III. Shell energy balances

A. the basic approach

B. examples from *FTI*

1. Fourier's law of heat conduction: Eq. 3.1, p. 105; note difference in symbols w *BSL*
2. steady conduction through a flat wall: pp. 105 ff.
3. steady conduction through a composite flat wall: pp. 107 ff.
   Analogy to electrical conduction through multiple resistances: Fig. 3.3
4. steady conduction through a cylindrical wall: pp. 109 ff.
5. steady conduction through spherical wall: pp. 111 ff.
6. Newton's law of cooling: sect. 3.2, pp. 117 ff. – note broader definition of Newton's Law than used in *BSL*
   a. example: heat transfer through flat wall with Newton's law of cooling on both sides pp. 118

C. new examples from *BSL*

1. steady heating of an electrical wire
2. conduction through composite walls
   a. rectangular
   b. cylindrical
3. heat conduction in a cooling fin
4. heat conduction and convection in a porous medium
5. Review of simplifications in examples 3 + 4
6. unsteady conduction in a solid

A. [redacted] Outline of Shell Energy Balance Approach [redacted]

APPROACH

1. SELECT COORDINATE SYSTEM; DEFINE CONTROL VOLUME
2. STATE BOUNDARY CONDITIONS *
3. PERFORM ENERGY BALANCE **
4. THICKNESS $\rightarrow 0$ ($\rightarrow$ dif. eq. for q)
   (optional): solve dif. eq. for q, apply b.c. - IF b.c. applies
   to q alone
5. RELATE q TO dT/dx (Fourier's law)
6. SOLVE DIF. EQ. FOR T; APPLY B.C.*
   (optional) COMPUTE Q, etc.

* - BOUNDARY CONDITIONS (see pp. 291-292)

1. SPECIFY T AT SURFACE
2. SPECIFY q AT SURFACE
   2a) q = 0 across surface ("perfectly insulated surface")
   2b) q specified in problem statement
   2c) q, T CONTINUOUS ACROSS SOLID/SOLID I.F.
3. "NEWTON'S LAW OF COOLING" AT SOLID/FLUID SURFACE:
   $q = \pm h\,(T - T_{fluid})$
   (sign of q depends on direction of coordinate system)
4. q, T NOT INFINITE ANYWHERE IN REGION OF INTEREST

"ALL BOUNDARY CONDITIONS ARISE FROM NATURE"
(i.e., from problem statement)

** - ELEMENTS OF ENERGY BALANCE

ENERGY FLUX (α area); sometimes called "e" vector in BSL
1. CONVECTION OF ENERGY THROUGH SURFACE ($v\rho C_pT$)
2. ENERGY CONDUCTION THROUGH SURFACE q
   ("molecular transport of energy")

ENERGY "GENERATION" or "SOURCE" (α volume)
3. ELECTRICAL, NUCLEAR, CHEMICAL OR VISCOUS
   HEAT "GENERATION" WITHIN VOLUME ("S")

ENERGY ACCUMULATION (α volume) (not at steady state!)
4. ACCUMULATION OF ENERGY IN SYSTEM ($\rho C_p \frac{dT}{dt}$)

11.6

conduction through composite walls (BSL sect. 10.6)

Initial notes

suppose there are n layers

each layer has different properties (e.g., thermal conductivity k)

- ∴ need separate shell balance on each layer
- → n second-order differential equations for T
-

Boundary conditions:

T continuous at contacts between layers

q continuous at contacts between layers

$q = h\,\Delta T$ at both outer surfaces

Total:

rectangular layers (BSL section 10.6)

what if T_o is specified instead of T_a?

analogy to Darcy flow in layered rock

5. Review of simplifications in examples 3 and 4

Cooling fin: complex reality: Newton's law of cooling is B.C. at x=+/-B. Variation of T in x direction is small compared to variation in z direction. Complete solute requires partial differential equation for T(x,z).

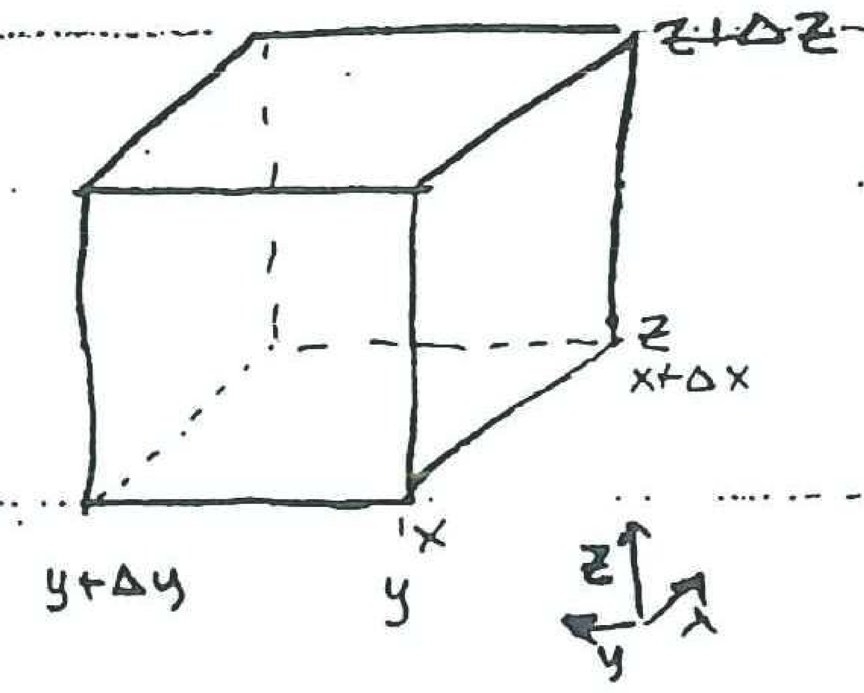
Simplified model: Assume uniform T across fin for any z. T=T(z). Control volume extends across fin, from x=B to x=-B. Newton's law of cooling now applies at boundary of control volume *enters energy balance*. Energy balance leads to ordinary differential equation for T(z).

Convection, reaction and conduction in porous medium. Complex reality: T differs slightly between solids and nearby fluid, but much less than it varies along length of medium. Complete solution requires partial differential equation for T as function of distance from particle and position along bed.

Simplified model: assume T uniform between catalyst and adjacent fluid for any z; T=T(z). Energy balance leads to ordinary differential equation.

Derivation of p.d.e. for Unsteady Conduction in Solid

Energy Balance on Rectangular Control Volume



- no convection
- no generation
- constant, uniform $\rho, k, \hat{C}_p$

Conduction into system at x: $(\Delta y \Delta z)\, q_x|_x$

" out of " $x+\Delta x$: $(\Delta y \Delta z)\, q_x|_{x+\Delta x}$

Conduction into system at y: $(\Delta x \Delta z)\, q_y|_y$

" out of " $y+\Delta y$: $(\Delta x \Delta z)\, q_y|_{y+\Delta y}$

Conduction into system at z: $(\Delta x \Delta y)\, q_z|_z$

" out of " $z+\Delta z$: $(\Delta x \Delta y)\, q_z|_{z+\Delta z}$

Accumulation of thermal energy in system: $(\Delta x \Delta y \Delta z)\, \rho \hat{C}_p \frac{\partial T}{\partial t}$

$$\rightarrow (\Delta y \Delta z)(q_x|_x - q_x|_{x+\Delta x}) + (\Delta x \Delta z)(q_y|_y - q_y|_{y+\Delta y}) + (\Delta x \Delta y)(q_z|_z - q_z|_{z+\Delta z}) = (\Delta x \Delta y \Delta z)\rho \hat{C}_p \frac{\partial T}{\partial t}$$

Divide by $\Delta x \Delta y \Delta z$; let $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$

$$\rightarrow -\frac{\partial}{\partial x}(q_x) - \frac{\partial}{\partial y}(q_y) - \frac{\partial}{\partial z}(q_z) \equiv -\nabla \cdot \vec{q} = \rho \hat{C}_p \frac{\partial T}{\partial t}$$

Fourier's law: $\vec{q} = -k \nabla T$

$$\rightarrow -\nabla \cdot (-k \nabla T) = k \nabla^2 T = \rho \hat{C}_p \frac{\partial T}{\partial t}$$

$$\boxed{\alpha \nabla^2 T = \frac{\partial T}{\partial t}}, \quad \alpha \equiv \frac{k}{\rho \hat{C}_p}$$

α called "a" in FT

(α is called a in FT)

If system is at steady state, $\frac{\partial T}{\partial t} = 0$ and $\nabla^2 T = 0$. This is identical to "diffusivity equation" for steady flow of incompressible fluid in isotropic, homogeneous porous medium (Caudle, "Fundamentals of Reservoir Engineering," p. III-3).