c. shell balances for non-Newtonian fluids

1. fluid properties:

non-Newtonian fluids

motivation:

sometimes want fluids that sometimes behave like solids:

sometimes want fluids that have low viscosity near well, high viscosity away from well:

Recall for Newtonian fluid: $\tau_{yx} = -\mu\left(\frac{dv_x}{dy}\right)$

a. **Bingham plastic**

equations:

(Beware confusion over ± sign in BSL eq. 1.2-2)

two parameters: μ_o, τ_o

μ_o:

τ_o:

<u>units</u>:

μ_o:

τ_o:

Note: if τ_o --> 0, Bingham plastic --> Newtonian fluid

examples of fluids with yield stress:

two asides:

why does honey tear bread, while mayonnaise does not?

why does ketchup get stuck in the neck of a ketchup bottle?

b. power-law fluid (Ostwald-de Waele fluid)

equation:

two parameters: m, n

m:

n:

If n = 1: Newtonian fluid, with $m \equiv \mu$

n < 1: "shear thinning", "pseudoplastic"

n > 1: "shear thickening", "dilatant"

(recall definition of $|x|$:

if $x \geq 0$, $|x| = x$

if $x \leq 0$, $|x| = -x$)

C. effective viscosity of non-Newtonian fluids

a. general definition:

The "effective viscosity" of a non-Newtonian fluid is the viscosity of a

hypothetical Newtonian fluid that would give the same

as the real fluid does in the same

(Definition applies even if have no idea of true nature of fluid or even of how viscometer works - see homework)

effective viscosity for shear flow between parallel plates

The "effective viscosity" of a non-Newtonian fluid in shear flow between parallel plates is the viscosity of a hypothetical Newtonian fluid that would

give the same as the real fluid does at the same

Note; the given value of "effective viscosity" may not apply to other situations:

2. Flow derivations for non-Newtonian fluids

3.6

a. Bingham plastic in tube - BSL Ex. 2.3-2 (*first edition of BSL*)

Notes:

- only the relation between τ_{rz} and (dv_z/dr) changes from Newtonian fluid in tube; geometry, momentum balance, boundary conditions are unchanged.
  Therefore everything up through Eq. 2.3-13 (Eq. 2.3-12 in 1st edition of BSL) still applies

- Recall definition of Bingham plastic:

$$\tau_{rz} = -\mu_o \frac{dv_z}{dr} + \tau_o \qquad \tau_{rz} \geq \tau_o$$

$$\frac{dv_z}{dr} = 0 \qquad -\tau_o \leq \tau_{rz} \leq \tau_o$$

$$\tau_{rz} = -\mu_o \frac{dv_z}{dr} - \tau_o \qquad \tau_{rz} \leq -\tau_o$$

- Drawing a picture of $\tau_{rz}(r)$ is essential (cf. Fig. 2.3-3)

i. calculating Q for Bingham plastic in tube

- calculate $\tau_R \equiv \frac{\Delta P\, R}{2\, L}$

- compare τ_R, τ_o:

  - if $\tau_R \leq \tau_o$, $Q = 0$. Don't use Eq. 2.3-30!

  - if $\tau_R \geq \tau_o$, Q given by Eq. 2.3-30

tips for homework:

2. Flow derivations, cont.

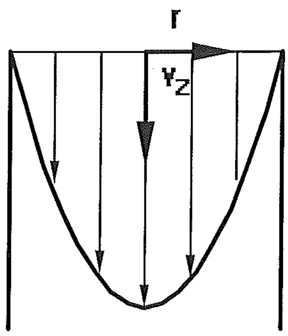
b. Power-Law Fluid

Key points in this derivation are as follows:

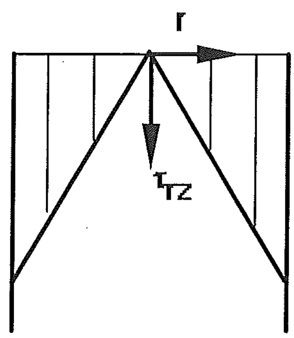
1) Because the system geometry, elements in the momentum balance, and boundary conditions are the same as for flow of a Newtonian fluid in a tube (BSL Section, 2.3), we can skip directly to Eq. 2.3-13, just before Newton's law is introduced in Section 2.3.
2) It is important to recall the mathematical definition of absolute value:

$$|x| = x \quad \text{if } x \geq 0$$
$$= -x \quad \text{if } x \leq 0. \qquad [1]$$

3) Recall that one cannot take arbitrary, fractional powers and roots of negative numbers (unless one is working with imaginary numbers).
4) Due to points (2) and (3), it is important to identify from the start the sign of (dv_z/dr). Of course, deriving $v_z(r)$ is the point of the exercise, so one has to sketch out the <u>expected</u> shape of $v_z(r)$ to guess in advance what the sign of (dv_z/dr) will be.



<u>Expect</u>
v_z = max at r = 0
v_z = 0 at r = R
therefore $(dv_z/dr) < 0$
<u>KNOW</u> (from BSL 2.3-13)
$\tau_{rz} > 0$



BSL Eq. 2.3-13, for flow of any fluid in a tube:

$$\tau_{rz} = \left(\frac{P_o - P_L}{2L}\right) r . \qquad [2]$$

For a power-law fluid,

$$\tau_{rz} = - m \left|\frac{dv_z}{dr}\right|^{n-1} \frac{dv_z}{dr} . \qquad [3]$$

Eventually we want to get rid of the absolute value. As a first step, we need to combine both derivative terms within the absolute value. Since $(dv_z/dr) < 0$ in this case (see diagram above), $(dv_z/dr) = - | dvz/dr |$; therefore

$$\tau_{rz} = - m \left|\frac{dv_z}{dr}\right|^{n-1} (-1) \left|\frac{dv_z}{dr}\right| = m \left|\frac{dv_z}{dr}\right|^{n} . \qquad [4]$$

Combining with Eq. [2],

$$\left|\frac{dv_z}{dr}\right|^{n} = \left(\frac{P_o - P_L}{2 L m}\right) r . \qquad [5]$$

Both sides of this equation are positive; therefore we can take the nth root of both sides:

$$\left|\frac{dv_z}{dr}\right| = \left(\frac{P_o - P_L}{2\,L\,m}\right)^{1/n} r^{1/n} \, . \qquad [5]$$

Now for the final time we use the definition of absolute value. Since $(dv_z/dr) < 0$ in this case, $(dv_z/dr) = -\,|\,dvz/dr\,|$, and

$$\frac{dv_z}{dr} = -\left(\frac{P_o - P_L}{2\,L\,m}\right)^{1/n} r^{1/n} \, . \qquad [6]$$

If one were less careful about handling the absolute values in this derivation, one would end up with the wrong sign upon arriving at Eq. [6]. Integrating Eq. [6] gives

$$v_z = -\left(\frac{P_o - P_L}{2\,L\,m}\right)^{1/n} \frac{1}{[1+(1/n)]}\, r^{[1+(1/n)]} + C_2 \qquad [7]$$

with C_2 a constant of integration. The final boundary condition is

$$v_z = 0 \quad \text{at } r = R \qquad [8]$$

which gives, after some algebraic manipulation,

$$v_z = \left(\frac{P_o - P_L}{2\,L\,m}\right)^{1/n} \frac{R^{[1+(1/n)]}}{[1+(1/n)]} \left(1 - \left(\frac{r}{R}\right)^{[1+(1/n)]}\right) . \qquad [9]$$

Note that if $n = 1$ and $m = \mu$, this is the same as BSL Eq. 2.3-18 for a Newtonian fluid.

$$Q = \int_0^R 2\,\pi\, r\, v_z\, dr \qquad [10]$$

$$Q = 2\,\pi\left(\frac{P_o - P_L}{2\,L\,m}\right)^{1/n} \frac{R^{[1+(1/n)]}}{[1+(1/n)]} \int_0^R r\left(1 - \left(\frac{r}{R}\right)^{[1+(1/n)]}\right) dr \qquad [11]$$

$$Q = 2\,\pi\left(\frac{P_o - P_L}{2\,L\,m}\right)^{1/n} \frac{R^{[1+(1/n)]}}{[1+(1/n)]} \left(\frac{R^2\,[1+(1/n)]}{2\,[3+(1/n)]}\right) \qquad [12]$$

$$Q = \pi\left(\frac{P_o - P_L}{2\,L\,m}\right)^{1/n} \frac{R^{[3+(1/n)]}}{[3+(1/n)]} = \frac{\pi}{(2\,m)^{1/n}} \frac{R^{[3+(1/n)]}}{[3+(1/n)]} \left(\frac{P_o - P_L}{L}\right)^{1/n} . \qquad [13]$$

Note that if $n = 1$ and $m = \mu$, this is the same as BSL Eq. 2.3-21 for a Newtonian fluid. Note that while $Q \sim \Delta P/L$ for a Newtonian fluid, $Q \sim (\Delta P/L)^{1/n}$ for a power-law fluid.

c. Definition of effective viscosity for tube flow

The "effective viscosity" of a non-Newtonian fluid in tube flow is the viscosity of a hypothetical Newtonian fluid that has the same as the given fluid in a tube with the same and .

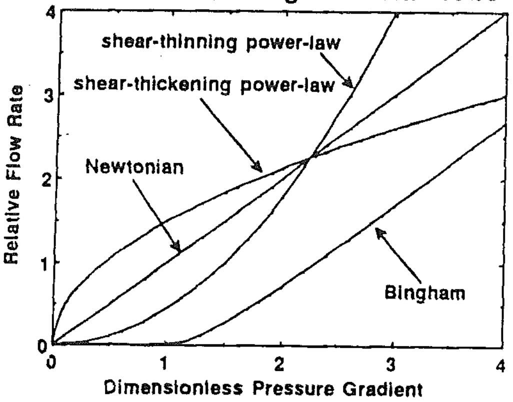
e. Summary of fluid behavior in tube flow

velocity profiles:

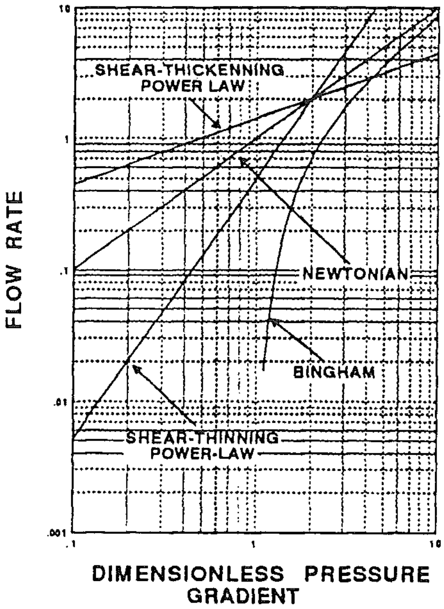
flow rates - on attached page

SUMMARY OF BEHAVIOR OF FLUIDS IN TUBE FLOW

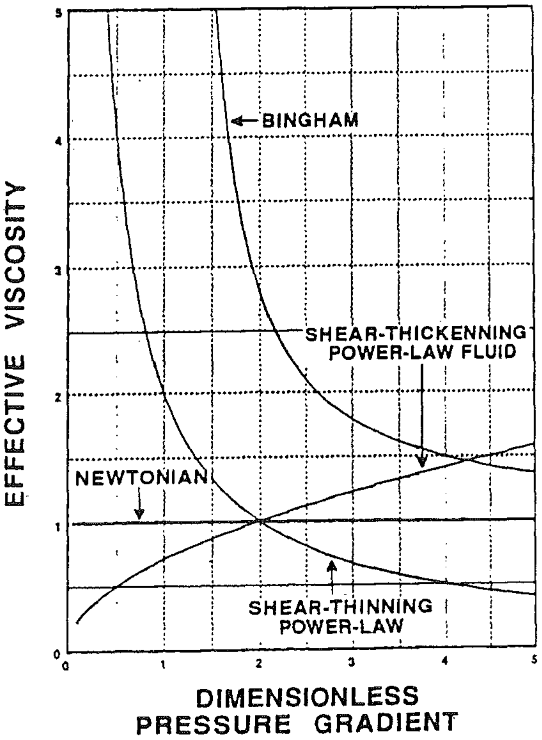
Non-Newtonian Fluids: Flow Rate Through Circular Tube



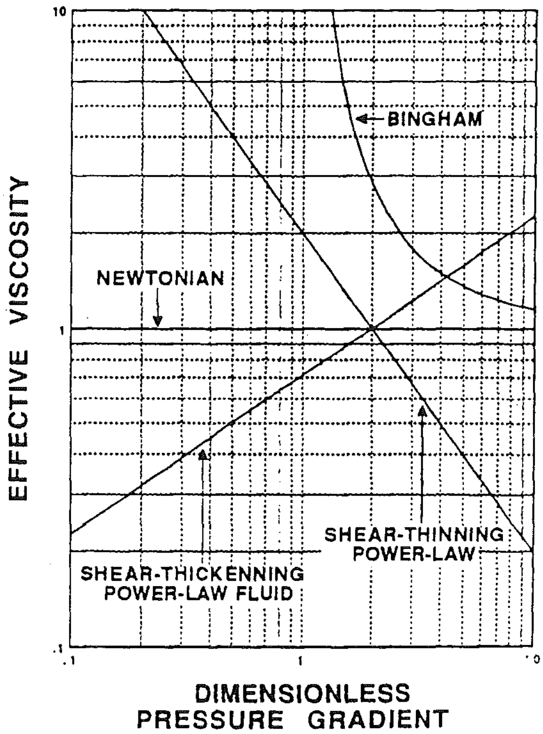
FLOW THROUGH A TUBE



FLOW THROUGH A TUBE



FLOW THROUGH A TUBE



d. rectangular slit

i. Newtonian fluid - done on homework; BSL p. 62

a note on boundary conditions

In the homework, we solved for $v_z(x)$ using the BC (1) $v_z = 0$ at $x = B$ and (2) $v_z = 0$ at $x = -B$. If one were clever, one could notice that the symmetry of the problem about $x = 0$ implies a different boundary condition: (3) $\tau_{xz} = 0$ at $x = 0$. (The justification is as follows: Since the problem is symmetric about $x = 0$, how can there by any momentum transport <u>across</u> the plane of symmetry? Therefore τ_{xz} must be zero at this plane, at $x = 0$.) For the Newtonian flow, then one could have solved for $v_z(x)$ for $x \geq 0$ using BC (1) and (3), and used symmetry to argue that, for $x < 0$, $v_z(x) = v_z(-x)$. For the Newtonian flow this approach is not really necessary, but it <u>greatly</u> simplifies the solution for Bingham plastics and power-law fluids. The reason is that in a slit τ_{xz} changes sign at $x = 0$, and for both Bingham and power-law fluids, the equations for τ_{xz} differ for $\tau_{xz} < 0$ and $\tau_{xz} > 0$. Therefore we use BC (1) and (3) below and solve for $x \geq 0$ only.

ii. Bingham plastic

Since the geometry and momentum balance are the same is in the homework solution for Newtonian flow, we can jump directly to "Eq. II" on the homework solution set:

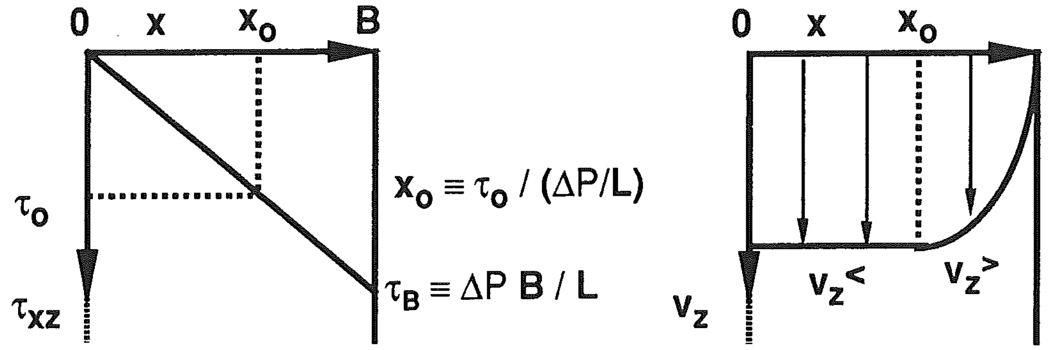
$$\tau_{xz} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L}\, x + C_1 \;. \qquad [1]$$

We limit our consideration to $0 \leq x \leq B$ and use BC (3): $\tau_{xz} = 0$ at $x = 0$. This implies

$$C_1 = 0$$

$$\tau_{xz} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L}\, x \;. \qquad [2]$$

At this point it pays to sketch τ_{xz} and the expected v_z profile.



We define x_o is the location where $\tau_{xz} = \tau_o$. That is,

$$x_o \equiv \frac{\tau_o}{\left(\frac{(\mathcal{P}_o-\mathcal{P}_L)}{L}\right)} \quad . \qquad [3]$$

For $x < x_o$, $t_{xz} < \tau_o$ and therefore $dv_z/dx = 0$, according to the Bingham plastic equation. For $x \geq x_o$,

$$\tau_{xz} = \frac{(\mathcal{P}_o-\mathcal{P}_L)}{L} x = -\mu_o \frac{dv_z}{dx} + \tau_o \qquad \text{for } x \geq x_o \ . \qquad [4]$$

Rearranging and integrating gives

$$v_z^{>} = -\frac{(\mathcal{P}_o-\mathcal{P}_L)}{\mu_o L}\frac{x^2}{2} + \frac{\tau_o}{\mu_o} x + C_1 \qquad \text{for } x \geq x_o \ . \qquad [5]$$

BC (1), $v_z = 0$ at $x = B$, implies

$$C_2 = \frac{\mathcal{P}_o-\mathcal{P}_L}{\mu_o L}\frac{B^2}{2} - \frac{\tau_o}{\mu_o} B \qquad [6]$$

$$v_z^{>} = \frac{(\mathcal{P}_o-\mathcal{P}_L) B^2}{2 \mu_o L}\left(1 - \left(\frac{x}{B}\right)^2\right) - \frac{\tau_o B}{\mu_o}\left(1 - \frac{x}{B}\right) \qquad \text{for } x \geq x_o \qquad [7]$$

For $x < x_o$, v_z is given by Eq. [6] with $x = x_o$:

$$v_z^{<} = \frac{(\mathcal{P}_o-\mathcal{P}_L) B^2}{2 \mu_o L}\left(1 - \left(\frac{x_o}{B}\right)^2\right) - \frac{\tau_o B}{\mu_o}\left(1 - \frac{x_o}{B}\right) \qquad \text{for } x < x_o \ . \qquad [8]$$

Q for the whole slit is twice the flow through the half defined by $x \geq 0$:

$$Q = W \, 2 \int_0^B v_z(x) \, dx \quad . \qquad [9]$$

Evaluating the equation for Q is easiest if one uses the trick in BSL Example 2.3-2 and integrates by parts. The result is

$$Q = \frac{2}{3} \frac{W (\mathcal{P}_o-\mathcal{P}_L) B^3}{\mu_o L}\left(1 - \frac{3}{2}\left(\frac{\tau_o}{\tau_B}\right) + \frac{1}{2}\left(\frac{\tau_o}{\tau_B}\right)^3\right) \text{ for } \tau_B \geq \tau_o \qquad [10]$$

$$Q = 0 \quad \text{for } \tau_B \leq \tau_o \qquad [11]$$

with

$$\tau_B \equiv \frac{(\mathcal{P}_o-\mathcal{P}_L)}{L} B \ . \qquad [12]$$

$$\tau_B \equiv \frac{(P_o-P_L)}{L} B \quad [12]$$

Note the similarity in the form of Eqs. [7], [8], and [10] to BSL Eqs. 2.3-25, 26 and 30. Note also that all of these equation revert to the Newtonian equations if $\tau_o = 0$. In Eq. [10], the first part of the equation matches the Newtonian equation (with μ_o substituted for μ) while the bracketed term reduces Q below the value for a Newtonian fluid.

The final equations for v_z and $w \equiv Q\rho$ for Bingham flow in a slit are found on pp. 259-260 of BSL 2nd Ed., though the derivation is not given there.

iii) power-law fluid

The derivation of $v_z(x)$ for a power-law fluid in a slit is given as a homework assignment. The final result is

$$v_z = \left(\frac{(P_o-P_L)}{m\,L}\right)^{1/n} \frac{B^{[1+(1/n)]}}{1+(1/n)} \left(1 - \left(\frac{x}{B}\right)^{[1+(1/n)]}\right) \quad \text{for } x > 0. \quad [13]$$

The total flow rate is derived by integrating v_z over x:

$$Q = W\, 2 \int_0^B v_z(x)\, dx \quad [14]$$

$$Q = \left(\frac{2\,W\,n}{2n+1}\right) \left(\frac{(P_o-P_L)\, B^{2n+1}}{m\,L}\right)^{1/n} . \quad [15]$$

Note that for flow of a power-law fluid through a slit, as for power-law flow through a tube, $Q \sim \Delta P^{1/n}$. Note also that Eq. [15] reverts to the Newtonian formula if n = 1 and m $= \mu$.

4. annulus

a. Newtonian fluid - BSL sect. 2.4

Notes:

- the basic geometry (i.e., cylindrical) and the momentum balance are same as for flow through circular tube
- BC differ:
  - $v_z = 0$ at $r = R$
  - $v_z = 0$ at $r = \kappa R$

  ($r = 0$ is not within system; therefore can't apply BC that τ_{rz} is finite at $r = 0$)
- BSL defines z pointing *up* this time, unlike sect. 2.3 - alters definition of P

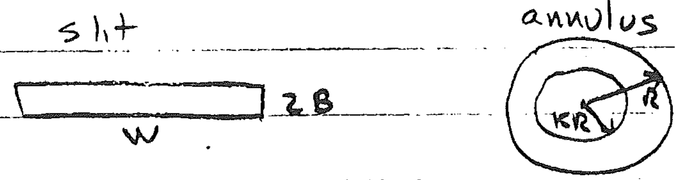
The math gets hairy. Don't let the math distract you from the following:

e) Bingham + Power-Law Fluids in Annulus

Math gets <u>really</u> hairy...

Instead, recall annulus → slit as $K \to 1$;

i.e., define "equivalent slit" and use slit eqs. for annulus:



slit	annulus
area = $2BW$	area = $\pi R^2(1-K^2)$
WIDTH = $2B$	width = $R(1-K)$

"Equivalent" slit has same area and width as annulus.

Use slit formulas substituting $R(1-K)$ for $2B$ and $\pi R^2(1-K^2)$ for $2BW$

For power-law fluid

$$Q = \pi R^2 (1-K^2) \left[\frac{\Delta P \left[R(1-K)\right]^{n+1}}{2^{n+1} \; m L} \right]^{1/n} \frac{n}{2n+1}$$

For Bingham plastic,

let $\tau_B \equiv \left| \frac{\Delta \mathcal{P}}{L} \frac{R(1-K)}{2} \right|$

if $\tau_B \le \tau_0$, $\quad Q = 0$

$\tau_B \ge \tau_0$ $\quad Q = \dfrac{\pi \Delta \mathcal{P} R^4 (1-K)^2 (1-K^2)}{12 \mu_0 L} \left[1 - \frac{3}{2}\left(\frac{\tau_0}{\tau_B}\right) + \frac{1}{2}\left(\frac{\tau_0}{\tau_B}\right)^3 \right]$

Formulas are "reasonably" accurate for $K > 0.3$.

4.8 **suspension of particles in Bingham plastic**

students not responsible for derivation, but for final equation:

Particle is completely suspended, as in a solid, if

$$\tau_o \geq \frac{4}{3\pi} R\,|\rho_s - \rho|\,g$$

applications to suspension of:

D. Final Notes

1. An aside: Definition of $\Delta\mathcal{P}$

$\Delta\mathcal{P}$ is always $(p_0 - p_L) + \rho\, g\, (\Delta \text{ vertical position})$

$$\Delta\mathcal{P} \equiv (p_0 - p_L) + \rho\, g\, [(\text{vertical position})_0 - (\text{vertical position})_L]$$

If z axis is defined as pointing *down*, as in BSL sect. 2.3, then

$$\mathcal{P} \equiv p - \rho\, g\, z$$

$$\Delta\mathcal{P} \equiv (p_0 - p_L) - \rho\, g\, (z_0 - z_L)$$

$$= (p_0 - p_L) + \rho\, g\, L$$

(same formula applies to Ex. 2.3-2 (p. 48) because z points down there, too)

If z points *up*, as in BSL sect. 2.4, then

$$\mathcal{P} \equiv p + \rho\, g\, z$$

$$\Delta\mathcal{P} \equiv (p_0 - p_L) + \rho\, g\, [(\text{vertical position})_0 - (\text{vertical position})_L]$$

$$= (p_0 - p_L) - \rho\, g\, L$$

Better than memorizing either formula is to realize the physical significance of the hydrostatic forces; they should *add* to $\Delta\mathcal{P}$ if flow is downwards and *reduce* $\Delta\mathcal{P}$ if flow is upwards, against gravity.

See next p.

2. An aside: physical meaning of $\tau < 0$

Now we have had two problems here $\tau < 0$ somewhere:

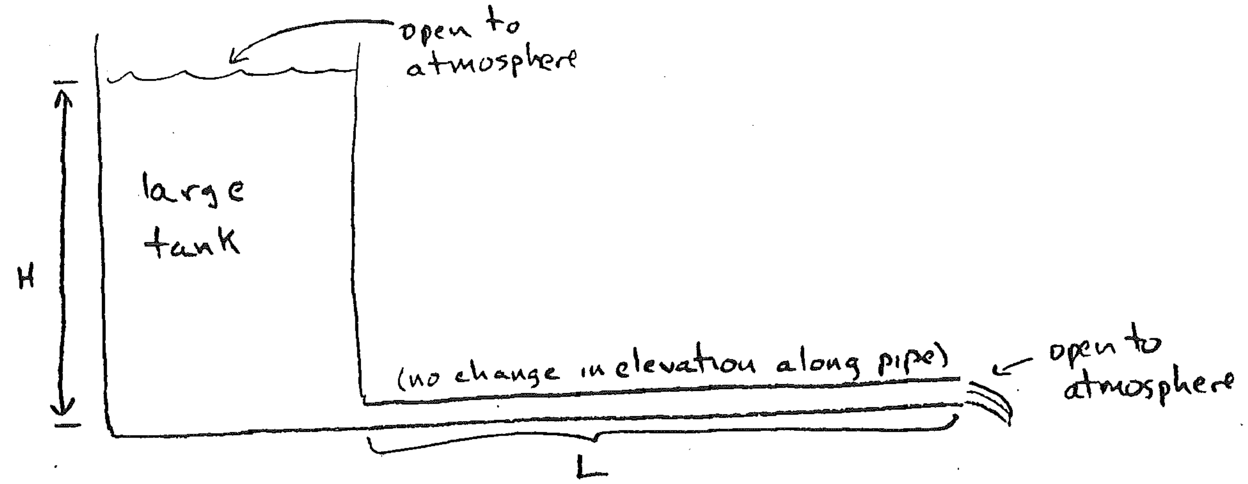
slit:

annulus:

Examples

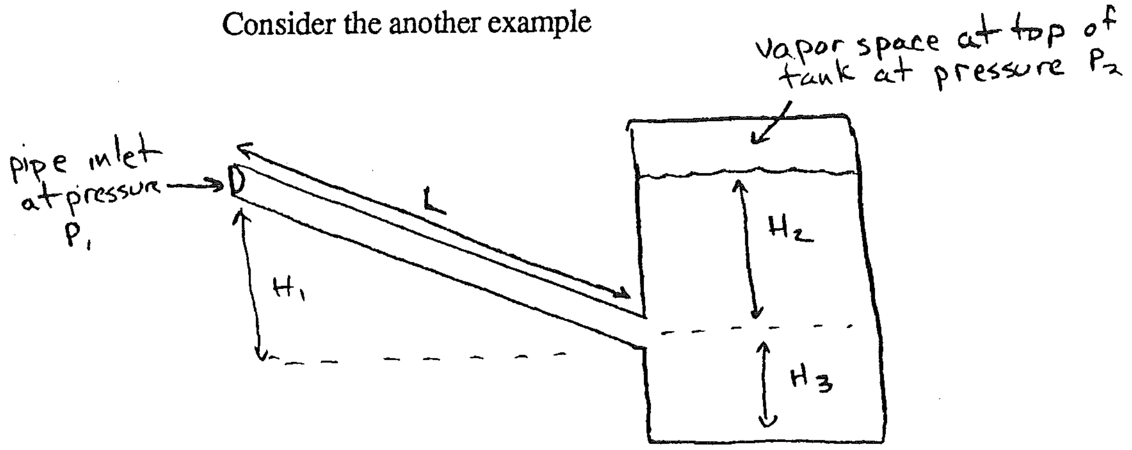
Correctly including hydrostatic pressure in flow equations

Consider the following example



What is $\Delta\mathcal{P}$ across the pipe?

Consider the another example



What is $\Delta\mathcal{P}$ across the pipe?

Moral: include hydrostatic contributions to inlet and outlet pressures in $\Delta\mathcal{P}$

physical significance: either

- momentum in positive z direction is carried in direction of *decreasing* x or r (both cases here), or

- momentum in the *negative* z direction is carried in the direction of *increasing* x or r

... how to handle this mathematically ... ? Which terms represent transport "into" and "out of" the control volume?

3. An aside: What if fluxes or velocities are *negative*?

(For instance, what if wall is moving in negative z direction; or positive z momentum is transported radially inward (in direction of decreasing r, i.e., $\tau_{rz} < 0$))

Principles:

1. Coordinate axes define directions of positive velocity, fluxes.
2. For purposes deriving shell balance, assume all fluxes and velocities are in positive directions, as defined by coordinate axes.
3. Apply boundary conditions consistently with physical constraints and coordinate-axis directions.
4. Negative velocities or fluxes (e.g., $v_z < 0$, $\tau_{rz} < 0$) will result naturally from application of boundary conditions.

(Don't try to out-smart the process.)

~~Non-Newtonian fluid flow in an annulus ... next page~~