C. Extending the 1D solutions

1. Zero-flux boundary conditions (such as perfectly insulated boundaries)

a. principle:

b. analogy to "method of images" in reservoir engineering

c. Specific examples

i. opposide flat sides perfectly insulated (q=0 across surface)
for example, finite cylinder
insulated on top and bottom

T(r) is
that for

consider how one could extend solution for finite-width slab . . .

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ii. **one flat side perfectly insulated (q=0 across surface)**

T(r) is
that for

2. **"Orthogonal conduction"** - see attached handout from F. M. White, *Heat Transfer*, 1984.

Requirements:

- system shape must be orthogonal intersection of shapes of 1D tabulated solutions (see handout for examples)
- system must be a uniform $T = T_0$ at time $t = 0$
- all surfaces must be changed to same T_1 at $t \geq 0$

Note must use "unrealized temperature change" $\frac{(T_1\text{-}T)}{(T_1\text{-}T_0)}$, *not* $\frac{(T\text{-}T_0)}{(T_1\text{-}T_0)}$

Then:

$\frac{(T_1\text{-}T)}{(T_1\text{-}T_0)}$ for any position in 3D solid is the product of $\left(\frac{(T_1\text{-}T)}{(T_1\text{-}T_0)}\right)_i$ values for the corresponding position in each of the 1D solutions.

! ! Note: must use "unrealized temperature change" $\frac{(T_1\text{-}T)}{(T_1\text{-}T_0)}$, *not* $\frac{(T\text{-}T_0)}{(T_1\text{-}T_0)}$! !

Please reread the preceding sentence!

Note differences in notation between handout and BSL. Many of these are corrected by hand in your handout:

	handout (White's book)	**BSL**
surface temperature	T_0	T_1
initial temperature	T_i	T_0
dimensionless temperature	$\theta \equiv \frac{(T\text{-}T_0)}{(T_i\text{-}T_0)}$	$\frac{(T_1\text{-}T)}{(T_1\text{-}T_o)}$
B.C. at surfaces for $t > 0$	Newton's law of cooling	$T = T_1$

D. Worked examples

Unsteady Heat conduction in Solids & Product Solutions

Example 1

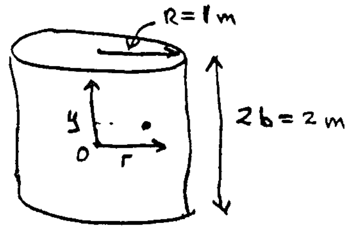
0.1 m dia steel ball, w/ $k = 16.3\ W/(m\,K)$, $\rho = 7700\ kg/m^3$, $\hat{C}_p = 500\ \frac{J}{kg\,K}$.
initial $T \equiv T_0 = 300\ K$. At time $t \geq 0$, surface temperature is $T_1 = 600\ K$.
How long until center rises to 500 K?

solution: compute $\alpha \equiv \frac{k}{\rho \hat{C}_p} = \frac{16.3}{(7700)(500)} = 4.23 \cdot 10^{-6}\ m^2/s$

want $\frac{T_1 - T}{T_1 - T_0} = \frac{600-500}{600-300} = 0.33$

From Fig 12.1-3, for $r/R = 0$ (center of sphere), $\frac{T_1-T}{T_1-T_0} = 0.33$ for
$\frac{\alpha t}{R^2} \cong 0.18 \rightarrow t = (0.18)\frac{(0.05)^2}{4.2 \cdot 10^{-6}} = 106$ (Note $r = \frac{0.1}{2}$)

Example 2



Cylinder of dimensions shown. Note height of cylinder corresponds to 2×(half-width of slab, b) = 2b. $T_0 = 300K$, $T_1 = 600K$.
$\alpha = 4.2 \cdot 10^{-6}$ Want T at point shown, at $t = 800\,min = 4.8 \cdot 10^4$ sec, 0.2m up and 0.5m out from center of cylinder. ($y = .2m$, $r = .5m$. Note origin is in center of solid.)
From handout, Table 4.2, $\frac{T_1 - T}{T_1 - T_0}$ (*) is given by item 9 (p. 185); it is product of P, solution for finite-width slab (or "plate"), and C, solution for cylinder.

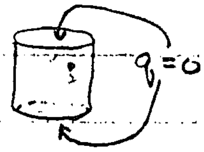
For plate, $\frac{y}{b} = \frac{0.2}{1} = 0.2$; $\alpha t / b^2 = \frac{(4.2 \cdot 10^{-6})(4.8 \cdot 10^4)}{(1)^2} \cong 0.2$, From Fig 12.1-1,
$P \equiv \frac{T_1 - T}{T_1 - T_0} \cong 0.74$ (note P is given by scale on right of Figure)

For cylinder, $\frac{r}{R} = \frac{0.5}{1} = 0.5$; $\alpha t / R^2 = \frac{(4.2 \cdot 10^{-6})(4.8 \cdot 10^4)}{(1)^2} \cong 0.2$. From Figure 12.1-2,
$C \cong 0.33$ (note again we use scale on right.)

$\frac{T_1 - T}{T_1 - T_0} = (0.74)(0.33) \cong 0.25 \rightarrow T \doteq 525\ K$

* this is equivalent to the defs given on the handout on p. 186; they use dif. notation

(Unsteady Conduction in Solids, cont.)



Example 2a

Same as example 2, but both top and bottom surfaces are insulated.

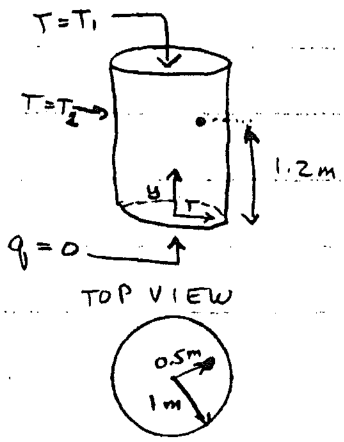
Solution: With both opposite surfaces insulated, there is no conduction in vertical direction, only radially.

Solution is that for infinite-length cylinder

$\frac{r}{R} = \frac{0.5}{1} = 0.5; \quad \frac{\alpha t}{R^2} \doteq 0.2$ as in example 2

$C = \frac{T_1 - T}{T_1 - T_0} = 0.33 \rightarrow T \doteq 500K$

Example 2b



Same as example 2, except bottom surface is insulated while top surface has $T = T_1$.

Solution: this is superposition or product of 2 solutions: a cylinder with $R = 1m$, and a finite-width slab with $b = 2m$.*

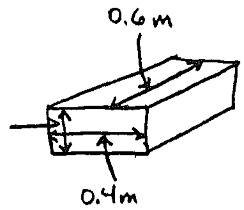
slab: $\frac{y}{b} = \frac{1.2}{2} = 0.6 \quad \frac{\alpha t}{b^2} = 0.05, \quad P = \left(\frac{T_1 - T}{T_1 - T_0}\right)_c = 0.8$

cylinder: $C = 0.33$ again

$$\left(\frac{T_1 - T}{T_1 - T_0}\right) = (0.8)(0.33) = 0.26 \quad T \doteq 522\,K$$

* Note origin is in center of insulated face!

Example 3. $T_0 = 300K$, $T_1 = 400K$



Temperature at center of rectangular steel solid.

$\alpha = 4.2\cdot10^{-6}$ again. Want temperature after 40 min = 2,400 sec. From Handout, Table 4.2, Item 7, this is a product of three "plate" 1-D problems.

For the first, $b = \frac{0.2\,m}{2}$ (note b is $\underline{\underline{\tfrac{1}{2}}}$-width of slab), $\frac{\alpha t}{b^2} = \frac{(4.2\cdot10^{-6})(2400)}{(0.1)^2} = 1.$

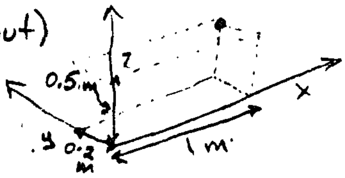
From Fig 12.1-1, $P \approx 0.11$ at $\frac{y}{b} = 0$.

For the second, $b = \frac{0.4}{2}$, $\frac{\alpha t}{b^2} = 0.25$; From Fig 12.1-1, $\frac{y}{b} = 0$, $P \approx 0.71$

For the third, $b = \frac{0.6}{2}$, $\frac{\alpha t}{b^2} \approx 0.11$, " , " , $P \approx 0.93$

$\Theta = \frac{T_1 - T}{T_1 - T_0} \approx (0.11)(0.71)(0.93) \approx 0.073$; $\rightarrow$ $T = 393\,K$. Note that the narrow dimension contributes most to the heating of the block.

Example 4 (cf Item 2, Table 4.2 of handout)



$T_0 = 300K$, $T_1 = 400K$. $\alpha = 4.2\cdot10^{-6}$ again.

Want temperature at a point 1m in x direction, 0.2m in y direction, 0.5m "up" (z direction) from the corner in a huge (semi-infinite) 3-D solid, after $2\frac{1}{2}$ hr (9000 sec) heating.

This is the product of 3 solutions to semi-infinite solid.

For the first (x direction), $\frac{y}{\sqrt{4\alpha t}} = \frac{1}{\sqrt{4(4.2\cdot10^{-6})(9000)}} = 2.57$. This doesn't even appear on Fig 4.1-2, but it's clear that Θ (given by the curve for "n=1") is nearly zero, or $S = 1 - \Theta \cong 1$

For the second (y direction) $\frac{y}{\sqrt{4\alpha t}} = \frac{0.2}{\sqrt{4(4.2\cdot10^{-6})(9000)}} = 0.51$. From Fig 4.1-2 (curve "n=1"), the vertical axis, Θ, is ≈ 0.46. $S = 1 - \Theta = 0.54$,

For the third, (z direction), $\frac{y}{\sqrt{4\alpha t}} = \frac{0.5}{\sqrt{4(4.2\cdot10^{-6})(9000)}} = 1.29$. From Fig 4.1-2, $\Theta \cong 0.06$, $S = 1 - \Theta = 0.94$.

For the 3-D probem, $\frac{T_1 - T}{T_1 - T_0} = (1)(0.54)(0.94) = 0.48$; $T = 349\,K$

3. time-varying boundary conditions: "superposition"

Principles illustrated on homework.

Example: cylinder of infinite length

Define:

Let:

Then

Note this definition of dimensionless T differs from that required for product method!

Can apply to more-complex BC; to be safe, verify assumptions as in homework example.

Example of Unsteady Surface B.C.

A semi-infinite solid is initially at temperature T_0. A $t=0$, its surface temperature is immediately raised to T_1 and held there. At $t=t_1$, the surface temperature is returned to T_0 and held there. Sketch the dimensionless temperature profile in this solid as a function of time.

Sample calculations. It is convenient to plot position as $y/\sqrt{4\alpha t_1}$. Since t_1 is a constant, this is equivalent to plotting a constant $*$ y.

The solution is that $\frac{T-T_0}{T_1-T_0} = \Theta(y,t) - \Theta(y,(t-t_1))$ where Θ is given by Fig 4.1-2.

For $t \le t_1$, the second term is zero and $\frac{T-T_0}{T_1-T_0}$ is given directly by Fig 4.1-2, with the y axis scaled according to t.

For $t > t_1$, one must subtract two terms.

For instance: $t = 1.5t_1$; $y = 0.2\sqrt{4\alpha t_1}$; $\Theta(y,t) = \Theta(y, 1.5t_1) = \Theta\left(\frac{y}{\sqrt{4\alpha t}} = \frac{0.2}{\sqrt{1.5}}\right)$; the value on the horizontal axis is $0.2/\sqrt{1.5} \cong 0.16$, and $\Theta \cong 0.82$.

$\Theta(y, t-t_1) = \Theta(y, \frac{1}{2}t_1) = \Theta\left(\frac{y}{\sqrt{4\alpha t}} = \frac{0.2}{\sqrt{1/2}}\right)$ value on horiz. axis is $(0.2) \times \sqrt{2} \cong 0.28$, and $\Theta \cong 0.67$

Thus $\frac{T-T_0}{T_1-T_0} = 0.82 - 0.67 \approx 0.15$

For $t = 2t_1$, $y = 1.0\sqrt{4\alpha t_1}$

$\Theta(y,t) = \Theta(y, 2t_1) = \Theta\left(\frac{y}{\sqrt{4\alpha t}} = \frac{1}{\sqrt{2}}\right)$, horiz axis value $= \frac{1}{\sqrt{2}} \cong 0.71$.

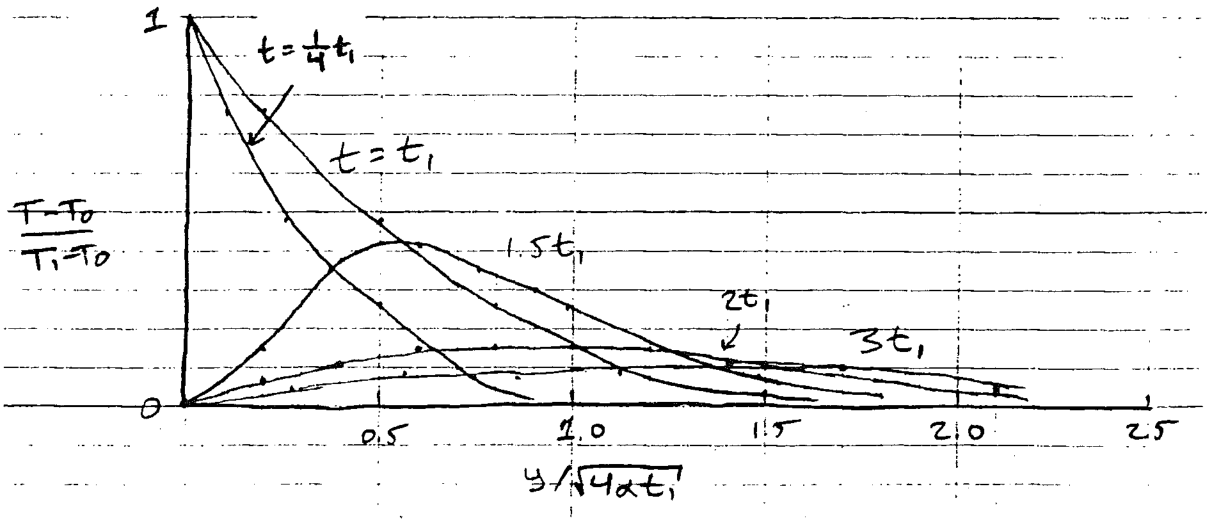
$\Theta \approx 0.33$

$\Theta(y, t-t_1) = \Theta(y, t_1) = \Theta\left(\frac{y}{\sqrt{4\alpha t}} = 1\right) \rightarrow$ horiz. axis value $= 1$

$\Theta \approx 0.17$

$\frac{T-T_0}{T_1-T_0} \cong 0.33 - 0.17 \approx 0.16$

A rough plot for $t = \frac{1}{2}t_1, t_1, 1.5t_1, 2t_1$, and $3t_1$ is shown below.



4. an aside: analogous methods in reservoir engineering

- see also Carslaw and Jaegar's book

Can also use "superposition" to derive the solution for a complex, time-varying boundary conditions from solutions with simpler boundary conditions - this is principle behind well-testing.

5. recap of unsteady conduction

Method:

1. Analyze geometry of solid into its 1D components
2. Deal with perfectly insulated surfaces according to rules in XII.C.2.
3. Determine position of given point in each of the 1D problems
4. If product method applies, then

$$\frac{(T_1-T)}{(T_1-T_0)} = \text{product of factors } \left(\frac{(T_1-T)}{(T_1-T_0)}\right)_i \text{ for each 1D problem}$$

NOTE: Must use $\frac{(T_1-T)}{(T_1-T_0)}$, *not* $\frac{(T-T_0)}{(T_1-T_0)}$, for product method

Product Method

4. a) Let $\theta_1(x,t)$ be the solution of $(\partial\theta_1/\partial t) = \alpha\,(\partial^2\theta_1/\partial x^2)$
   with boundary conditions
   1) $\theta_1 = 1$ at $t = 0$, $-a \le x \le a$ *
   2) $\theta_1 = 0$ at $x = a$, $t > 0$
   3) $\theta_1 = 0$ at $x = -a$, $t > 0$;

   * "unrealized T change" = 1 at start, 0 on surfaces

   and let $\theta_2(y,t)$ be the solution of $(\partial\theta_2/\partial t) = \alpha\,(\partial^2\theta_2/\partial y^2)$
   with boundary conditions
   1) $\theta_2 = 1$ at $t = 0$, $-b \le y \le b$
   2) $\theta_2 = 0$ at $y = b$, $t > 0$
   3) $\theta_2 = 0$ at $y = -b$, $t > 0$;

   and let $\theta_3(z,t)$ be the solution of $(\partial\theta_3/\partial t) = \alpha\,(\partial^2\theta_3/\partial z^2)$
   with boundary conditions
   1) $\theta_3 = 1$ at $t = 0$, $-c \le z \le c$
   2) $\theta_3 = 0$ at $z = c$, $t > 0$
   3) $\theta_3 = 0$ at $z = -c$, $t > 0$.

   Show that $\theta(x,y,z,t) \equiv \theta_1(x,t)\,\theta_2(y,t)\,\theta_3(z,t)$ satisfies
   $$(\partial\theta/\partial t) = \alpha\,((\partial^2\theta/\partial x^2) + (\partial^2\theta/\partial y^2) + (\partial^2\theta/\partial z^2))$$
   with boundary conditions
   1) $\theta = 1$ at $t = 0$, for $-a \le x \le a$, $-b \le y \le b$, $-c \le z \le c$
   2) $\theta = 0$ at $x = a$, $-b \le y \le b$, $-c \le z \le c$, $t > 0$
   3) $\theta = 0$ at $x = -a$, $-b \le y \le b$, $-c \le z \le c$, $t > 0$
   4) $\theta = 0$ at $y = b$, $-a \le x \le a$, $-c \le z \le c$, $t > 0$
   5) $\theta = 0$ at $y = -b$, $-a \le x \le a$, $-c \le z \le c$, $t > 0$
   6) $\theta = 0$ at $z = c$, $-a \le x \le a$, $-b \le y \le b$, $t > 0$
   7) $\theta = 0$ at $z = -c$, $-a \le x \le a$, $-b \le y \le b$, $t > 0$.

   b) Are θ_1, θ_2 and θ_3 given by the left-hand side of Fig. 12.1-1, or the right-hand side?

Superposition

5. a) Let $\theta_1(x,t)$ be the solution of $(\partial\theta_1/\partial t) = \alpha\,(\partial^2\theta_1/\partial x^2)$
   with boundary conditions
   1) $\theta_1 = 0$ at $t \le 0$, $-b \le x \le b$
   2) $\theta_1 = 1$ at $x = b$, $t > 0$
   3) $\theta_1 = 1$ at $x = -b$, $t > 0$.

   dim'less T = 0 at start, = 1 on surfaces

   Show that $\theta(x,t) \equiv \theta_1(x,t) - \theta_1(x,(t-t_1))$ satisfies $(\partial\theta/\partial t) = \alpha\,(\partial^2\theta/\partial x^2)$
   with boundary conditions
   1) $\theta = 0$ at $t \le 0$, $-b \le x \le b$
   2) $\theta = 1$ at $x = b$, $0 < t < t_1$,
      $\theta = 0$ at $x = b$, $t > t_1$
   3) $\theta = 1$ at $x = -b$, $0 < t < t_1$,
      $\theta = 0$ at $x = -b$, $t > t_1$.

   [i.e., the surfaces of the slab are raised to T_1 from $t = 0$ to $t = t_1$ and then returned to T_0 for $t > t_1$.]

   b) Is θ_1 given by the left-hand side of Fig. 12.1-1, or the right?

Solutions continued

4. a) Differential Equation:

$$\frac{\partial\theta}{\partial x} = \frac{\partial\theta_1}{\partial x}\theta_2\theta_3 \quad \text{(since } \theta_2 \text{ and } \theta_3 \text{ do not depend on } x\text{)} \quad \frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta_1}{\partial x^2}\theta_2\theta_3$$

Similarly, $\frac{\partial^2\theta}{\partial y^2} = \theta_1\frac{\partial^2\theta_2}{\partial y^2}\theta_3$, and $\frac{\partial^2\theta}{\partial z^2} = \theta_1\theta_2\frac{\partial^2\theta_3}{\partial z^2}$

According to the chain rule, $\frac{\partial\theta}{\partial t} = \frac{\partial\theta_1}{\partial t}\theta_2\theta_3 + \theta_1\frac{\partial\theta_2}{\partial t}\theta_3 + \theta_1\theta_2\frac{\partial\theta_3}{\partial t}$

But $\frac{\partial\theta_1}{\partial t} = \alpha\frac{\partial^2\theta_1}{\partial x^2}$, $\frac{\partial\theta_2}{\partial t} = \alpha\frac{\partial^2\theta_2}{\partial y^2}$, $\frac{\partial\theta_3}{\partial t} = \alpha\frac{\partial^2\theta_3}{\partial z^2}$;

$$\therefore \frac{\partial\theta}{\partial t} = \alpha\frac{\partial^2\theta_1}{\partial x^2}\theta_2\theta_3 + \theta_1\alpha\frac{\partial^2\theta_2}{\partial y^2}\theta_3 + \theta_1\theta_2\alpha\frac{\partial^2\theta_3}{\partial z^2} = \alpha\left[\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2}\right] ✓$$

BC 1: at $t=0$, $\theta_1=1, \theta_2=1, \theta_3=1 \rightarrow \theta = 1\cdot1\cdot1 = 1$ ✓

2: at $x=a$, $\theta_1=0 \rightarrow \theta = 0\cdot\theta_2\cdot\theta_3 = 0$ ✓

3: at $x=-a$, $\theta_1=0 \rightarrow \theta = 0\cdot\theta_2\cdot\theta_3 = 0$ ✓

4: at $y=b$, $\theta_2=0 \rightarrow \theta = \theta_1\cdot0\cdot\theta_3 = 0$ ✓

5: at $y=-b$, $\theta_2=0 \rightarrow \theta = \theta_1\cdot0\cdot\theta_3 = 0$ ✓

6: at $z=c$, $\theta_3=0 \rightarrow \theta = \theta_1\cdot\theta_2\cdot0 = 0$ ✓

7: at $z=-c$, $\theta_3=0 \rightarrow \theta = \theta_1\theta_2\cdot0 = 0$ ✓

5 a) Differential eq: $\frac{\partial\theta}{\partial t} = \frac{\partial\theta_1}{\partial t}\Big|_t - \frac{\partial\theta_1}{\partial t}\Big|_{t-t_1} = \alpha\frac{\partial^2\theta_1}{\partial x^2}\Big|_t - \alpha\frac{\partial^2\theta_1}{\partial x^2}\Big|_{t-t_1}$ $\boxed{\text{I}}$

$\frac{\partial\theta}{\partial x} = \frac{\partial\theta_1}{\partial x}\Big|_t - \frac{\partial\theta_1}{\partial x}\Big|_{t-t_1}$; $\frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta_1}{\partial x^2}\Big|_t - \frac{\partial^2\theta_1}{\partial x^2}\Big|_{t-t_1}$ $\boxed{\text{II}}$

$$\boxed{\text{I}} \rightarrow \frac{\partial\theta}{\partial t} = \alpha\left(\frac{\partial^2\theta_1}{\partial x^2}\Big|_t - \frac{\partial^2\theta_1}{\partial x^2}\Big|_{t-t_1}\right) = \alpha\frac{\partial^2\theta}{\partial x^2} ✓$$

B.C.: 1) at $t=0$, $\theta_1(t) = \theta_1(0) = 0$; $\theta_1(t-t_1) = 0$ as well

$\therefore \theta = 0 - 0 = 0$ ✓

2) for $x=b$, $0<t<t_1$, $\theta_1(t)=1$, $\theta_1(t-t_1)=0$; $\theta=1-0=1$ ✓

for $x=b$, $t>t_1$, $\theta_1(t)=1$, $\theta_1(t-t_1)=1$; $\theta=1-1=0$ ✓

3) for $x=-b$, $0<t<t_1$, $\theta_1(t)=1$, $\theta_1(t-t_1)=0$; $\theta=1-0=1$ ✓

for $x=-b$, $t>t_1$, $\theta(t)=1$, $\theta_1(t-t_1)=1$; $\theta=1-1=0$ ✓

4 b) In this problem, θ_1, θ_2 and θ_3 are 1 initially and are later changed to 0 at the boundaries. This is the "unrealized temperature change," $(T_1-T)/(T_1-T_0)$, found on the right side of Fig. 11.1-1.

5 b) In this problem, θ_1 is initially 0 and is changed to 1 at the boundary for $t>0$. This is $(T-T_0)/(T_1-T_0)$, found on the left side of Fig 11.1-1.