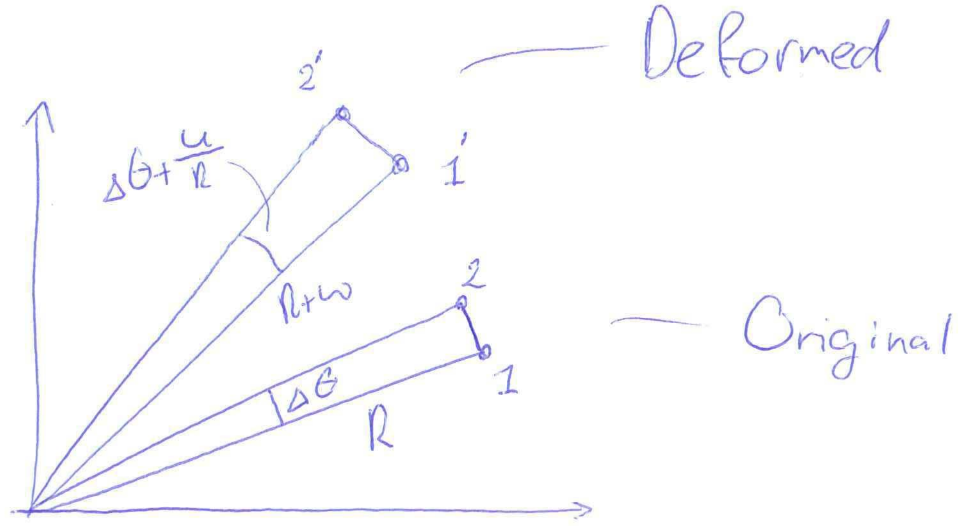
Exercise 1

1.1)



Assume $w_1 = w_2 = w$

then $|\vec{r}_{12}| = R\Delta\theta$

$$|\vec{r}_{12}'| = (R+w)\left(\Delta\theta + \frac{u}{R}\right)$$

$$\Rightarrow \varepsilon = \frac{|\vec{r}_{12}'| - |\vec{r}_{12}|}{|\vec{r}_{12}|} = \frac{(R+w)\left(\Delta\theta + \frac{u}{R}\right) - R\Delta\theta}{R\Delta\theta} = \frac{u + w\Delta\theta + \underbrace{\cancel{\frac{wu}{R}}}_{\simeq 0}}{R\Delta\theta}$$

$$\varepsilon = \frac{u + w\Delta\theta}{R\Delta\theta} = \frac{\frac{u}{\Delta\theta} + w}{R} = \frac{u_\theta + w}{R}$$

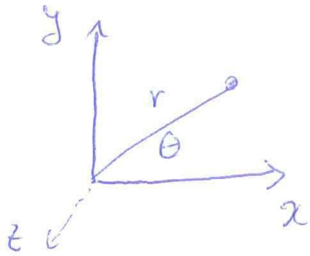
1.2) $$W_{in} = -\int_0^\theta N\delta\varepsilon R d\theta = -\int_0^\theta N(\delta u_{,\theta} + \delta w) d\theta = -\int_0^\theta N\delta w d\theta$$

$\delta u_{,\theta} = 0$ since u can only be constant along θ

$$W_{ex} = \int_0^\theta pR\delta w d\theta$$

$W_{in} + W_{ex} = 0 \Rightarrow -N\delta w + pR\delta w = 0 \Rightarrow N = pR$

2.1) Any point $\underline{r}$ can be written as:



$$\underline{r} = x\,\underline{e}_x + y\,\underline{e}_y + z\,\underline{e}_z$$
$$= \hat{r}\,\underline{e}_r + \hat{\theta}\,\underline{e}_\theta + \hat{z}\,\underline{e}_z$$

Then:

$$\begin{pmatrix}\hat{r}\\ \hat{\theta}\\ \hat{z}\end{pmatrix} = \underline{\underline{R}}\begin{pmatrix}x\\ y\\ z\end{pmatrix} \quad \text{with } \underline{\underline{R}} = \begin{pmatrix} \underline{e}_x\cdot\underline{e}_r & \underline{e}_y\cdot\underline{e}_r & \underline{e}_z\cdot\underline{e}_r \\ \underline{e}_x\cdot\underline{e}_\theta & \underline{e}_y\cdot\underline{e}_\theta & \underline{e}_z\cdot\underline{e}_\theta \\ \underline{e}_x\cdot\underline{e}_z & \underline{e}_y\cdot\underline{e}_z & \underline{e}_z\cdot\underline{e}_z \end{pmatrix}$$

Use: $\underline{e}_r = \begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix}$; $\underline{e}_\theta = \begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix}$; $\underline{e}_z = \begin{pmatrix}0\\0\\1\end{pmatrix}$; $\underline{e}_x = \begin{pmatrix}1\\0\\0\end{pmatrix}$; $\underline{e}_y = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$$\Rightarrow \underline{\underline{R}} = \begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix} = \frac{1}{r}\begin{pmatrix}x & y & 0\\ -y & x & 0\\ 0 & 0 & r\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\hat{r}\\ \hat{\theta}\\ \hat{z}\end{pmatrix} = \underline{\underline{R}}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \frac{1}{r}\begin{pmatrix}x^2+y^2\\ 0\\ rz\end{pmatrix} = \begin{pmatrix}r\\ 0\\ z\end{pmatrix}$$

$$\Rightarrow \underline{r} = r\,\underline{e}_r + z\,\underline{e}_z$$

2.2) $d\underline{r} = dr\,\underline{e}_r + r\,d\underline{e}_r + \cancel{d\theta\,\underline{e}_\theta +} + dz\,\underline{e}_z + z\,\cancelto{0}{d\underline{e}_z}$

$= dr\,\underline{e}_r + r\,d\theta\,\underline{e}_\theta + dz\,\underline{e}_z$ (using: $d\underline{e}_r = d\theta\,\underline{e}_\theta$)

2.3) $\underline{u} = u\,\underline{e}_r + v\,\underline{e}_\theta + w\,\underline{e}_z$

$$d\underline{u} = du\,\underline{e}_r + u\,d\underline{e}_r + dv\,\underline{e}_\theta + v\,d\underline{e}_\theta + dw\,\underline{e}_z + w\,d\underline{e}_z$$

Use:

$$du = u_{,r}\,dr + u_{,\theta}\,d\theta + u_{,z}\,dz$$
$$dv = v_{,r}\,dr + v_{,\theta}\,d\theta + v_{,z}\,dz$$
$$dw = w_{,r}\,dr + w_{,\theta}\,d\theta + w_{,z}\,dz$$

$$d\underline{e}_r = d\theta\,\underline{e}_\theta$$

$$d\underline{e}_\theta = -d\theta\,\underline{e}_r$$

So:

$$d\underline{u} = \left(u_{,r}\,dr + u_{,\theta}\,d\theta + u_{,z}\,dz - v\,d\theta\right)\underline{e}_r$$
$$+ \left(v_{,r}\,dr + v_{,\theta}\,d\theta + v_{,z}\,dz + u\,d\theta\right)\underline{e}_\theta$$
$$+ \left(w_{,r}\,dr + w_{,\theta}\,d\theta + w_{,z}\,dz\right)\underline{e}_z$$

$$\Rightarrow \quad d\underline{u} = \underbrace{\begin{pmatrix} u_{,r} & \frac{u_{,\theta} - v}{r} & u_{,z} \\ v_{,r} & \frac{v_{,\theta} + u}{r} & v_{,z} \\ w_{,r} & \frac{w_{,\theta}}{r} & w_{,z} \end{pmatrix}}_{\underline{\underline{H}}} \underbrace{\begin{pmatrix} dr \\ r\,d\theta \\ dz \end{pmatrix}}_{d\underline{r}}$$

2.4) Small strains: $\underline{\underline{\varepsilon}} = \frac{1}{2}(\underline{\underline{H}} + \underline{\underline{H}}^T)$

$$\Rightarrow \underline{\underline{\varepsilon}} = \begin{pmatrix} u_{,r} & \frac{1}{2}\left(v_{,r} + \frac{u_{,\theta} - v}{r}\right) & \frac{1}{2}\left(w_{,r} + u_{,z}\right) \\ & \frac{v_{,\theta} + u}{r} & \frac{1}{2}\left(\frac{w_{,\theta}}{r} + v_{,z}\right) \\ \text{symm} & & w_{,z} \end{pmatrix}$$

3.1) $W_{in} = -\int_V \underline{\underline{\sigma}} : \delta\underline{\underline{\varepsilon}} \, dV = -\int_V \underline{\sigma}^T \delta\underline{\varepsilon} \, dV$

because 1D bar only carries normal stresses

$$= -\int_V \sigma_x \delta\varepsilon_x + \cancel{\tau_{xy}\delta\gamma_{xy}}^{\,0} + \dots \, dV = -\int_V \sigma_x \delta\varepsilon_x \, dV$$

$$= -\int_0^L \int_A \sigma_x \delta\varepsilon_x \, dA \, dx = \int_0^L \int_A \sigma_x \, dA \, \delta\varepsilon_x \, dx$$

$$= -\int_0^L N \delta\varepsilon_x \, dx \quad \text{with } N = \int_A \sigma_x \, dA$$

3.2) $W_{ex} = \int_V \underbrace{\cancel{\underline{b}^T \delta\underline{u}}}_{0 \text{ (no dist. load)}} dV + \int_{\partial V} \underline{t}^T \delta\underline{u} \, dA = F_L \delta u_L$ (tip load)

$$W_{in} + W_{ex} = 0 \Rightarrow -\int_0^L N \delta\varepsilon \, dx + F_L \delta u_L = 0$$

$$-\int_0^L N \delta \frac{du}{dx} \, dx + F_L \delta u_L = 0 \quad \left(\text{since } \varepsilon = \frac{du}{dx}\right)$$

$$-\left[N \delta u\right]_0^L + \int_0^L N_{,x} \delta u \, dx + F_L \delta u_L = 0 \quad \left(\text{integration by parts}\right)$$

$$-N_L \delta u_L + F_L \delta u_L + \int_0^L N_{,x} \delta u \, dx = 0$$

This has to be satisfied for arbitrary δu:

$$\left.\begin{array}{l} N_L = F_L \\ N_x = 0 \end{array}\right\} \rightarrow$$ The bare has a constant normal force equal to the tip load F_L