XIII. Analysis of complex transport problems

A. Motivation

Although you learn (I hope!) a number of specific analytical and mathematical tools for solving transport problems in this course, most problems in the real world don't fit any of the tools you have learned - or will ever learn. Consider two simple examples:

1. Cooling of hot metal ball in cold water

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2. Cooling of cup of coffee

-

B. method:

1. Diagram overall transport process in terms of its individual "modes"

   •

   •

2. Analyze the problem

   • for modes , assuming any mode is always gives on of the rate of heat transfer

     - i.e., in the estimate than in the real system

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3. Simplify the problem:

   For modes in , focus on , "controlling" mode

   - assume all other modes are - infinitely fast in transport

   For modes , focus on , "controlling" mode

   - ignore transport from slower modes - assume they are

4. Solve simplified transport problem

5. If possible, check solution against
   - Complete solution if available;
   - Solutions assuming <u>other</u> modes control transport

     - for modes in series
       solution giving lowest rate of transport (slowest equilibration) is best approximate solution
       if solution is *much* slower than all other approximate solutions (assuming other modes control transport), then it is probably close to complete solution

     - for modes in parallel
       solution giving <u>highest</u> rate of transport (fastest equilibration) is best approximate solution
       if solution is *much* faster than all other approx solutions (assuming other modes control transport), it is probably close to complete solution

   - experiment

6. Estimate qualitative effect of modes for which no analytical solution is available
   - Can one say whether computed solution an "upper bound" or "lower bound" to true behavior?
     - if modes excluded are in series with mode studied, true transport rate is slower than computed solution
     - if modes excluded are in parallel with mode studied, true transport rate is faster than computed solution

C. Previously encountered examples of simplified solutions

1. "Lumped-Parameter Analysis" - assume system, or part of system, has uniform T. In other words, heat transfer with fluid at surface is <u>**in series**</u> with internal conduction; internal conduction is assumed to offer negligible resistance to heat transfer compared to convective heat transfer at the surface, and can therefore be ignored. Determine rate of change of T from macroscopic energy balance, since internal temperature is uniform.
   See I.C.2.d, "cooling of metal ball," and
   XI.D.2 - "example of macroscopic energy balance"
   Related case: in BSL 10.7 (cooling fin), Newton's law at $x = \pm B$ is in series with conduction in x direction; conduction is assumed to offer negligible resistance and is ignored.
2. Assume surface T at equilibrium with surroundings. In other words, heat transfer at surface is <u>**in series**</u> with internal conduction; because it is assumed to offer negligible resistance in these examples, it can therefore be ignored. used in
   - BSL section 10.2 (conduction in wire),
   - BSL section 10.3 (conduction in spherical nuclear fuel element),
   - BSL section 12.1 (unsteady conduction)).

   In BSL sect. 10.5 (convection and conduction in porous medium), convective heat transfer between catalyst particles and surrounding fluid is in series with transport along axis of bed; radial heat transfer is assumed to be much faster, and is ignored; particles are assumed to be at equilibrium with surrounding fluid.
3. Ignore one direction of conduction in multi-directional heat-conduction problem. In other words, conduction in one direction is <u>**in parallel**</u> with conduction in other directions. <u>Slowest</u> mode of heat transfer can therefore be ignored. (used in BSL section 10.8 (convection and conduction in laminar tube flow))

1. **an aside: Why use $\frac{(T_1-T)}{(T_1-T_0)}$ or $\frac{(T-T_0)}{(T_1-T_0)}$ in the "product method"?**

<u>Brute-force answer</u>:

You will never go wrong in this course using $\frac{(T_1-T)}{(T_1-T_0)}$ (except for two semi-infinite slabs in contact, for which dimensionless temperature is defined differently). It's OK to use it in all other cases, and it's <u>required</u> to use it for product method.

<u>Better answer</u>:

Product method describes conduction modes "in parallel" rather than in series. In other words, conduction in the x, y and z directions (or r and z directions) each <u>separately</u> transport energy - none acts as a bottleneck for the others.

Using $\frac{(T_1-T)}{(T_1-T_0)}$ as dimensionless temperature in the product method means that thinnest dimension dominates the heat-transfer problem, as it should for modes in parallel. In other words, if the thinnest dimension gives a dimensionless T near zero (T near T_1), the product of dimensionless T's is near zero (T near T_1). This means that conduction into a solid in 3D proceeds at least as fast as conduction in any one direction. This is appropriate for modes in "parallel."

If one used $\frac{(T-T_0)}{(T_1-T_0)}$ by mistake as dimensionless T in the product method, the <u>thickest</u> dimension would control heat-transfer process. In other words, if the dimensionless T for the thickest dimension were near zero (T near T_0), the product of dimensionless T's would be near zero (T near T_0). In essence, this says that the whole solid must wait for the slowest mode to approach equilibrium. This is inappropriate for modes "in parallel."

D. worked examples

1. **Iron ball,** Dia. = 0.1 m. initial T = 0°C at t = 0. At t ≥ 0, ball is dropped into boiling water with T maintained at 100°C. h = 500 W/(m^2 K) at surface of ball. How long until T reaches 90°C at center of ball? For iron, k = 73 W/(m K), ρ = 7880 kg/m^3, and $\hat{C}_p$ = 511 J/(kg K).

method: **steps (1) and (2): Diagram and analyze problem:**

modes:

1)

2)

- modes "in series"

step 3) Simplify problem.

Approximate solution (#1). Assume mode 1 "controls." For modes in series, this means that mode 2 (internal conduction) is at equilibrium. Ball T is uniform at all times. T(r,t) = T(t) in ball.

step 4) Solve simplified problem. Perform macroscopic energy balance

back to step (3) Approximate solution (#2). Assume mode (2) "controls." For modes in series, that means that mode (1) (convective heat transfer to surface) is at equilibrium: T of water = T at surface of ball.

step 4) Solve simplified problem.

step 5) check solutions.

Approximate solution (#1) predicts slower heat transfer. For modes in series, *any* approximate solution that leaves out one mode gives *faster* equilibration than the real process. Therefore, approximate solution (#1), which evidently errs less in this direction, is the better answer.

step 6) estimate effect of simplifying problem.

For modes in series, leaving out *any* mode can only err towards giving faster equilibration than is correct. Therefore, the true answer is *slower* equilibration, or a longer time to come to 90°C, than estimated above.

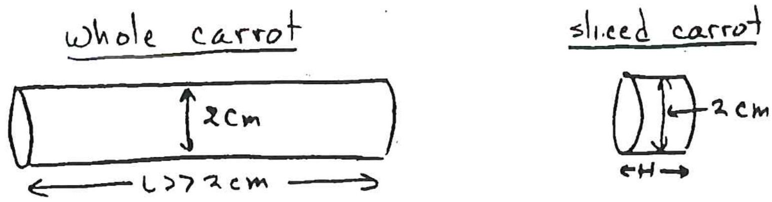
However, since approximate solution (#1) predicts *much* slower equilibration than the other solution, mode 2 is evidently relatively unimportant in this overall process. Therefore approximate solution (#1), which errs only in leaving out insignificant mode 2, is probably nearly correct.

13.7

Rocky is cooking carrots in boiling water. The carrots are initially at 0°C (the refrigerator temperature) and must be heated to 80°C at the center, in water that is maintained at 100°C. Rocky usually prepares the carrots by boiling them whole, and they then take 10 minutes to reach the desired temperature. A whole carrot is a long cylinder, about 2 cm in diameter. Rocky wants to slice the carrots into cylindrical disk shapes, with diameter still 2 cm, but with thickness H, so that the carrot pieces will reach the desired temperature in only 5 minutes. What value of H should he pick if

a) he assumes that convective heat-transfer to the surface controls the process of heating of the carrot?

b) he assumes that conduction within the solid carrot controls the process of heating of the carrot?



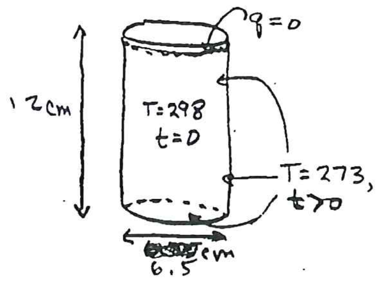
a) An energy balance for both cases is $V\rho\hat{C}_p \frac{dT}{dt} = -h(T-T_w)A$
$\rightarrow dT/dt = -[hA/(V\rho\hat{C}_p)](T-T_w) \rightarrow \ln(T-T_w) = -[hAt/(V\rho\hat{C}_p)] + C$, or
$(T-T_w) = C'\exp\{-[hAt/(V\rho\hat{C}_p)]\}$. Since $T-T_w$ is initially 100K, $C' = 100$.
If one wants the same $(T-T_w)$ in half the time, one must compensate so the factor $hAt/(V\rho\hat{C}_p)$ is unchanged. h and $\rho\hat{C}_p$ are fixed, so the change has to come in the ratio of A to V.
For the whole carrot, $A = 2\pi R^2 + 2\pi RL \cong 2\pi RL$ if $L \gg R$; $V = \pi R^2 L$, so
$hAt/V\rho\hat{C}_p = (h/\rho\hat{C}_p)[2\pi RLt/(\pi R^2 L)] = (h/\rho\hat{C}_p)(2t/R)$.
For the sliced carrot, $A = 2\pi R^2 + 2\pi RH$, $V = \pi R^2 H$, and $t' = \frac{1}{2}t$.
$hAt/(V\rho\hat{C}_p) = (h/\rho\hat{C}_p)[(2\pi R^2 + 2\pi RH)\frac{1}{2}t/(\pi R^2 H)]$. The factor in brackets should equal $2t/R$ as for the whole carrot:

$$\frac{(2\pi R^2 + 2\pi RH)t}{2\pi R^2 H} = \frac{2t}{R} \rightarrow \frac{(R/H+1)t}{R} = \frac{2t}{R} \rightarrow R/H = 1 \text{ or } H = R = 0.01\,m$$
(1 cm)

b) For the whole carrot, since $L \gg R$, this is an "infinite" cylinder, and conduction reflects Fig 12.1-2. Since we want $T = 80°C$, with $T_1 = 100°C$ and $T_0 = 0°C$, we want either $(T-T_0)/(T_1-T_0) = 0.8$ or $(T_1-T)/(T_1-T_0) = 0.2$ at $r = 0$.
From Fig 12.1-2, we need $\alpha t/R^2 \cong 0.36$.
Now with the sliced carrot, the solution will be a product of slab + cylinder. For the cylinder, α and R are unchanged, and t decreases by $\frac{1}{2}$. Therefore $\alpha t/R^2 \cong 0.18$. From Fig 12.1-2, $(T_1-T)/(T_1-T_0) \cong 0.54$. (For this case, since product method is involved, one must use $(T_1-T)/(T_1-T_0)$.) We still need $\frac{T_1-T}{T_1-T_0} = 0.2$ for carrot slice. Therefore $(0.2) = (0.54)\left(\frac{T_1-T}{T_1-T_0}\right)_{slab} \rightarrow \left(\frac{T_1-T}{T_1-T_0}\right)_{slab} = 0.37$.
From Fig 12.1-1, $(T_1-T)/(T_1-T_0) = 0.37$ at $b = 0$ if $\alpha t/b^2 \cong 0.5$
Now if $\frac{\alpha t}{b^2} = 0.5$ and $\frac{\alpha t}{R^2} = 0.18$, then $0.5b^2 = 0.18R^2 \rightarrow b = (0.18/0.5)^{1/2}R$
$b = 0.6R$; $2b = H = (1.2)(0.01\,m) = 0.012\,m = 1.2\,cm$

13.8

Rocky is trying to cool off a can of soda in ice water. The can is 12 cm high and 6.5 cm in diameter. The can is originally at 298°K, and the ice water is maintained at 273°K. There is an air gap in the can above the top surface of the soda, and Rocky assumes that q=0 there. Rocky assumes that the other surfaces of the can are immediately reduced (at t=0) to 273°K, and that the liquid in the can does not flow around as it cools. The walls of the aluminum can itself are so thin that they can be ignored. For the soda he estimates the properties are roughly those of water ($\rho \sim 1000$ kg/m³, $\hat{C}_p \sim 4200$ J/(kg K), $k \sim 0.6$ W/(m K)).



a) What will be the temperature of the liquid at the warmest spot in the can after 30 min, according to these assumptions?

b) Where is this location?

c) Suppose instead that Rocky had assumed that convective heat transfer to the surface of the can controlled its rate of cooling, with a heat-transfer coefficient of 100 W/(m² K). What would be the temperature of the soda after 30 minutes using this assumption?

d) Given your answers to (a) and (c), which assumption is most accurate?

a) This is product of slab + cylinder. Because $q=0$ at top, b for slab = 12 cm = 0.12 m

For cylinder, $R = 6.5\text{ cm}/2 = 0.0325\text{ m}$.

$\alpha = k/\rho \hat{C}_p = (0.6)/[(1000)(4200)] = 1.43\cdot 10^{-7}$

$t = 30 \cdot 60 = 1800$ sec

slab: $\alpha t/b^2 = (1.43\cdot10^{-7})(1800)/(0.12)^2 = 0.0178$. For $y/b=0$, $\frac{T_1-T}{T_1-T_0} = 1$. (Fig 12.1-1)

cylinder: $\alpha t/R^2 = (1.43\cdot10^{-7})(1800)/(0.0325)^2 = 0.243$. For $r/R=0$, $\frac{T_1-T}{T_1-T_0} \simeq 0.4$

(very rough estimate from Fig 12.1-2)

$\frac{T_1-T}{T_1-T_0} = (1)(0.4) = 0.4 = \frac{273-T}{273-298}$, $T \approx 283\text{ K}$.

b) The warmest location is in the center at the top.

c) $\pi R^2 H \rho \hat{C}_p \, dT/dt = h(T_w - T)(\pi R^2 + 2\pi R H)$

$-\frac{d(T_w-T)}{(T_w-T)} = \frac{h(\pi R^2 + 2\pi R H)}{\pi R^2 H \rho \hat{C}_p} dt$; let "K" $\equiv \frac{h(\pi R^2 + 2\pi RH)}{\pi R^2 H \rho \hat{C}_p} = \frac{h(R+2H)}{RH\rho \hat{C}_p}$

$\ln(T_w - T) = -Kt + C_1$ BC: $T_w - T = 273-298$ at $t=0 \rightarrow C_1 = \ln(273-298)$

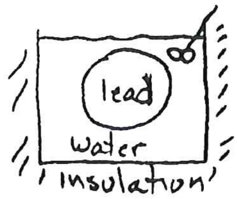
$\ln\left(\frac{273-T}{273-298}\right) = -Kt$ $K = 100\,\frac{(0.0325 + 2(0.12))}{(0.0325)(0.12)(1000)(4200)} = 0.00166$

at $t = 1800$ sec,

$\ln\left(\frac{273-T}{273-298}\right) = (-2.99)$; $T = 274.3$ K.

d) These are 2 resistances in "series". Conduction mode gives slower equilibration and is best approximation (assuming there is no convection in can!).

A solid sphere of lead, 10 cm in diameter, is initially at 100°C. It is placed in an insulated, stirred water bath with 1 liter of water, initially at 0°C. The water completely covers the sphere. Assume that convective heat-transfer between the solid and the water controls the rate of heating of the solid; in other words, ignore conduction within the solid. In this case, because the volume of water is limited, <u>the water heats up as the solid cools down</u>. The heat-transfer coefficient at the surface of the solid is 250 W/(m^2 K).



a) Perform separate energy balances on the water and the solid to derive an equation for $U \equiv (T_S - T_L)$, the temperature <u>difference</u> between the solid and liquid, as a function of time.

b) What is the final temperature of the solid and liquid?

<u>properties of lead</u>

$k = 34.6$ W/(m K) $\quad \hat{C}_p = 130$ J/(kg K) $\quad \rho = 11{,}340$ kg/m^3

<u>properties of water</u>

$\rho = 1000$ kg/m^3 $\quad k = .680$ W/(m K) $\quad \hat{C}_p = 4190$ J/(kg K)

a) Energy balance on sphere: $\underbrace{\underbrace{\frac{4}{3}\pi R^3}_{\text{volume}} \rho_s \hat{C}_{ps} \frac{dT_s}{dt}}_{\text{accumulation of energy}} = \underbrace{-\underbrace{4\pi R^2}_{\text{area}} h (T_s - T_w)}_{\text{rate of energy flow in}}$

The minus sign accounts for the fact that if $T_s > T_w$ the sphere <u>cools</u> down.

Energy balance on water: $V_w \rho_w \hat{C}_{pw} \frac{dT_w}{dt} = 4\pi R^2 h (T_s - T_w)$

In this case area for heat transfer is still spherical surface, but if $T_s > T_w$, the water heats up.

Now since $U \equiv T_s - T_w$, $\frac{dU}{dt} = \frac{dT_s}{dt} - \frac{dT_w}{dt} = 4\pi R^2 h (T_s - T_w)\left[-\left(\frac{4}{3}\pi R^3 \rho_s \hat{C}_{ps}\right)^{-1} - \left(V_w \rho_w \hat{C}_{pw}\right)^{-1}\right]$

Plugging in numbers, $\frac{dU}{dt} = -4\pi(0.05)^2 250 (T_s - T_w)\left\{\left[\frac{4}{3}\pi(0.05)^3(11{,}340)(130)\right]^{-1} - \left[(0.001)(1000)(4190)\right]^{-1}\right\}$

$\frac{dU}{dt} = -(7.854)(T_s - T_w)\left[(772)^{-1} + (4190)^{-1}\right] = (-0.0120)(T_s - T_w) = -0.0120\, U$

$\frac{dU}{U} = -0.0120\,t \rightarrow \ln U = -0.0120\,t + c$ or $U = c' \exp(-0.0120\,t)$

Since $U = 100$ K at $t = 0$, $c' = 100$, and $U = 100 \exp(-0.0120\,t)$

b) Energy lost by sphere $= \frac{4}{3}\pi R^3 \rho_s \hat{C}_{ps} (T_s^\circ - T_f)$

" gained by water $= V_w \rho_w \hat{C}_{pw} (T_f - T_w^\circ)$

These must be equal. Therefore $\frac{4}{3}\pi R^3 \rho_s \hat{C}_{ps} (T_s^\circ - T_f) = V_w \rho_w \hat{C}_{pw} (T_f - T_w^\circ)$

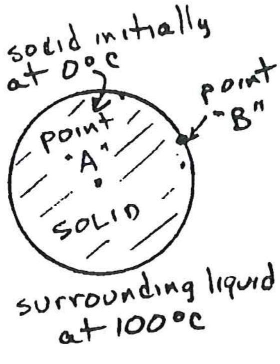
$$T_f \left(V_w \rho_w \hat{C}_{pw} + \frac{4}{3}\pi R^3 \rho_s \hat{C}_{ps}\right) = \frac{4}{3}\pi R^3 \rho_s \hat{C}_{ps} T_s^\circ + V_w \rho_w \hat{C}_{pw} T_w^\circ$$

Actually, one could start with this equation, which says that the total energy of sphere + water at the start (right-hand side) = total energy at end (left side). Plugging in numbers,

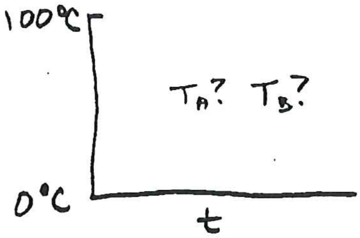
$$T_f (4190 + 772) = 772(373) + 4190(273) \rightarrow T_f = 288.6 \text{ K or } 15.6\,^\circ\text{C}$$

(It would be OK to leave T's in °C here, since only T <u>differences</u> enter problem.)

An engineer is interested in the heating of a spherical solid in hot water. The solid is initially at 0°C and the water is maintained at 100°C. The engineer doesn't know the properties of the solid or whether the heating process is controlled by internal conduction within the solid or by convective heat transfer to the surface. To investigate further, she has placed two thermocouples in the solid, to measure temperature T_A at point A, at the center, and temperature T_B at point B, at the edge, as functions of time after the sphere enters the hot water. From a comparison of the plots of T_A and T_B v. time, she plans to determine which process controls the heating. Show how one could do this as follows:



a) Assume that convective heat transfer to the surface of the solid controls the heating. Draw a rough sketch of how temperature T_A would vary with time. On this same plot, sketch T_B v. time. You cannot calculate exact values here: just make clear the general shapes of the curves - whether the curves are convex upwards or downwards, or "S shaped" - and roughly how the two curves compare with each other. On your answer sheet, clearly mark which curve is for T_A and which for T_B.



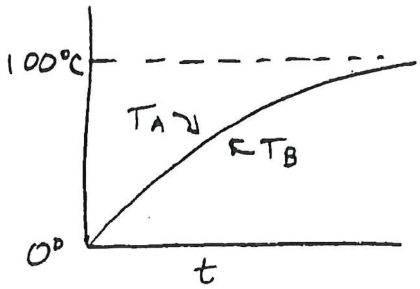
b) Now assume that internal conduction controls the heat-transfer process and make a plot of T_A and T_B v. time for this case.

a) If convective heat transfer controls the process, the we can assume the solid T is uniform and do a macroscopic energy balance:

$$\rho \hat{C}_p (\text{Vol.}) \frac{dT}{dt} = -h(\text{Area})(T - T_w)$$

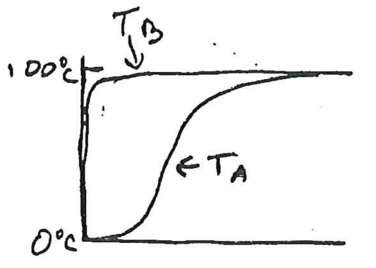
$$\frac{d(T - T_w)}{T_w - T} = -\frac{hA}{\rho \hat{C}_p V} dt$$

$$\rightarrow \frac{T - T_w}{(T - T_w)_{t=0}} = \exp\left(-\frac{hAt}{\rho \hat{C}_p V}\right)$$



In this case $T_A = T_B$ (the solid is assumed at uniform T) and both T_A and T_B approach 100°C with an exponential decay.

b) In this case T_B immediately reaches 100°C - the solid heats from the outside in. Meanwhile, T_A does not change at all at first (See Fig 11.1-3 for $r/R = 0$: no change in T until $\alpha t/R^2 \approx 0.04$); then it changes pretty rapidly, then asymptotically approaches final T. Thus a rough sketch looks as at right.

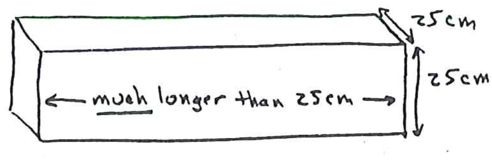


The two types of behavior are sharply different - especially in whether T_A and T_B are similar at short times. This would give a way to tell which explanation (or something in between, perhaps) is better.

1. a) Dr. Kluge, the famous TV police doctor, has a murder case on his hands. The naked body was found at midnight completely surrounded by ice-cold water (0°C); before the murder, its temperature was, of course, about 37°C. When found, the body's internal temperature was 5°C at its warmest point. Assuming the body has the same thermal properties as water, and can be treated as a long rectangular solid roughly 25 cm deep × 25 cm wide, what is the <u>shortest</u> possible time the body could have been in the water, consistent with these assumptions? Justify your answer.

<u>Properties of water:</u>

$\rho = 1000$ kg/m^3 $\qquad \hat{C}_p = 4190$ J/(kg °K) $\qquad k = 0.67$ W/(m °K)



b) When the murderer confesses, Dr. Kluge is stunned to discover that the murder took place 48 hours before the body was found. Given the fact that the body's internal temperature was 5°C after 48 hr in the cold water, what is the <u>lowest</u> possible value of the heat-transfer coefficient h between the water and the surface of the body? Assume the body shape and thermal properties are those given in part (a).

1. a) There are two modes of heat transport here: convection to the surface + unsteady conduction within. Given the parameters governing unsteady conduction, the <u>shortest</u> time is that for infinitely fast convection heat transfer: i.e., assuming surface T = 0°C starting at t=0.

T at the center = 5°C; $\frac{T_1 - T}{T_1 - T_0} = \frac{5}{38} = 0.135$. Since one dimension is much greater than the others, we can ignore it. We have then a 2D product-method problem: $\left(\frac{T_1 - T}{T_1 - T_0}\right)^2_{1D} = 0.135$; $\left(\frac{T_1 - T}{T_1 - T_0}\right)_{1D} = 0.368$.

For $y/b = 0$, this corresponds to roughly $\frac{\alpha t}{b^2} \cong 0.50$. $\alpha = \frac{0.67}{(1000)\,4190}$

$\to \alpha = 1.60 \cdot 10^{-7}$ m^2/s, $t = (0.50)(0.125)^2/(1.60 \cdot 10^{-7}) = 4.9 \cdot 10^4$ sec, or about 13 hr 30 min.

b) The body cooled off a <u>lot</u> more slowly than estimated ignoring convection. Therefore convection must have dominated the process. The <u>lowest</u> possible value of h assumes that internal conduction offered no resistance at all; i.e., that T was uniform in the body.

Energy balance on body: $\rho C_p V \frac{dT}{dt} = -h A (T - T_w)$

with V = volume = $(0.25)^2 L$; $A = (0.25) L\, 4$ (ignoring ends);

$T_w = 0$°C.

$\to \frac{d(T - T_w)}{(T - T_w)} = \frac{-h L \, dt}{(1000)(4190)(0.25)^2 L} = -3.82 \cdot 10^{-6} h \, dt$

$\ln (T - T_w) = -(3.82 \cdot 10^{-6} h) t + C_1$

$\to \frac{T - T_w}{(T - T_w)_{t=0}} = \exp[-3.82 \cdot 10^{-6} h \, t] \to \frac{5}{37} = \exp[-3.82 \cdot 10^{-6} h \; 48 (60)^2]$

$\ln \frac{5}{37} = -0.66 h \to h = 3.03$ W/m^2 K

E. a final note

Determining which mode is fastest or slowest can be ambiguous.

For instance, consider a solid sphere dropped in water. There are two modes in series.

At $t \rightarrow 0$, by itself, the internal conduction formula (Fig. 12.1-3) predicts an infinite rate of heat transfer

(recall eq. 12.1-10:)

∴ Newton's law of cooling at the surface *always* controls in the first instant of contact. At these short times, Newton's law isn't able to keep up with the huge capacity of internal conduction with a sharp temperature gradient at the surface. Therefore at short time, Newton's law is the "bottleneck,", and controls the heat-transfer process.

If convective heat transfer to the surface is relatively efficient, then control soon shifts to internal conduction mode; at long times, Fig. 12.1-3 is reasonably accurate.

Lesson: approach outlined here for "complex" problems is heuristic, not rigorous.