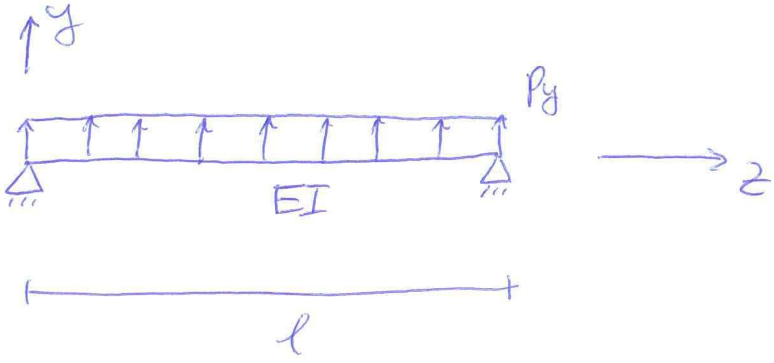
# Exercise 3




Governing beam equation:

$$EIv'''' - p_y = 0 \quad \Rightarrow \quad v'''' = \frac{p_y}{EI}$$

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Use direct integration:

$$v''' = \frac{p_y}{EI} z + C_1 \qquad (1)$$

$$v'' = \frac{p_y}{2EI} z^2 + C_1 z + C_2 \qquad (2)$$

$$v' = \frac{p_y}{6EI} z^3 + \frac{C_1}{2} z^2 + C_2 z + C_3 \qquad (3)$$

$$v = \frac{p_y}{24EI} z^4 + \frac{C_1}{6} z^3 + \frac{C_2}{2} z^2 + C_3 z + C_4 \qquad (4)$$

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Boundary conditions: $v(0) = v(L) = 0$, $v''(0) = v''(L) = 0$

$$v(0) = 0 \quad \Rightarrow \quad C_4 = 0$$

$$v''(0) = 0 \quad \Rightarrow \quad C_2 = 0$$

$$v''(L) = 0 \quad \Rightarrow \quad \frac{p_y L^2}{2EI} + C_1 L = 0 \quad \Rightarrow C_1 = -\frac{p_y L}{2EI}$$

$$v(L) = 0 \quad \Rightarrow \quad \frac{p_y L^4}{24EI} - \frac{p_y L^4}{12EI} + C_3 L = 0 \Rightarrow C_3 = \frac{p_y L^3}{24EI}$$

Substitute these coefficients into:

$$(4) \Rightarrow v = \frac{p_y}{24\,EI}\left(z^4 - 2Lz^3 + L^3 z\right)$$

$$(2) \Rightarrow M = -EIv'' = \frac{p_y}{2}\left(-z^2 + Lz\right)$$

At midspan:
$$v\left(\frac{L}{2}\right) = \frac{5}{384}\,\frac{p_y L^4}{EI}$$

$$M\left(\frac{L}{2}\right) = \frac{p_y L^2}{8}$$

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Approximate solution:

Assume:
$$\hat{v} = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi z}{L} \qquad (5)$$

$$p_y = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi z}{L} \qquad (6)$$

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Substitute assumption in beam equation to get:

$$\sum_{n=1}^{\infty} EI\left(\frac{n\pi}{L}\right)^4 A_n \sin\frac{n\pi z}{L} = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi z}{L}$$

$$\Rightarrow A_n = \frac{a_n}{EI\left(\frac{n\pi}{L}\right)^4} = \frac{a_n L^4}{EI\,n^4\pi^4} \qquad (7)$$

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Compute $a_n$ starting from (6)

$$p_y = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi z}{L}$$

$$\int_0^L p_y \sin\frac{m\pi z}{L}\,dz = \sum_{n=1}^{\infty} a_n \underbrace{\int_0^L \sin\frac{n\pi z}{L}\sin\frac{m\pi z}{L}\,dz}_{= \begin{cases} \frac{L}{2} & \text{if } m=n \\ 0 & \text{otherwise} \end{cases}}$$

So:
$$p_y \int_0^L \sin\frac{n\pi z}{L}\,dz = a_n \cdot \frac{L}{2}$$

$$\Rightarrow a_n = \frac{2p_y}{L}\int_0^L \sin\frac{n\pi z}{L}\,dz = \frac{2p_y}{L}\,\frac{L}{n\pi}\left(1-\cos(n\pi)\right)$$

$$a_n = \begin{cases} \dfrac{4p_y}{n\pi} & \text{if } n \text{ odd} \\ 0 & \text{if otherwise} \end{cases}$$

$\Rightarrow$ into (7):
$$A_n = \frac{a_n L^4}{EI n^4 \pi^4} = \begin{cases} \dfrac{4p_y L^4}{EI n^5 \pi^5} & \text{if } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ into (5)
$$\hat{v} = \sum_{n=1,3,5,\dots} \frac{4p_y L^4}{EI n^5 \pi^5} \sin\frac{n\pi z}{L} = \frac{4p_y L^4}{EI\pi^5} \sum_{n=1,3,5,\dots} \frac{1}{n^5}\sin\frac{n\pi z}{L}$$

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So: $$\hat{m} = -EI\hat{v}'' = \frac{4p_y L^2}{\pi^3} \sum_{n=1,3,5,\dots} \frac{1}{n^3}\sin\frac{n\pi z}{L}$$

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For convergence, use:
$$\varepsilon_v = \left|\frac{\hat{v}-v}{v}\right|_{\text{at } z=\frac{L}{2}}$$

$$\varepsilon_m = \left|\frac{\hat{m}-m}{m}\right|_{\text{at } z=\frac{L}{2}}$$

$\Rightarrow$