ta 3220 Practice Problem Solutions: Shell balances, heat transfer

[redacted]

Shell Energy balance: no convection, no generation, no accumulation

Define control volume as spherical shell of radius r and thickness Δr.

Energy in by "molecular transport": $(q_r\, 4\pi r^2)|_r$

" out " $(q_r\, 4\pi r^2)|_{r+\Delta r}$

Energy balance: $(q_r\, 4\pi r^2)_r - (q_r\, 4\pi r^2)_{r+\Delta r} = 0$

Divide by $4\pi\Delta r$; let $\Delta r \to 0 \to \lim_{\Delta r\to 0} \frac{(r^2 q_r)|_r - (r^2 q_r)|_{r+\Delta r}}{\Delta r} = -\frac{d}{dr}(r^2 q_r) = 0$

Integrate $\to r^2 q_r = C_1$; since $(r^2 q_r)$ is constant, it must be the same everywhere as at the inner surface at r_0; i.e., $C_1 = r_0^2 q_0$; (q_0 yet to be determined).

Thus in each of the three layers of insulation $r^2(-k\frac{dT}{dr}) = r_0^2 q_0$; i.e.,

$-k^{01} r^2 \frac{dT^{01}}{dr} = r_0^2 q_0$ $\boxed{\mathrm{I}}$ in innermost layer.

$-k^{12} r^2 \frac{dT^{12}}{dr} = r_0^2 q_0$ in middle layer (cf Eqs 10.6-21 to 23)

$-k\, r^2 \frac{dT^{23}}{dr} = r_0^2 q_0$ in outer layer.

Integrating first eq($\boxed{\mathrm{I}}$), $\to \frac{dT^{01}}{dr} = -\frac{r_0^2 q_0}{k}\frac{1}{r^2} \to T = +\frac{r_0^2 q_0}{k}\frac{1}{r} + C_2$; evaluating at r_0 and r_1 gives $T_0 - T_1 = \frac{r_0^2 q_0}{k^{01}}\left(\frac{1}{r_0} - \frac{1}{r_1}\right)$ (i.e., C_2 drops out of this expression). Note that if $q_0 > 0$ (i.e., if heat flows outward in positive r direction), $T_0 > T_1$, as expected. (since $r_0 < r_1$, $1/r_0 > 1/r_1$). Similarly, for the other 2 layers

$$T_1 - T_2 = \frac{r_0^2 q_0}{k^{12}}\left(\frac{1}{r_1} - \frac{1}{r_2}\right), \quad T_2 - T_3 = \frac{r_0^2 q_0}{k^{23}}\left(\frac{1}{r_2} - \frac{1}{r_3}\right). \quad \text{(cf Eqs 10.6-24 to 26)}$$

At the inner surface the BC1 is $q_0 = h_0(T_a - T_0)$, which $\to T_a - T_0 = q_0/h_0$.

At the outer surface, $q_3 = h_3(T_3 - T_b)$. (Note heat flows out (in positive r direction) if $T_3 > T_b$) $\to T_3 - T_b = q_3/h_3$. But since everywhere in the system $r^2 q_r = r_0^2 q_r$, $q_3 = q_0 \frac{r_0^2}{r_3^2}$,

$\to T_3 - T_b = \frac{q_0}{h_3}\left(\frac{r_0^2}{r_3^2}\right)$. cf Eqs. 10.6-27, 28.

Adding $(T_a - T_0) + (T_0 - T_1) + (T_1 - T_2) + (T_2 - T_3) + (T_3 - T_b) = T_a - T_b$ gives

$$T_a - T_b = r_0^2 q_0\left[\frac{1}{r_0^2 h_0} + \frac{1}{k^{01}}\left(\frac{1}{r_0} - \frac{1}{r_1}\right) + \frac{1}{k^{12}}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{1}{k^{23}}\left(\frac{1}{r_2} - \frac{1}{r_3}\right) + \frac{1}{h_3 r_3^2}\right]$$

$$Q_0 = 4\pi r_0^2 q_0 = 4\pi(T_a - T_b)\left[\frac{1}{r_0^2 h_0} + \frac{r_0^{-1} - r_1^{-1}}{k^{01}} + \frac{r_1^{-1} - r_2^{-1}}{k^{12}} + \frac{r_2^{-1} - r_3^{-1}}{k^{23}} + \frac{1}{r_3^2 h_3}\right]^{-1} \quad \text{(cf Eq. 10.6-29)}$$

Q_0 is total heat transfer at inner surface. (Since system is at steady state, heat in at inner surface = heat out at outer surface.)

If we define "Heat Transfer coefficient based on innersurface" by

$$Q_0 = U_0(4\pi r_0^2)(T_a - T_b), \text{ then}$$

$$U_0 = \frac{1}{r_0^2}\left[\frac{1}{r_0^2 h_0} + \frac{r_0^{-1} - r_1^{-1}}{k^{01}} + \frac{r_1^{-1} - r_2^{-1}}{k^{12}} + \frac{r_2^{-1} - r_3^{-1}}{k^{23}} + \frac{1}{r_3^2 h_3}\right]^{-1}$$

$$= \frac{1}{r_0^2}\left[\frac{1}{r_0^2 h_0} + \sum_{i=1}^{3}\left(\frac{r_{i-1}^{-1} - r_i^{-1}}{k^{i-1,i}}\right) + \frac{1}{r_3^2 h_3}\right]^{-1}$$

The extension to an arbitrary number of layers is straight-forward.

(Note if Inner dia. = 6 in., radius = 3 in., $r_0 = 3(0.0254)$, $r_1 = 4(0.0254)$.)

2. a) The solution is given by Eqs. 10.6-30+31, modified for 1 layer and for the fact that the inside surface temperature is given, rather than an inner heat-transfer coefficient: $h_0 \to \infty$. What the problem statement calls "h_0" is the heat-transfer coefficient for the outer wall, h_1.

Eq. 10.6-31 → $U_o = \frac{1}{r_o}\left(\cancel{\frac{1}{\infty}}^{\,0} + \frac{\ln r_1/r_o}{k} + \frac{1}{r_1 h_1}\right)^{-1}$

$= \frac{1}{(3)(0.0254)}\left(\frac{\ln(4/3)}{16.3} + \frac{1}{(4)(0.0254)(100)}\right)^{-1}$

$= \frac{1}{0.0762}\left(0.0176 + 0.0984\right)^{-1} = 113 \ \ W/(m^2 K)$

(note that heat-transfer at wall, represented by this term, provides most of the resistance to heat transfer). (Eqs 13.1-8+9, w/ $h_o \to \infty$, give same result.)

Eq. 10.6-30: $Q_o = U_o(2\pi r_o L)(T_o - T_b) = (113)(2\pi\, 3(0.0254)\, 100)[(300-70)/1.8]$

$= (113)(47.9)(127.8) = 6.92 \cdot 10^5 \ W$

b) Eq 10.6-31 → $U_o = \frac{1}{r_o}\left(\cancel{\frac{1}{\infty}}^{\,0} + \frac{\ln r_1/r_o}{k_1} + \frac{\ln r_2/r_1}{k_2} + \frac{1}{r_2 h_2}\right)^{-1}$

$U_o = \frac{1}{(3)(0.0254)}\left(\frac{\ln(4/3)}{16.3} + \frac{\ln(5/4)}{0.02} + \frac{1}{(5)(0.0254)(100)}\right)^{-1} = \frac{1}{0.0762}\left(0.0176 + 11.16 + 0.0787\right)^{-1}$

$= 1.166 \ W/(m^2 K)$ (note that now insolation provides most resistance)

$Q_o = (1.166)(47.9)(127.8) = 7140 \ W$ (reduction by a factor of 97!)

The geometry is the same as in Example 10.6-1 (BSL p. 305 ff.), but the boundary conditions differ and the outer layer has also a generation term.

Inner layer: Energy balance: $q_r^{01}|_r\, 2\pi r L - q_r^{01}|_{r+\Delta r}\, 2\pi(r+\Delta r)L = 0$ (cf Eq. 10.6-1?)

Divide by $2\pi L\Delta r$; let $\Delta r \to 0$; → $-\frac{d}{dr}(r q_r^{01}) = 0$ (cf. 10.6-19)

Integrating → $r q_r^{01} = C_1$; since $r q_r^{01}$ in inner layer is constant, it equals its value at $r = r_o$: $r q_r^{01} = r_o q_o$ $\boxed{\text{I}}$ (q_o is not yet known). Fourier's law →

$-k^{01} r \frac{dT^{01}}{dr} = r_o q_o$; integrating → $T = -(r_o q_o/k^{01})\ln r + C_2$

From $T = T_o$ at $r = r_o$ → $T_o = -(r_o q_o/k^{01})\ln r_o + C_2 \to C_2 = T_o + (r_o q_o/k^{01})\ln r_o$

→ $T - T_o = (r_o q_o/k^{01})\ln(r_o/r)$ $\boxed{\text{III}}$ [q_o is still unspecified.]

Outer layer: Energy balance: $q_r^{12}|_r\, 2\pi r L - q_r^{12}|_{r+\Delta r}\, 2\pi(r+\Delta r)L + S_e\, 2\pi r \Delta r L = 0$

→ $\frac{d}{dr}(r q^{12}) = S_e r \to r q^{12} = \frac{S_e r^2}{2} + C_1$

B.C.: $q = 0$ at $r = r_2$ → $C_1 = -S_e r_2^2/2 \to r q^{12} = \frac{S_e}{2}(r^2 - r_2^2)$ $\boxed{\text{II}}$

Now, since $q^{01} = q^{12}$ at $r = r_1$, $\boxed{\text{I}}$ → $r_1 q_r^{01} = r_o q_o \to q_r^{01} = q_o(r_o/r_1)$

Eq $\boxed{\text{II}}$ → $r_1 q^{12} = r_1 q^{01} = r_1 q_o(r_o/r_1) = \frac{S_e}{2}(r_1^2 - r_2^2) = q_o r_o$

Eq $\boxed{\text{III}}$ → $T - T_o = \frac{S_e(r_1^2 - r_2^2)}{2k^{01}}\ln(r_o/r) = \frac{S_e(r_2^2 - r_1^2)}{2k^{01}}\ln(r/r_o)$

Note that problem statement asks only for T(r) in inner layer. You do not need to complete solution to T(r) in outer layer.

b) $Q/L = -2\pi r_o q_o = -2\pi \frac{S_e}{2}(r_1^2 - r_2^2) = -S_e \pi(r_1^2 - r_2^2) = S_e\pi(r_2^2 - r_1^2)$. (This value is algebraically positive; the negative sign is required because q_r is negative, because heat flows inward, in the minus r direction.)

3. Consider a cylindrical shell of thickness Δr, and height L.

Conduction heat into shell: $2\pi r L\, q_r|_r$

" out of shell: $2\pi r L\, q_r|_{r+\Delta r}$ — v_r here is physical velocity here, not Darcy velocity. *

Convection into shell : $2\pi r L \phi \rho \hat{C}_p T v_r|_r$

" out of shell : $2\pi r L \phi \rho \hat{C}_p T v_r|_{r+\Delta r}$

} note v_r here is a negative number because flow is in.

No generation, no accumulation at steady state. Energy balance:

$$2\pi L(r q_r|_r - r q_r|_{r+\Delta r}) + 2\pi L \phi \rho \hat{C}_p (r T v_r|_r - r T v_r|_{r+\Delta r}) = 0$$

For an incompressible fluid in steady flow, $-2\pi r v_r \phi = q_0$ † (assuming $q_0 > 0$);

$v_r = -\frac{q_0}{2\pi r}$ for all r. Thus

$$2\pi\,(r q_r|_r - r q_r|_{r+\Delta r}) - \rho \hat{C}_p q_0 (T|_r - T|_{r+\Delta r}) = 0$$

Divide by $2\pi\ \Delta r$; let $\Delta r \Rightarrow 0$.

$$-\frac{d}{dr}(r q_r) + \frac{\rho \hat{C}_p q_0}{2\pi}\frac{dT}{dr} = 0$$

Insert Fourier's law: $q_r = -k\frac{dT}{dr}$

$$\rightarrow \frac{d}{dr}\left(r\frac{dT}{dr}\right) + K\frac{dT}{dr} = 0$$

with: $K \equiv \frac{\rho \hat{C}_p}{k}\frac{q_0}{2\pi}$

* or, could assume v_r is Darcy velocity; then ϕ never appears here or at "†"