XII. Multivariate and Unsteady Conduction

A. governing partial differential equation and assumptions

- see example 9 of XI.C

Assumptions:

* ...

same mathematical equation is called "diffusivity equation" in PGE 323, because

* ...

all obey the same partial differential equation

B. Tabulated 1D solutions

1. Semi-infinite slab (BSL Sect. 4.1, Ex. 2.1-1)

   a. initial and boundary conditions

   \[
   T = T_0 \text{ at } t = 0, \quad \text{for } y \geq 0
   \]

   \[
   T = T_1 \text{ at } y = 0 \quad \text{for } t > 0
   \]

   \[
   T = T_0 \text{ at } y \to \infty \quad \text{for } t > 0
   \]

   b. solution given by Eq. 12.1-8 and Fig. 4.1-2

   c. heat flux at \( y = 0 \)
Velocity distribution in dimensionless form for flow in the neighborhood of a wall suddenly set in motion. (The function is a complementary error function.)
2. Two semi-infinite slabs brought together
   - slabs have identical properties, different initial temperatures

     \[ T = \frac{T_{ob} + T_{of}}{2} \]

     \[ T_{of} \rightarrow 0 \text{ at } y \rightarrow -\infty \]

     \[ T_{ob} \rightarrow 0 \text{ at } y \rightarrow \infty \]

     \[ T(y \rightarrow 0^+) = T(y \rightarrow 0^-) \text{ for } t > 0 \]

     \[ q_y(y \rightarrow 0^+) = q_y(y \rightarrow 0^-) \text{ for } t > 0 \]

a. boundary and initial conditions

   \[ T = T_{of} \text{ at } t = 0, \text{ for } y > 0 \]
   \[ T = T_{of} \text{ at } y \rightarrow \infty, \text{ for } t > 0 \]
   \[ T = T_{ob} \text{ at } t = 0, \text{ for } y < 0 \]
   \[ T = T_{ob} \text{ at } y \rightarrow -\infty, \text{ for } t > 0 \]

b. solution

(note erf(-x) = - erf(x); thus equation applies to y > 0 and y < 0)

solution is plotted on next page
XII.B.2.b Unsteady Conduction Between Two Semi-Infinite Slabs in Contact

\[
y / \sqrt{4 \alpha t}
\]
3. Slab of finite thickness (BSL Ex. 1.1-2)
   a. boundary and initial conditions

   $T = T_0$ at $t = 0, \quad$ for $-b \leq y \leq b$

   $T = T_1$ at $y = b \quad$ for $t > 0$

   $T = T_1$ at $y = -b \quad$ for $t > 0$

   b. solution: BSL Figure. 1.1-1

   (note curves for small $\alpha t/B^2$ do not go below horizontal axis)

c. an aside: how to estimate heat flux at surface from this chart?

4. Conduction within cylinder ($0 \leq r \leq R$) (BSL Ex. 1.2-2)
   a. boundary and initial conditions

   $T = T_0$ at $t = 0, \quad$ for $0 \leq r \leq R$

   $T = T_1$ at $r = R \quad$ for $t > 0$

   $T$ finite at $r = 0$ for $t > 0$

   b. solution: BSL Figure. 1.1-2

   (note curves for small $\alpha t/R^2$ do not go below horizontal axis)
5. Conduction within sphere \((0 \leq r \leq R)\) (BSL Ex. 12.1-2)

a. boundary and initial conditions
\[
\begin{align*}
T &= T_0 \text{ at } t = 0, \quad \text{for } 0 \leq r \leq R \\
T &= T_1 \text{ at } r = R, \quad \text{for } t > 0 \\
T &= \text{finite at } r = 0 \text{ for } t > 0
\end{align*}
\]

b. solution: BSL Figure. 12.1-3

(note curves for small \(\alpha t/R^2\) do not go below horizontal axis)

6. Conduction outwards from cylinder \((R_{\text{well}} \leq r < \infty)\)

a. boundary and initial conditions
\[
\begin{align*}
T &= T_0 \text{ at } t = 0, \quad \text{for } r \geq R_{\text{well}} \\
T &= T_1 \text{ at } r = R_{\text{well}}, \quad \text{for } t > 0 \\
T &= T_0 \text{ at } r \to \infty, \quad \text{for } t > 0
\end{align*}
\]

b. solution: Carslaw and Jaeger, *Conduction of Heat in Solids*, Figs. 41 and 42 (see attached)

7. Comments on 1D solutions
The trend shown here, in which dimensionless temperature \([T-T_0/(T_1-T_0)]\) is a function of dimensionless position and dimensionless time, which is proportional to \((t/L^2)\), where \(L\) is the characteristic length of the solid, applies to any homogeneous solid of any shape, with uniform initial temperature \(T_0\) and a sudden change in surface temperature to \(T_1\).
Fig. 42. The flux $f$ at the surface of the region bounded internally by a circular cylinder of radius $a$, with zero initial temperature and constant surface temperature $V$.

Fig. 43. Surface temperature $v$, of the region bounded internally by a circular cylinder of radius $a$, with constant initial temperature $V$ and radiation at its surface into a medium at zero. The numbers on the curves are values of $\alpha h$.

Fig. 44. Surface temperature $v$, of the region bounded internally by a circular cylinder of radius $a$, with zero initial temperature and constant flux $Q$ at the surface.

Fig. 41. Temperatures in the region bounded internally by the cylinder $r = a$, with zero initial temperature and constant surface temperature $V$. The numbers on the curves are the values of $\alpha/\alpha^2$. 