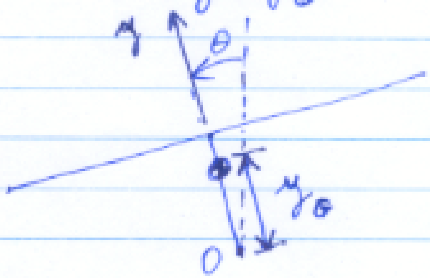
$$= \frac{1}{2} m l^2 \left(\frac{13}{3}\omega^2 \sin^2\theta + \frac{16}{3}\omega^2\cos^2\theta + \frac{29}{3}\dot\theta^2 \right) =$$

$$= \frac{1}{2} m l^2 \left(\frac{13}{3}\omega^2 + \omega^2\cos^2\theta + \frac{29}{3}\dot\theta^2 \right)$$

$$= m l^2 \left[\frac{29}{6}\dot\theta^2 + \left(\frac{13}{6} + \frac{1}{2}\cos^2\theta \right)\omega^2 \right]$$

c.

$$L = T - V \qquad V = 5mg\, y_G \cos\theta$$



$$y_G = \left(4ml + m\frac{l}{2}\right) \Big/ 5m$$

$$= \frac{9l}{10}$$

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$$V = 5mg\frac{9l}{10}\cos\theta = \frac{9}{2}mgl\cos\theta$$

$$L = ml^2\left[\frac{29}{6}\dot{\theta}^2 + \left(\frac{13}{6} + \frac{1}{2}\cos^2\theta\right)\omega^2\right] - \frac{9}{2}mgl\cos\theta$$

Equation of motion:

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{29}{3}ml^2\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{29}{3}ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -ml^2\sin\theta\cos\theta\,\omega^2 + \frac{9}{2}mgl\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{29}{3}ml^2\ddot{\theta} + ml^2\sin\theta\cos\theta\,\omega^2 - \frac{9}{2}mgl\sin\theta = 0$$

$$\boxed{\frac{29}{3}\ddot{\theta} + \frac{1}{2}(\sin 2\theta)\,\omega^2 - \frac{9}{2}\frac{g}{l}\sin\theta = 0}$$

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d. For $\theta = 0$ $\ddot{\theta} = 0$, thus $\theta = 0$ is an equilibrium position indeed.

Stability:

Equation of motion was found in c, stability analysis by linearisation is convenient.

$$\left.\begin{aligned} \dot{\theta} &= \varphi \\ \dot{\varphi} &= \frac{3}{2q}\left(-\frac{1}{2}(\sin 2\theta)\omega^2 + \frac{9g}{2\ell}\sin\theta\right) \end{aligned}\right\}$$

Linearisation for $\theta = 0$:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\varphi} \end{bmatrix} = \left.\begin{bmatrix} 0 & 1 \\ \frac{3}{2q}\left(-(\cos 2\theta)\,\omega^2 + \frac{9g}{2\ell}\cos\theta\right) & 0 \end{bmatrix}\right|_{\theta=0} \begin{pmatrix} \theta \\ \varphi \end{pmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ \frac{3}{2q}\left(\frac{9g}{2\ell} - \omega^2\right) & 0 \end{bmatrix} \begin{pmatrix} \theta \\ \varphi \end{pmatrix}$$

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$$\begin{vmatrix} -\lambda & 1 \\ \frac{3}{2a}\left(\frac{9g}{2l}-\omega^2\right) & -\lambda \end{vmatrix} = \lambda^2 - \frac{3}{2a}\left(\frac{9g}{2l}-\omega^2\right) = 0$$

$$\lambda \in \mathrm{Im} \Leftrightarrow \frac{9g}{2l} - \omega^2 < 0$$

Stable Equilibrium $\Leftrightarrow \omega^2 > \dfrac{9g}{2l}$

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4 a.

$$\omega_1 = \Omega_1 \qquad \omega_2 = \Omega_2 \qquad \omega_3 = \Omega_3 + p$$

$$\dot{\omega}_1 = \dot{\Omega}_1 \qquad \dot{\omega}_2 = \dot{\Omega}_2 \qquad \dot{\omega}_3 = \dot{\Omega}_3 + \dot{p}$$

$$\dot{p} = 0$$

$$\underline{0} = \frac{d\underline{\Omega}}{dt} = \dot{\Omega}_1 \underline{i} + \dot{\Omega}_2 \underline{j} + \dot{\Omega}_3 \underline{k} + \underline{\omega} \times \underline{\Omega}$$

$$= \dot{\Omega}_1 \underline{i} + \dot{\Omega}_2 \underline{j} + \dot{\Omega}_3 k + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \Omega_1 & \Omega_2 & \Omega_3 + p \\ \Omega_1 & \Omega_2 & \Omega_3 \end{vmatrix}$$

$$= (\dot{\Omega}_1 - \Omega_2 p)\underline{i} + (\dot{\Omega}_2 + \Omega_1 p)\underline{j} + \dot{\Omega}_3 \underline{k} = \underline{0}$$

$$\dot{\Omega}_1 = \Omega_2 p \qquad \dot{\Omega}_2 = -\Omega_1 p \qquad \dot{\Omega}_3 = 0$$

Consequently

$$\dot{\omega}_1 = \Omega_2 p \qquad \dot{\omega}_2 = -\Omega_1 p \qquad \dot{\omega}_3 = 0$$

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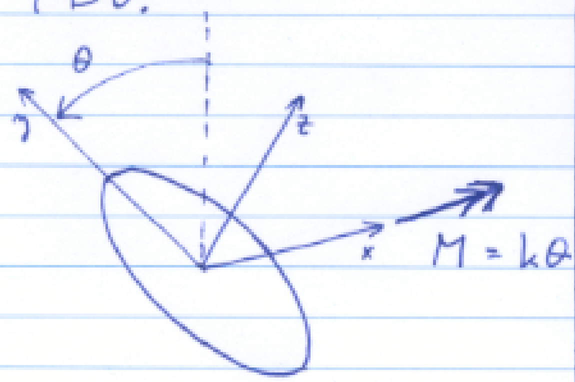
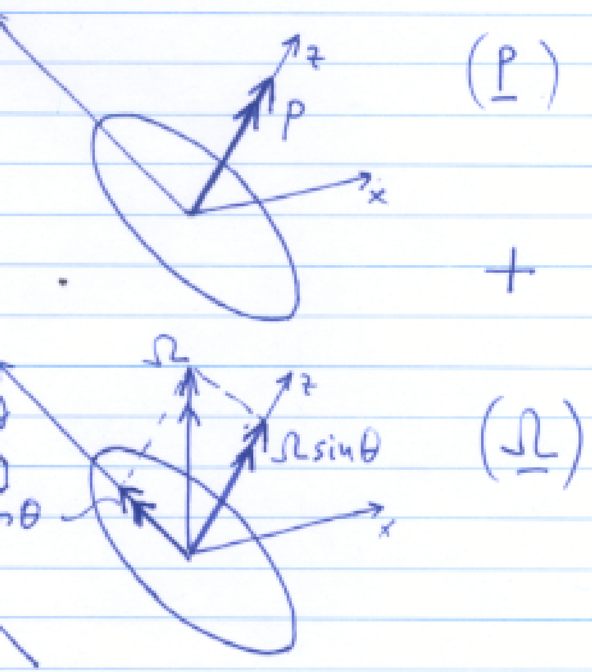
b.

FBD:



ANGULAR VELOCITIES:



$\Omega_1 = 0; \quad \Omega_2 = \Omega\cos\theta; \quad \Omega_3 = \Omega\sin\theta$

$\omega_1 = 0; \quad \omega_2 = \Omega\cos\theta; \quad \omega_3 = \Omega\sin\theta + p$

$\dot{\omega}_1 = \Omega\cos\theta\, p; \quad \dot{\omega}_2 = 0; \quad \dot{\omega}_3 = 0$

$M_1 = k\theta; \quad M_2 = 0; \quad M_3 = 0; \quad I_1 = I_2 = \frac{1}{4}mR^2; \quad I_3 = \frac{1}{2}mR^2$

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = M_1$$
$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_1\omega_3 = M_2 \longrightarrow 0 = 0$$
$$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = M_3 \longrightarrow 0 = 0$$

$$\hookrightarrow \frac{1}{4}mR^2\Omega\cos\theta\, p - \left[\left(\frac{1}{4} - \frac{1}{2}\right)mR^2\right]\Omega\cos\theta(\Omega\sin\theta + p) = k\theta$$

$$\frac{1}{2}mR^2\Omega\cos\theta\, p + \underbrace{\frac{1}{4}mR^2\Omega^2\sin\theta\cos\theta}_{\text{neglected because of } \Omega^2} = k\theta$$

In de Onderwijs- en examenregeling is vastgelegd dat tentamenuitslagen binnen 20 werkdagen zullen worden gepubliceerd.
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$$\frac{1}{2} m R^2 \Omega \cos\theta \, p = k\theta$$

$$\Omega = \frac{k\theta}{\frac{1}{2} m R^2 p \cos\theta}$$

c.

$m = 0{,}2$ kg $\quad R = 0{,}05$ m $\quad p = 1000\ s^{-1}$

$k = 6{,}25 \times 10^{-3}$ Nm and $\theta = 60°$

$\Longrightarrow \quad \Omega = 3°\ s^{-1}$ (standard turn!)

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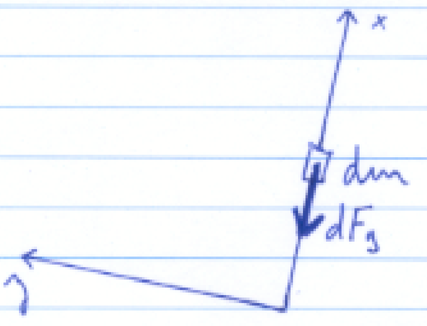
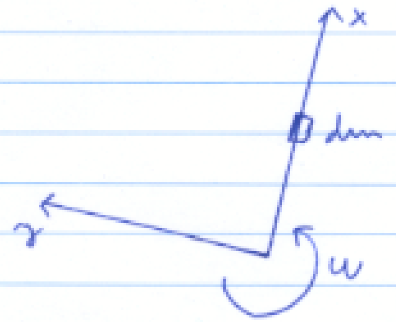
(5) a. Actual forces: Gravity



$$d\underline{F}_g = -dm \cdot g\,\underline{i}$$

$$= -\rho A \frac{k}{x^2}\,\underline{i}\,dx$$

Fictitious forces: Centrifugal force



$$\underline{\omega} = \omega\underline{k}$$

$$\underline{r}_{rel} = x\,\underline{i}$$

$$\underline{v}_{rel} = 0$$

$$d\underline{F}_{fict} = -dm\left(\underbrace{\dot{\underline{\omega}}\times\underline{r}_{rel}}_{0} + \underline{\omega}\times(\underline{\omega}\times\underline{r}_{rel}) + \underbrace{2(\underline{\omega}\times\underline{v}_{rel})}_{0}\right)$$

$$\underline{\omega}\times(\underline{\omega}\times\underline{r}_{rel}) = \underline{\omega}\times\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 0 & 0 & \omega\\ x & 0 & 0\end{vmatrix} = \underline{\omega}\times(\omega x\,\underline{j})$$

$$= \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 0 & 0 & \omega\\ 0 & \omega x & 0\end{vmatrix} = -\omega^2 x\,\underline{i}$$

$$d\underline{F}_{fict} = +dm\,\omega^2 x\,\underline{i} = \rho A\omega^2 x\,\underline{i}\,dx$$

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Total forces (as measured on Earth-tower system)

$$d\underline{F} = d\underline{F}_g + d\underline{F}_{fict} = \left(-\rho A \frac{k}{x^2} + \rho A \omega^2 x\right) dx\, \underline{i}$$

which is in the x-direction.

$$dF(x) = dF_x(x) = \rho A\left(\omega^2 x - \frac{k}{x^2}\right) dx$$

b. The generalised coordinate for the motion of a point of the tower is $u(x,t)$.

For a differential of mass one has

$$-\frac{\partial}{\partial u}\left(dV_{gen}\right) = dF$$

dF does not depend on $u(x,t)$, so we can write:

$$dV_{gen} = -dF \cdot u$$

in other words

$$dV_{gen} = -\rho A\left(\omega^2 x - \frac{k}{x^2}\right) u \, dx$$

5

c.

$$dL = dT - dV$$

$$dT = \frac{1}{2} v_{rel}^2 \, dm = \frac{1}{2} \rho A u_t^2 \, dx \qquad \left[u_t = \frac{\partial u}{\partial t} \right]$$

$$dV = dV_{el} + dV_{gen}$$

$$dV_{el} = \frac{1}{2} EA \varepsilon^2 dx = \frac{1}{2} EA u_x^2 dx \qquad \left[u_x = \frac{\partial u}{\partial x} \right]$$

$$dV = \left[\frac{1}{2} EA u_x^2 - \rho A \left(\omega^2 x - \frac{k}{x^2} \right) u \right] dx$$

$$\boxed{dL = \left[\frac{1}{2} \rho A u_t^2 - \frac{1}{2} EA u_x^2 + \rho A \left(\omega^2 x - \frac{k}{x^2} \right) u \right] dx}$$

d.

The basis of the tower is fixed to the Earth, which is considered to be rigid. Consequently,

$$u(R_e, t) = 0$$

which implies $\delta u(R_e, t) = 0$.

This is an ESSENTIAL boundary condition

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e.

$$I(u) = \int_{t_a}^{t_b}\int_{R_e}^{R_g} dL\, dt$$

$$= \int_{t_a}^{t_b}\int_{R_e}^{R_g} \left[\frac{1}{2}\rho A u_t^2 - \frac{1}{2} EA u_x^2 + \rho A\left(\omega^2 x - \frac{K}{x^2}\right) u\right] dx\,dt$$

$$= \int_{t_a}^{t_b}\int_{R_e}^{R_g} \phi(x, u, u_x, u_t)\, dx\,dt$$

$$\delta I(u) = \int_{t_a}^{t_b}\int_{R_e}^{R_g} \delta\phi(x, u, u_x, u_t)\, dx\,dt =$$

$$= \int_{t_a}^{t_b}\int_{R_e}^{R_g} \left[\frac{\partial\phi}{\partial u}\delta u + \frac{\partial \phi}{\partial u_x}\delta u_x + \frac{\partial\phi}{\partial u_t}\delta u_t\right] dx\,dt =$$

1

$$= \int_{t_a}^{t_b}\int_{R_e}^{R_g} \left[\frac{\partial\phi}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial u_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial u_t}\right)\right]\delta u\, dx\,dt +$$

$$+ \underbrace{\int_{t_a}^{t_b} \frac{\partial\phi}{\partial u_x}\delta u \Big|_{R_e}^{R_g} dt}_{2} + \underbrace{\int_{R_e}^{R_g} \frac{\partial\phi}{\partial u_t}\delta u \Big|_{t_a}^{t_b} dx}_{3}$$

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① $= 0$ for any variation δu $\Rightarrow$

$$\frac{\partial \phi}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial u_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial u_t}\right) = 0$$

Substituting ϕ:

$$\rho A u_{tt} - EA u_{xx} = \rho A\left(\omega^2 x - \frac{k}{x^2}\right)$$

$$\boxed{\rho u_{tt} - E u_{xx} = \rho\left(\omega^2 x - \frac{k}{x^2}\right)}$$

Equation of motion

② $= 0$ $\Rightarrow$

$$0 = \int_{t_a}^{t_b}\left[\left.\frac{\partial \phi}{\partial u_x}\right|_{x=R_g} \underbrace{\delta u(R_g, t)}_{\neq 0,\ \text{free}} - \left.\frac{\partial \phi}{\partial u_x}\right|_{x=R_e} \underbrace{\delta u(R_e, t)}_{0}\right] dt$$

see d.

$$\left.\frac{\partial \phi}{\partial u_x}\right|_{x=R_g} = 0 \Rightarrow EA\, u_x(R_g, t) = 0$$

$$\boxed{u_x(R_g, t) = 0}$$

This is a NATURAL boundary condition

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③ $= 0$ would provide natural boundary conditions for $u(x, t_a)$ and $u(x, t_b)$, which are __not__ asked for.

f. Additional fictitious force:

$$d\underline{F}_{fict} = -dm\left(\dot{\underline{\omega}} \times \underline{r}_{rel} + \underline{\omega} \times (\underline{\omega} \times \underline{r}_{rel}) + 2(\underline{\omega} \times \underline{v}_{rel})\right)$$

The component coming from $\underline{\omega} \times (\underline{\omega} \times \underline{r}_{rel})$ was already found, and $\dot{\underline{\omega}} = 0$.

Now we also have $\underline{v}_{rel} = u_t \, \underline{i}$, so

$$2\,\underline{\omega} \times \underline{v}_{rel} = 2 \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & \omega \\ u_t & 0 & 0 \end{vmatrix} =$$

$$= 2\omega u_t \underline{j}$$

thus $d\underline{F}_{fict} = -dm\left(-\omega^2 x \, \underline{i} + 2\omega u_t \underline{j}\right)$

so the additional term is $\boxed{-\rho A \, 2\omega u_t \, dx}$

which is a Coriolis force, in the y-direction