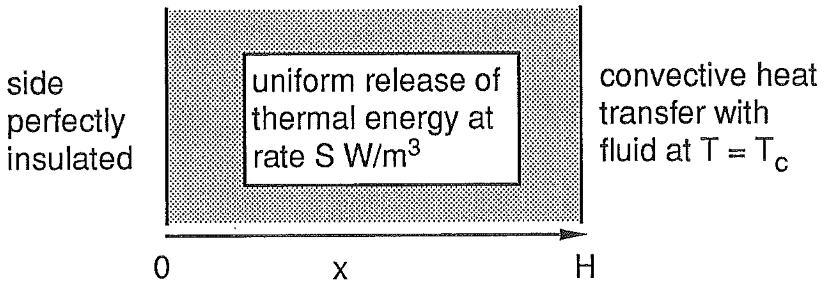
Example of Shell Balance Problem with Newton's Law of Cooling as Boundary Condition

A layer of solid of thickness H in the x direction is infinite in extent in the y and z directions. On one side (x = 0), the solid is perfectly insulated. On the other side, there is convective heat transfer with fluid at temperature T_C; the heat-transfer coefficient is h. There is uniform heating within the solid, with thermal energy released at a rate S (in units W/m^3).

Derive an equation for T(x) within this solid at steady state. You do not have to repeat any part of any derivation from BSL that applies here; however, if you use a part of any derivation in BSL, identify by number the equation from BSL that you use.
(35 points)



∃ generation + conduction, no convection (it's a solid), or accum (steady state).
Consider region of lengths L + W in y + z directions.

Shell balance: conduction in $q_x LW|_x$

out $q_x LW|_{x+\Delta x}$

generation $S LW\Delta x$

$$q_x LW|_x - q_x LW|_{x+\Delta x} + S LW\Delta x = 0$$

Divide by $LW\Delta x$; let $\Delta x \to 0 \rightarrow -\frac{dq_x}{dx} + S = 0$

$\frac{dq_x}{dx} = S$ integrate; $\rightarrow q_x = Sx + C$

B.C.: $q_x = 0$ at $x = 0$ $\rightarrow C = 0 \rightarrow q_x = Sx \; \boxed{I} = -k\frac{dT}{dx}$

$-\frac{Sx}{k} = \frac{dT}{dx}$; integrate; $\rightarrow T = -\frac{Sx^2}{2k} + C$ $\boxed{II}$

B.C.: $q_x = h(T - T_c)$ at $x = H$; Plug in eq. $\boxed{I}$ for q_x and $\boxed{II}$ for T

$$q_x|_{x=H} = SH = h\left(-\frac{SH^2}{2k} + C - T_c\right)$$

$$\frac{SH}{h} + T_c - \frac{SH^2}{2k} = C$$

$$\rightarrow T = -\frac{Sx^2}{2k} + \frac{SH}{h} + T_c - \frac{SH^2}{2k} \Rightarrow T - T_c = \frac{SH^2}{2k}\left(1 - \left(\frac{x}{H}\right)^2\right) + \frac{SH}{h}$$