Dynamics and Stability AE3-914

Sample problem—Week 1

Foucault's pendulum

Statement

Consider a Foucault's pendulum with mass m and length l oscillating at latitude φ on Earth. The mass is passing the vertical position with speed v at the considered instant, as seen by an observer on Earth's surface on the pendulum site. Calculate the fictitious forces applied to the mass at the considered instant, as observed from the pendulum site.

Reference frames

The pendulum is referred to the local xyz-axes with the basis vectors $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and the velocity vector forms an arbitrary angle ψ with the x-axis at the instant considered. An inertial reference system is considered with origin in Earth's centre and the basis vectors $\{\mathbf{I}, \mathbf{J}, \mathbf{K}\}$. The xyz-system would be placed on a point described by the angles θ and φ, where the latitude is recognised in the latter angle.

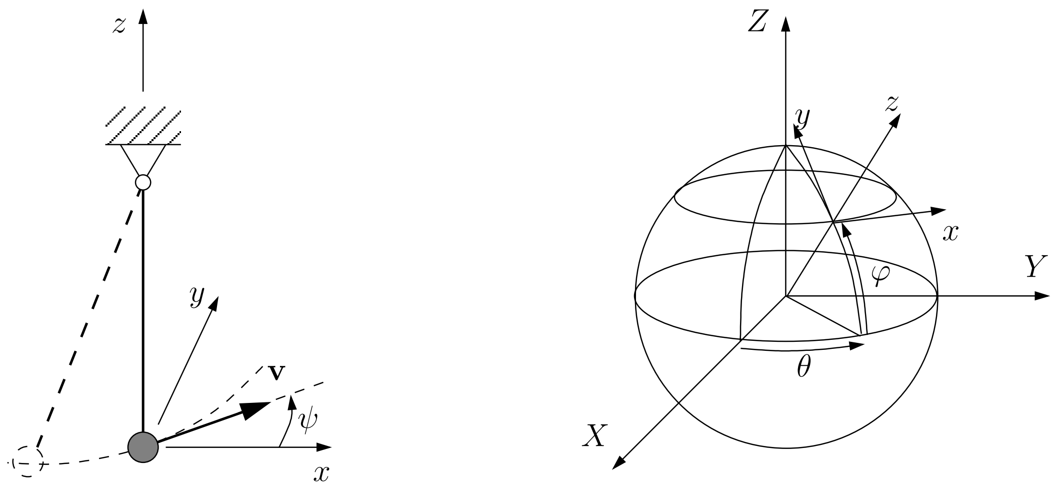

Figure 1: Representation of the non-inertial (xyz) and the inertial (XYZ) reference frames

Kinematics of the non-inertial reference frame

The xyz-system is attached to Earth and therefore it is rotating with the same constant angular velocity,

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{K}; \qquad \dot{\boldsymbol{\omega}} = \mathbf{0}. \tag{1}$$

Moreover, the origin of the xyz-frame experiences a centripetal acceleration

$$\mathbf{a}_{xyz} = \dot{\theta}^2 R \cos\varphi \,(-\cos\theta\,\mathbf{I} - \sin\theta\,\mathbf{J}), \tag{2}$$

where R is the radius of Earth and, consequently, $R\cos\varphi$ is the radius of the local parallel (see Figure 2).

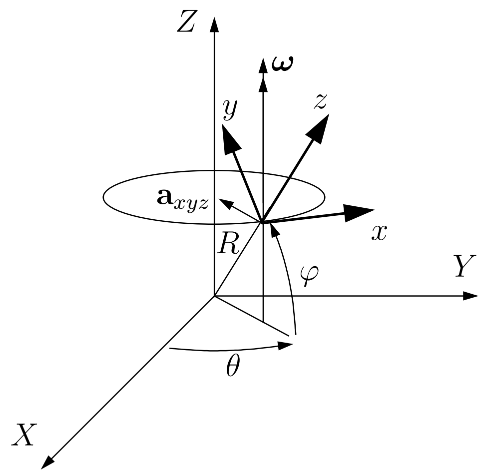

Figure 2: Kinematics of the non-inertial reference frame

Fictitious forces in the non-inertial reference frame

From the expression for the absolute acceleration it was derived that the fictitious forces read

$$\mathbf{F}_{fict} = -m\big(\mathbf{a}_{xyz} + \dot{\boldsymbol{\omega}} \times \mathbf{r}_{rel} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{rel}) + 2\boldsymbol{\omega} \times \mathbf{v}_{rel}\big). \tag{3}$$

In our case we have the situation that (see Figure 1)

$$\mathbf{r}_{rel} = \mathbf{0}; \qquad \mathbf{v}_{rel} = v(\cos\psi\,\mathbf{i} + \sin\psi\,\mathbf{j}), \tag{4}$$

while $\boldsymbol{\omega}$, $\dot{\boldsymbol{\omega}}$ and $\mathbf{a}_{xyz}$ are given by equations (1) and (2). However, everything should be expressed in the same coordinate system before elaborating equation (3). This is not inconsistent with the fact that one system is inertial and the other system is non-inertial. The time derivatives were taken when deriving the absolute velocities and accelerations. Once the vectors involved have been identified, the motion is considered to be frozen and all vectors are expressed in the non-inertial reference frame in order to elaborate (3). In particular one gets

$$\boldsymbol{\omega} = \dot{\theta}(\cos\varphi\,\mathbf{j} + \sin\varphi\,\mathbf{k}); \qquad \mathbf{a}_{xyz} = \dot{\theta}^2 R\cos\varphi\,(\sin\varphi\,\mathbf{j} - \cos\varphi\,\mathbf{k}), \tag{5}$$

as it can be easily deduced from Figure 2. Substitution in (3) provides

$$\begin{aligned}\mathbf{F}_{fict} \;&= -m\big(\dot{\theta}^2 R\cos\varphi\,(\sin\varphi\,\mathbf{j} - \cos\varphi\,\mathbf{k}) \\ &\qquad + 2\dot{\theta}v(\cos\varphi\,\mathbf{j} + \sin\varphi\,\mathbf{k}) \times (\cos\psi\,\mathbf{i} + \sin\psi\,\mathbf{j})\big).\end{aligned} \tag{6}$$

The cross product is evaluated as

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \cos\varphi & \sin\varphi \\ \cos\psi & \sin\psi & 0 \end{vmatrix} = -\sin\varphi\,\sin\psi\,\mathbf{i} + \sin\varphi\,\cos\psi\,\mathbf{j} - \cos\varphi\,\cos\psi\,\mathbf{k} \tag{7}$$

and terms can be rearranged as

$$\begin{aligned}\mathbf{F}_{fict} = m\big(&2\dot{\theta}v\sin\varphi\,\sin\psi && \mathbf{i} \\ &-(2\dot{\theta}v\sin\varphi\,\cos\psi && +\dot{\theta}^2 R\cos\varphi\sin\varphi)\mathbf{j} \\ &+(2\dot{\theta}v\cos\varphi\,\cos\psi && +\dot{\theta}^2 R\cos^2\varphi)\mathbf{k}\big).\end{aligned} \tag{8}$$

The left column in this equation represents the Coriolis force which is always orthogonal to the relative velocity vector, as can be observed from the contribution of ψ to the components in **i** and **j**. This Coriolis force always points to the right (with respect to the sense of motion) in the northern hemisphere, as can be deduced from the contribution of φ. The right column represents the centrifugal force due to the rotation of Earth.

Considering the actual values for Earth one gets

$$\begin{aligned}\dot{\theta} \quad &= \frac{2\pi}{24\cdot 3600\cdot \frac{365}{366}} = 7,292129\cdot 10^{-5}\,\mathrm{s}^{-1} \\ \dot{\theta}^2 \quad &= 5,317514\cdot 10^{-9}\,\mathrm{s}^{-2} \\ R \quad &= 6,371\cdot 10^{6}\,\mathrm{m} \\ \dot{\theta}^2 R \quad &= 3,387788\cdot 10^{-2}\,\mathrm{m/s}^2\end{aligned} \tag{9}$$

If the speed v is large enough the term due to the centrifugal forces can be neglected, especially for large values of the latitude φ.