Newtonian fluid in a slit

a) Control volume: (see Fig 2.6) Rectangular region of thickness ΔX, width W and length L.

Momentum balance: on z component of momentum:

convection of momentum in $\{\underset{(\text{area})}{(\Delta X\, W)}\ \underset{(\text{momentum})}{\rho V_z}\ \ V_z\}|_{z=0}$

" out $\{(\Delta X\, W)\ \rho V_z\ \ V_z\}|_{z=L}$

shear stress at surface at x: $\{\underset{(\text{area})}{(LW)}\ \underset{(\text{shear stress})}{\tau_{xz}}\}|_x$

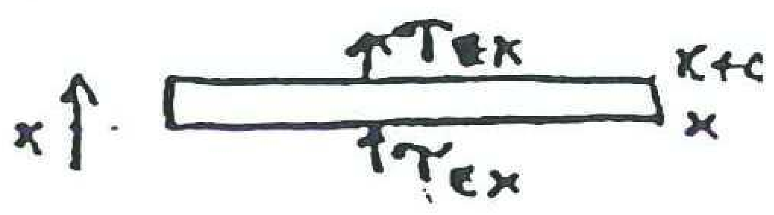
" $x+\Delta x$: $\{LW\ \ \tau_{xz}\}|_{x+\Delta x}$

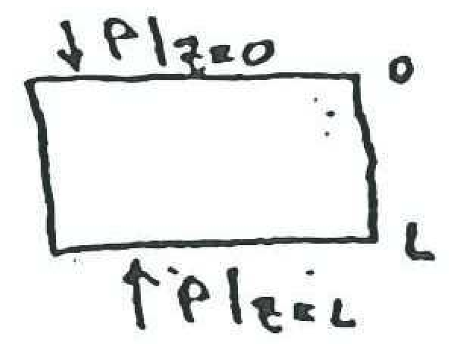
gravity force : $\underset{(\text{VOLUME})}{(LW\Delta X)}\ \rho g$

pressure force on surface at $z=0$: $\{\underset{(\text{area})}{(\Delta X W)}\ P\}|_{z=0}$

" $z=L$: $\{\Delta X W\ P\}|_{z=L}$

note that shear stress (defined as positive in the positive x direction) is $\underset{\text{flux}}{\text{out}}$ of the system at the surface $(x+\Delta x)$. Hence it appears as negative term in momentum balance. Similarly, pressure force at $z=0$ surface is in the positive z direction, while at $z=L$ pressure acts in negative z direction. Hence $z=L$ term appears as negative term in momentum balance. Because this is steady-state, there is no accumulation term in balance. Momentum balance is





$$(\Delta x\, W)\,\rho v_z^2|_{z=0} - (\Delta x\, W)\rho v_z^2|_{z=L} + (LW)\tau_{xz}|_x - (LW)\,\tau_{xz}|_{x+\Delta x} + (LW\Delta x)\rho g$$
$$+ (\Delta x W) P|_{z=0} - (\Delta x W) P|_{z=L} = 0 .$$

b) Since the fluid is incompressible, V_z does not change with z and the first two terms cancel. Divide rest of equation by $LW\Delta x$:

$$\frac{\tau_{xz}|_x - \tau_{xz}|_{x+\Delta x}}{\Delta x} + \rho g + \frac{P|_{z=0} - P|_{z=L}}{L} = 0 . \quad \boxed{\text{I}}$$

Let $\mathcal{P} \equiv P - \rho g z$; then $\mathcal{P}_0 - \mathcal{P}_L = (P|_{z=0} - \rho g(0)) - (P|_{z=L} - \rho g L) = \rho g L + P_0 - P_L \equiv \Delta\mathcal{P}$

$$\boxed{\text{I}} \rightarrow \frac{\tau_{xz}|_x - \tau_{xz}|_{x+\Delta x}}{\Delta x} = -\frac{\Delta\mathcal{P}}{L} \quad (\text{or } -(\mathcal{P}_0 - \mathcal{P}_L)/L)$$

Let $\Delta x \to 0$; $-\frac{d}{dx}(\tau_{xz}) = -\frac{\Delta\mathcal{P}}{L}$ or $\frac{d}{dx}(\tau_{xz}) = \frac{\Delta\mathcal{P}}{L}$

Integrate → $\tau_{xz} = \frac{\Delta\mathcal{P}}{L}x + C_1$ $\boxed{II}$

c) Newton's law: $\tau_{xz} = -\mu\frac{dv_z}{dx}$; $\boxed{II} \to \frac{dv_z}{dx} = -\frac{\Delta\mathcal{P}}{\mu L}x - \frac{C_1}{\mu}$

d) Integrate: $v_z = -\frac{\Delta\mathcal{P}}{\mu L}\frac{x^2}{2} - \frac{C_1}{\mu}x + C_2$ $\boxed{III}$

e) b.c.: 1) $v_z = 0$ at $x = B$ 2) $v_z = 0$ at $x = -B$

f) Subtract Eq. III for $x=B$ from that for $x=-B$:

$$v_z(x=B) - v_z(x=-B) = 0-0 = -\frac{\Delta\mathcal{P}B^2}{2\mu L} - \frac{C_1}{\mu}B + C_2 + \frac{\Delta\mathcal{P}B^2}{2\mu L} - \frac{C_1}{\mu}B - C_2 = \frac{2C_1}{\mu}B$$

$$\to C_1 = 0$$

Then evaluate III at $x=B$: $v_z = 0 = -\frac{\Delta\mathcal{P}}{2\mu L}B^2 + C_2$; $C_2 = \frac{\Delta\mathcal{P}}{2\mu L}B^2$

$\therefore$ III $\to$ $v_z = \frac{\Delta\mathcal{P}B^2}{2\mu L}\left(1-\left(\frac{x}{B}\right)^2\right)$ IV

g) Since $C_1 = 0$, II $\to$ $\tau_{xz} = \frac{\Delta\mathcal{P}}{L}x$

h) $Q = W\int_{-B}^{B} v_z\,dx = W\left[\int_{-B}^{B} v_z\,dx\right] = W\frac{\Delta\mathcal{P}B^2}{2\mu L}\int_{-B}^{B}\left(1-\left(\frac{x}{B}\right)^2\right)dx = \frac{W\Delta\mathcal{P}B^2}{2\mu L}\left(x - \frac{x^3}{3B^2}\right)\Big|_{-B}^{B}$

$= \frac{W\Delta\mathcal{P}B^2}{2\mu L}\left(B - \frac{B^3}{3B^2} + B - \frac{B^3}{3B^2}\right) = \frac{W\Delta\mathcal{P}B^2}{2\mu L}\left(2B - \frac{2B}{3}\right) = \frac{W\Delta\mathcal{P}B^2}{2\mu L}\left(\frac{4}{3}B\right)$

$= \frac{2\Delta\mathcal{P}B^3W}{3\mu L}$

i) differentiate v_z with respect to x: IV $\to$ $\frac{dv_z}{dx} = \frac{\Delta\mathcal{P}B^2}{2\mu L}\,2\frac{x}{B^2} = \frac{\Delta\mathcal{P}}{\mu L}x$

$\frac{dv_z}{dx} = 0$ at $x = 0$. At $x = 0$, $v_z = v_{z,max} = \frac{\Delta\mathcal{P}B^2}{2\mu L}$

j) $\langle v_z\rangle = \frac{2}{3}\frac{\Delta\mathcal{P}B^3W}{\mu L} \div (2BW) = \frac{\Delta\mathcal{P}B^2}{3\mu L} = \frac{2}{3}\left(\frac{\Delta\mathcal{P}B^2}{2\mu L}\right) = \frac{2}{3}v_{z,max}$

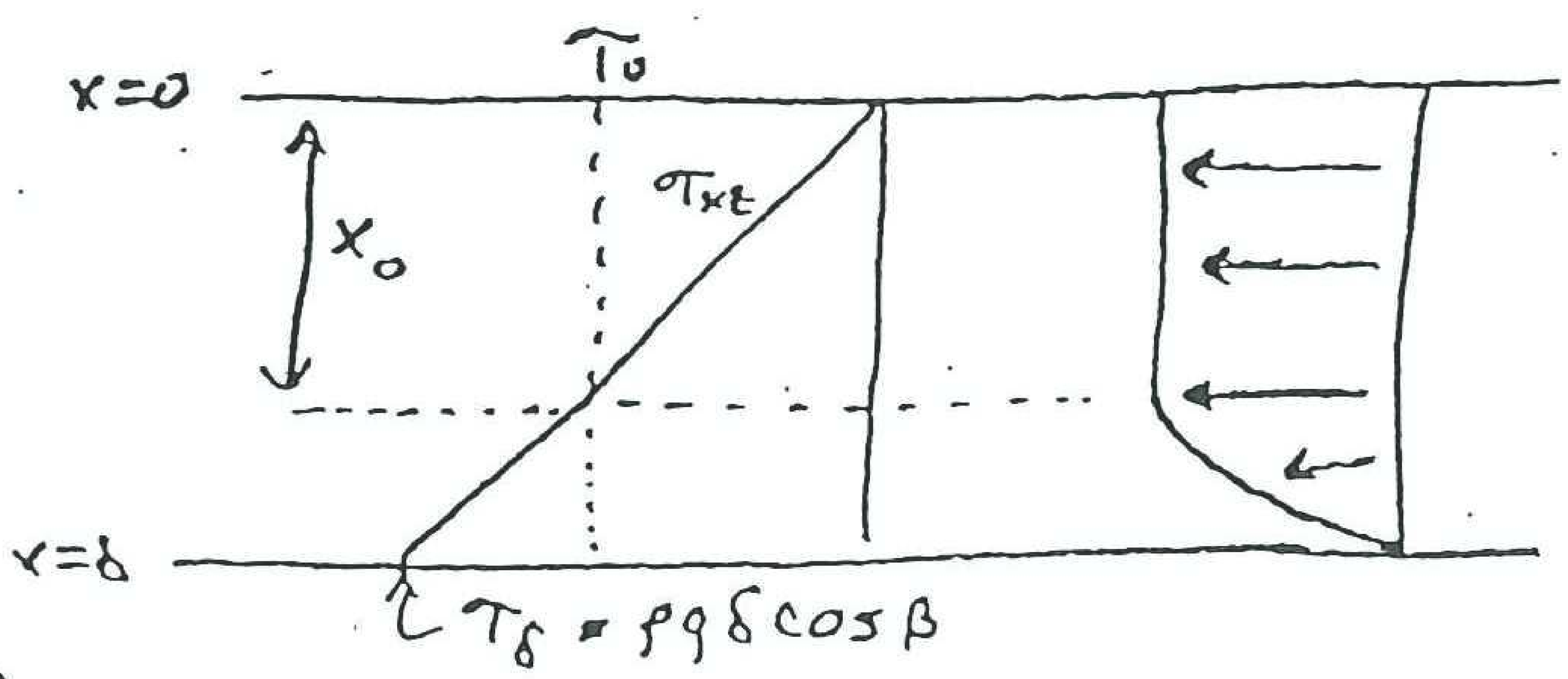
3. The momentum balance is the same as for a Newtonian fluid in the same geometry. The boundary condition $\tau_{xz} = 0$ at $x = 0$ also still applies, so Eq. 2.2-13 still holds:

$$\tau_{xz} = \rho g x\cos\beta \quad \boxed{V}$$

For a Bingham plastic, $\tau_{xz} = -\mu_0\frac{dv_z}{dx} + \tau_0$ $\boxed{VI}$ $\tau_{xz} \geq \tau_0$

$\frac{dv_z}{dx} = 0$ $0 \leq \tau_{xz} \leq \tau_0$

It is clear from (V) that $\tau_{xz} > 0$. Therefore one can draw a diagram similar to Fig 2.3-3:



For $x < x_0$, $\frac{dv_z}{dx} = 0$;

$v_z = \text{const} \equiv$ "v_z^c", $x < x_0$.

(cf. Eq. 2.3-26 (1st ed. BSL)).

x_0 satisfies $\rho g\cos\beta\, x_0 = \tau_0$; $x_0 = \tau_0/(\rho g\cos\beta)$