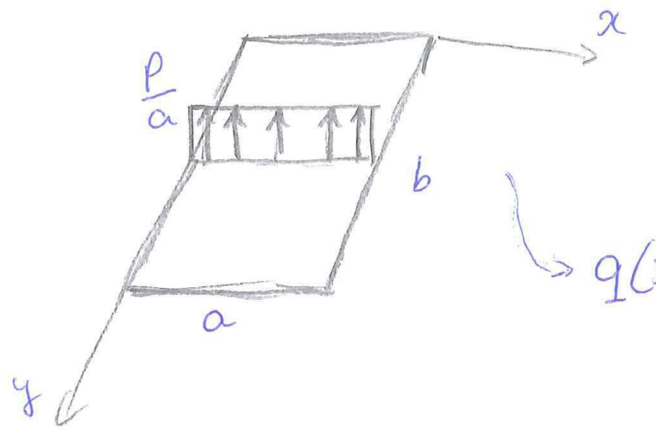
Exercise 2

Problem 1



$$q(x,y) = -\frac{P}{a}\delta\left(y-\frac{b}{2}\right)$$

Governing equation: $w_{xxxx} + 2w_{xxyy} + w_{yyyy} = \frac{q}{D}$ (1)

with $D = \frac{Et^3}{12(1-\nu^2)}$

Assume:

$$w = \sum_{m,n} A_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \quad (2)$$

$$q = \sum_{m,n} a_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \quad (3)$$

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(2) & (3) → (1) ⇒ $\cancel{a_{mn}}$ $A_{mn} = \frac{a_{mn}}{D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2}$ (4)

Compute a_{mn}: (from lecture notes)

$$a_{mn} = \frac{4}{ab}\int_0^a\int_0^b q(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dy\,dx$$

$$= -\frac{4P}{a^2b}\int_0^a \sin\left(\frac{m\pi x}{a}\right)dx \int_0^b \sin\left(\frac{n\pi y}{b}\right)\delta\left(y-\frac{b}{2}\right)dy$$

$$= -\frac{4P}{a^2b}\cdot\frac{a}{m\pi}\left(1-(-1)^m\right)\cdot\sin\left(\frac{n\pi}{2}\right) \qquad \left(\cos(m\pi) = (-1)^m\right)$$

P.T.O.

$$\Rightarrow \quad a_{mn} = \begin{cases} \dfrac{-\delta P}{ab\, m\pi} \sin\left(\dfrac{n\pi}{2}\right) & \text{if } m,n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Substitute into (4) & (2) and use $x=\frac{a}{2}$, $y=\frac{b}{2}$ at center of the plate:

$$w\left(\frac{a}{2},\frac{b}{2}\right) = -\frac{\delta P}{Dab\pi} \sum_{m,n=1,3,5,\dots}\sum \frac{1}{m} \frac{\sin\left(\frac{m\pi}{2}\right) \cancelto{=1}{\sin\left(\frac{n\pi}{2}\right)^2}}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2}$$

$$= -\frac{\delta P}{Dab\pi} \sum_{m,n=1,3,5,\dots}\sum \frac{1}{m} \frac{\sin\left(\frac{m\pi}{2}\right)}{\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2}$$

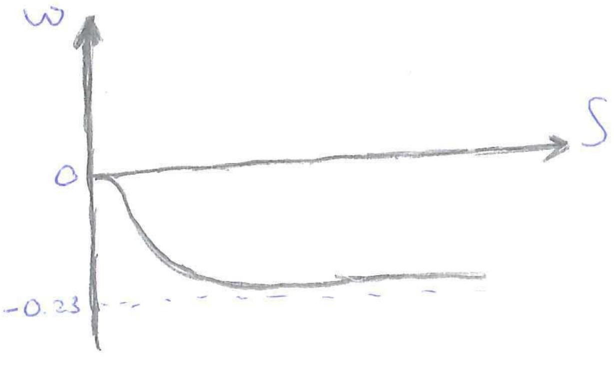
Use aspect ratio $S=\frac{a}{b} \Rightarrow b=\frac{a}{S}$ to give:

$$w_{center} = \frac{-\delta P S a^2}{D\pi} \sum_{m,n=1,3,5,\dots}\sum \frac{1}{m} \frac{\sin\left(\frac{m\pi}{2}\right)}{\left[(m\pi)^2+(n\pi S)^2\right]^2}$$

Write out expression for D, and normalize ~~wrt~~ with respect to $Pb^3/Et^3a = Pa^2/Et^3S^3$:

$$-\frac{\delta P S a^2}{D\pi}\cdot\frac{Et^3S^3}{Pa^2} = -\frac{\delta E t^3 S^4}{D\pi} = -\frac{96(1-\nu^2)S^4}{\pi}$$

So: $$w_{center,\ normalized} = -\frac{96(1-\nu^2)S^4}{\pi} \sum_{m,n=1,3,5,\dots}\sum \frac{1}{m} \frac{\sin\left(\frac{m\pi}{2}\right)}{\left[(m\pi)^2+(n\pi S)^2\right]^2}$$



assuming $\nu=0.3$

Problem 2

Governing equation for plate deflection:

$$D\left[w_{xxxx} + 2w_{xxyy} + w_{yyyy}\right] + N_x w_{xx} + N_y w_{yy} = q$$

NOTE: N_x & N_y assumed positive in <u>compression</u> for convenience

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Assume w and q as in (2) and (3) in problem 1.
This gives the buckling condition:

$$D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 - N_x\left(\frac{m\pi}{a}\right)^2 - N_y\left(\frac{n\pi}{b}\right)^2 = 0$$

Use $a/b = 2$ to get:

$$N_x (m\pi)^2 + 4N_y(n\pi)^2 = \frac{D}{a^2}\left[(m\pi)^2 + 4(n\pi)^2\right]^2$$

Normalize normal forces w.r.t $\frac{D}{a^2}$ to get:

$$\hat{N}_x (m\pi)^2 + 4\hat{N}_y (n\pi)^2 = \left[(m\pi)^2 + 4(n\pi)^2\right]^2 \qquad (5)$$

Compute critical modes:

case I: $\hat{N}_x$ only, $\hat{N}_y = 0$

into (5) $\rightarrow$ $\hat{N}_x = \frac{1}{(m\pi)^2}\left[(m\pi)^2 + 4(n\pi)^2\right]^2 = \left[m\pi + \frac{4n^2\pi}{m}\right]^2$

$\Rightarrow$ minimized for $m = 2$, $n = 1$

set $m=2$, $n=1$ in (5) $\rightarrow$ $\hat{N}_x \cdot 4\pi^2 + \hat{N}_y \cdot 4\pi^2 = (8\pi^2)^2$

$$\boxed{\hat{N}_y = 16\pi^2 - \hat{N}_x} \qquad (6)$$

Case II: $\hat{N}_y$ only, $\hat{N}_x = 0$

into (5) $\rightarrow$ $\hat{N}_y = \dfrac{1}{4(n\pi)^2}\left[(m\pi)^2 + 4(n\pi)^2\right]^2 = \left[\dfrac{m^2\pi}{2n} + 2n\pi\right]^2$

$\Rightarrow$ minimized for $m = 1$, $n = 1$

set $m=1$, $n=1$ in (5) $\rightarrow$ $\hat{N}_x \pi^2 + \hat{N}_y \cdot 4\pi^2 = (5\pi^2)^2$

$$\boxed{\hat{N}_y = \frac{25\pi^2 - \hat{N}_x}{4}} \qquad (7)$$

Plot eq (6) & (7) to obtain buckling interaction curve: