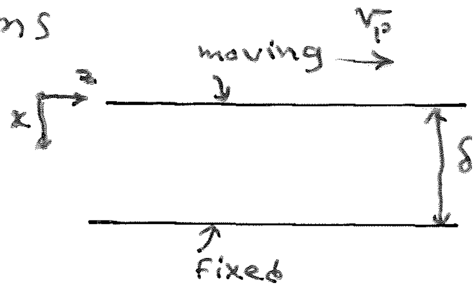
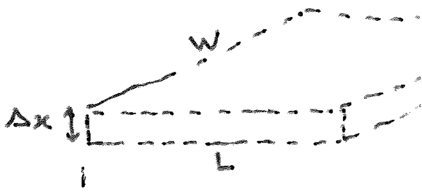
Shell Momentum Balance Problems



A)

2 large, horizontal, parallel plates

Newtonian fluid



(a) Do the momentum balance on C.V.

- geometry similar to § 2.2 in BSL

• rate of z-momentum in ■ accross a surface at x : $(LW)\,\tau_{xz}\Big|_{x}$

• rate of z-momentum out ■ accross a surface at $x+\Delta x$: $(LW)\,\tau_{xz}\Big|_{x+\Delta x}$

• rate of z-momentum in ■ accross surface at $z=0$: $W\Delta x\, v_z\, (\rho v_z)\Big|_{z=0}$

• rate of z-momentum out ■ accross surface at $z=L$: $W\Delta x\, v_z\, (\rho v_z)\Big|_{z=L}$

• No gravity force since plates are horizontal ($\cos\beta = 0$)

• No pressure term

⇒ Momentum Balance is:

$$LW\tau_{xz}\Big|_{x} - LW\tau_{xz}\Big|_{x+\Delta x} + \cancel{W\Delta x v_z \rho v_z} - \cancel{W\Delta x v_z \rho v_z} = 0 \qquad (1)$$

incompressible fluid ⇒ these two terms cancel each other.

b) $\Delta x \rightarrow 0$

Divide (1) by $(LW\Delta x)$: $\dfrac{\tau_{xz}\big|_x - \tau_{xz}\big|_{x+\Delta z}}{\Delta x} = 0$

$\lim\limits_{\Delta x \to 0}$: $-\dfrac{d\,\tau_{xz}}{dx} = 0 \quad \rightarrow \quad \tau_{xz} = C_1$ (2)

c) Insert Newton's Law of viscosity

Newton's law : $\tau_{xz} = -\mu \dfrac{dv_z}{dx}$ (3)

(3) in (2) : $-\mu \dfrac{dv_z}{dx} = C_1$ (4)

d) Solve (4) & evaluate C_1

integrate (4) : $v_z = \dfrac{-C_1}{\mu} x + C_2$ (7)

B.C. 1 : $v_z = v_p$ at $x = 0 \Rightarrow C_2 = v_p$ (5)

B.C. 2 : $v_z = 0$ at $x = \delta \Rightarrow C_1 = \dfrac{v_p \mu}{\delta}$ (6)

(5), (6) in (7) $\Rightarrow$

Velocity profile : $v_z = v_p \left(1 - \dfrac{x}{\delta}\right)$

C) Velocity profile (v_z) in a cylindrical tube of Newtonian fluid that "SLIPS" at the wall:

$\tau_{rz} = k\, v_z$ at $r = R$

Similar to equations in B.S.L. up through eq. 2.3-15:

- Momentum balance unchanged
- First BC at $r=0$ unchanged (τ_{rz} finite @ $r=0$)
- Newton's law of viscosity " .

B.S.L. 2.3-15: $v_z = -\left(\dfrac{P_0 - P_L}{4\mu L}\right) r^2 + C_2$ (1)

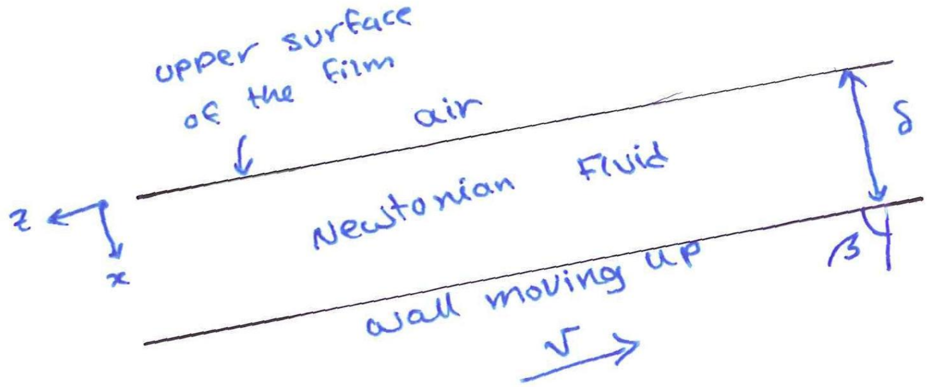
$$\text{B.C. 2:} \quad \underset{\text{Newton's Law}}{\tau_{rz}} = k\, \underset{(1)}{v_z} \quad \text{at} \quad r = R$$

$$\Rightarrow \left(\frac{P_0 - P_L}{2L}\right) R = k\left[-\left(\frac{P_0 - P_L}{4\mu L}\right) R^2 + C_2\right]$$

$$\Rightarrow C_2 = \frac{P_0 - P_L}{2Lk} R + \frac{P_0 - P_L}{4\mu L} R^2$$

$$\Rightarrow v_z = \frac{P_0 - P_L}{4\mu L} R^2 \left[1 - \left(\frac{r}{R}\right)^2\right] + \frac{P_0 - P_L}{2Lk} R$$

D)



* Note : $V < 0$ (opposite to positive z direction)

a) what is v_z ?

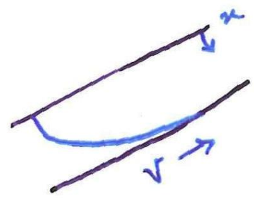
- Similar to equations in section 2.2 of B.S.L. up through Eq. 2.2.14 :

B.C.2: $v_z = v$ at $x = \delta$ (1)

(1) in Eq. 2.2.14 : $V = -\left(\frac{\rho g \cos\beta}{2\mu}\right)\delta^2 + C_2 \Rightarrow C_2 = V + \left(\frac{\rho g \cos\beta}{2\mu}\right)\delta^2$

$$\Rightarrow v_z = \left(\frac{\rho g \cos\beta}{2\mu}\right)\delta^2\left[1 - \left(\frac{x}{\delta}\right)^2\right] + v.$$

b) The general shape of v_z profile :



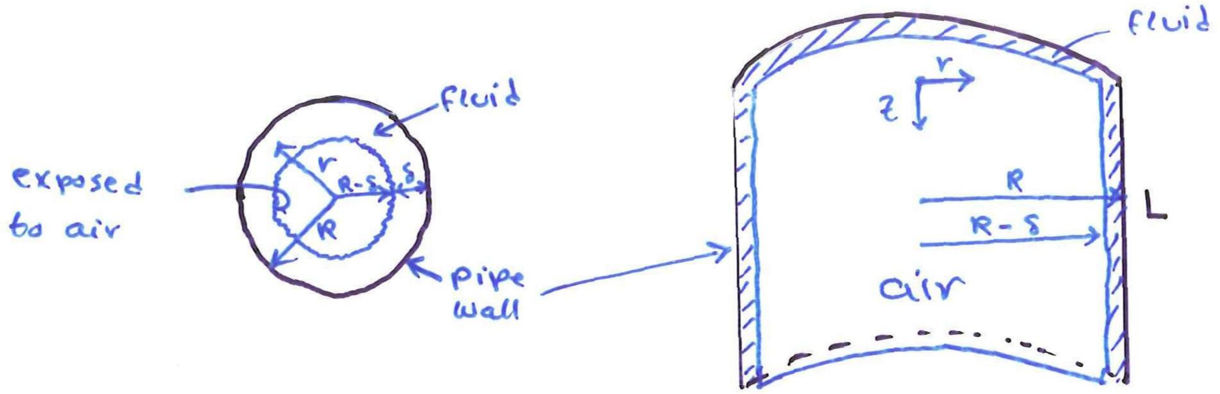
$v_z = V < 0$ at the wall ($x = \delta$)

All the flow is upward if $v_z \leq 0$ at $x = 0$, i.e. if

$v_z\big|_{x=0} = \left(\frac{\rho g \cos\beta}{2\mu}\right)\delta^2 (1 - 0) + v \leq 0$ or

$v \leq -\left(\frac{\rho g \cos\beta}{2\mu}\right)\delta^2$

F)



- Newtonian fluid
- flowing down under gravity force
- no pressure difference

what is v_z for $(R-\delta) \leq r \leq R$?

- Similar geometry like sec. 2.3 B.S.L.
- Different B.C. :

B.C.1) $\tau_{rz} = 0$ at $r = R - \delta$

B.C.2) $v_z = 0$ at $r = R$

BSL(2.3.11) : $\tau_{rz} = \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{2L}\right) r + \frac{C_1}{r}$ (1)

$$\mathcal{P}_0 - \mathcal{P}_L = (\cancel{p} - \rho g \overset{0}{\cancel{z}})\Big|_{z=0} - (\cancel{p} - \rho g \overset{L}{\cancel{z}})\Big|_{z=L} \quad \Rightarrow \mathcal{P}_0 - \mathcal{P}_L = \rho g L \quad (2)$$

no pressure difference $\Rightarrow P_0 = P_L$

(2) in (1) : $\tau_{rz} = \frac{\rho g}{2} r + \frac{C_1}{r}$

B.C.1 $\rightarrow C_1 = \frac{-\rho g}{2}(R-\delta)^2$

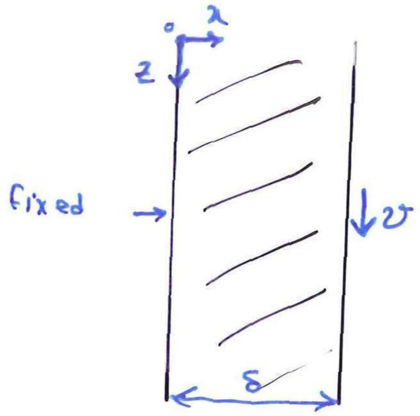
B.C.2 (First plug-in Newton's Law)

Newton's Law in (1) : $\frac{dv_z}{dr} = -\frac{\rho g}{2\mu}\left[\frac{r}{R-\delta} - \frac{R-\delta}{r}\right] \rightarrow$

$v_z = -\frac{\rho g}{2\mu}(R-\delta)\left[\frac{r^2}{2(R-\delta)} - (R-\delta)\ln r\right] + C_2$, B.C.2 $\Rightarrow$

$C_2 = \frac{\rho g}{2\mu}(R-\delta)\left[\frac{R^2}{2(R-\delta)} - (R-\delta)\ln R\right] \Rightarrow v_z = \frac{\rho g}{2\mu}(R-\delta)\left[\frac{R^2}{2(R-\delta)}\left(1-\left(\frac{r}{R}\right)^2\right) + (R-\delta)\ln(r/R)\right]$

G)



- Newtonian Fluid
- 2 parallel vertical plates
- $v > 0$ (in positive z direction)
- no pressure applied to flow

what is $v_z(x) = ?$

- Similar geometry to section 2.2. BSL ($\cos\beta = 1$)

BSL, Eq. 2.2-9: $\tau_{xz} = \rho g x + C_1$ (1)

(We can't use BSL any further, because B.C. at $x=0$ is different.

In this problem, both BC involve v_z, not τ_{xz} ⇒ We can't evaluate C_1 yet.)

Insert Newton's Law in (1): $\dfrac{dv_z}{dx} = -\dfrac{\rho g}{\mu} x - \dfrac{C_1}{\mu} \xrightarrow{\text{Integrate}}$

$$v_z = -\frac{\rho g}{2\mu} x^2 - \frac{C_1}{\mu} x + C_2$$

B.C. 1 : $v_z = 0$ at $x = 0$ ⇒ $C_2 = 0$

B.C. 2 : $v_z = v$ at $x = \delta$ ⇒ $C_1 = -\dfrac{v\mu}{\delta} - \dfrac{\rho g \delta}{2}$ ⇒

$$v_z = \left(\frac{\rho g \delta^2}{2\mu}\right)\left[\left(\frac{x}{\delta}\right) - \left(\frac{x}{\delta}\right)^2\right] + v\frac{x}{\delta}$$