Dynamics and Stability AE3-914

Sample problem—Week 6

Functional with several basic variables and higher-order derivatives

Statement

Find the Euler-Lagrange equation and the natural boundary conditions for the variational problem

$$I(u(x,t)) = \int_{t_a}^{t_b}\int_{x_a}^{x_b} F(x,t,u,u_t,u_{xx})\,dxdt \tag{1}$$

with the essential boundary conditions

$$u(x,t_a) = f_a(x); \qquad u(x,t_b) = f_b(x), \tag{2}$$

where $u_t = \dfrac{\partial u}{\partial t}$ and $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$.

Variation of the functional

An extremal of the functional is found when the variation vanishes,

$$\delta I = \int_{t_a}^{t_b}\int_{x_a}^{x_b} \delta F(x,t,u,u_t,u_{xx})\,dxdt = 0. \tag{3}$$

The variation δI is elaborated as

$$\delta I = \int_{t_a}^{t_b}\int_{x_a}^{x_b} \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_t}\delta u_t + \frac{\partial F}{\partial u_{xx}}\delta u_{xx}\right) dxdt. \tag{4}$$

In order to get the variation of I in terms of that of u only, and not those of u_t and u_{xx}, integration by parts is carried out, paying special attention to the actual integration variable,

$$\begin{aligned}\delta I = &\int_{t_a}^{t_b}\int_{x_a}^{x_b} \frac{\partial F}{\partial u}\delta u\,dxdt \\ &+ \int_{x_a}^{x_b}\left(\frac{\partial F}{\partial u_t}\delta u\bigg|_{t_a}^{t_b} - \int_{t_a}^{t_b}\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial u_t}\right)\delta u\,dt\right)dx \\ &+ \int_{t_a}^{t_b}\left(\frac{\partial F}{\partial u_{xx}}\delta u_x\bigg|_{x_a}^{x_b} - \int_{x_a}^{x_b}\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_{xx}}\right)\delta u_x\,dx\right)dt,\end{aligned} \tag{5}$$

in which the last integral is further developed to get

$$\begin{aligned}\delta I = &\int_{t_a}^{t_b}\int_{x_a}^{x_b}\frac{\partial F}{\partial u}\delta u\,dxdt\\&+\int_{x_a}^{x_b}\left(\frac{\partial F}{\partial u_t}\delta u\bigg|_{t_a}^{t_b}-\int_{t_a}^{t_b}\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial u_t}\right)\delta u\,dt\right)dx\\&+\int_{t_a}^{t_b}\left[\frac{\partial F}{\partial u_{xx}}\delta u_x\bigg|_{x_a}^{x_b}-\left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_{xx}}\right)\delta u\bigg|_{x_a}^{x_b}-\int_{x_a}^{x_b}\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial u_{xx}}\right)\delta u\,dx\right)\right]dt.\end{aligned} \tag{6}$$

Regrouping terms one gets

$$\begin{aligned}\delta I = &\int_{t_a}^{t_b}\int_{x_a}^{x_b}\left[\frac{\partial F}{\partial u}-\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial u_t}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial u_{xx}}\right)\right]\delta u\,dxdt\\&+\int_{x_a}^{x_b}\frac{\partial F}{\partial u_t}\delta u\bigg|_{t_a}^{t_b}dx\\&+\int_{t_a}^{t_b}\frac{\partial F}{\partial u_{xx}}\delta u_x\bigg|_{x_a}^{x_b}dt\\&-\int_{t_a}^{t_b}\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_{xx}}\right)\delta u\bigg|_{x_a}^{x_b}dt,\end{aligned} \tag{7}$$

which should vanish to provide an extremal to the variational problem.

Euler-Lagrange equation

For the variation (7) to vanish, i.e. $\delta I = 0$, each of the involved integrals must vanish separately. For the double integral one has

$$\int_{t_a}^{t_b}\int_{x_a}^{x_b}\left[\frac{\partial F}{\partial u}-\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial u_t}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial u_{xx}}\right)\right]\delta u\,dxdt = 0. \tag{8}$$

Since condition (8) must be fulfilled for any variation δu, the fundamental lemma of the calculus of variations provides

$$\frac{\partial F}{\partial u}-\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial u_t}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial u_{xx}}\right) = 0, \tag{9}$$

which is the Euler-Lagrange equation for this variational problem

Boundary conditions

The three boundary integrals in equation (7) must vanish to ensure that $\delta I = 0$. The boundary integral with respect to the variable x,

$$\int_{x_a}^{x_b}\frac{\partial F}{\partial u_t}\delta u\bigg|_{t_a}^{t_b}dx \tag{10}$$

is developed as

$$\int_{x_a}^{x_b} \left(\left.\frac{\partial F}{\partial u_t}\right|_{t=t_b} \delta u(x,t_b) - \left.\frac{\partial F}{\partial u_t}\right|_{t=t_a} \delta u(x,t_a) \right) dx. \tag{11}$$

From the boundary conditions (2)

$$u(x,t_a) = f_a(x); \qquad u(x,t_b) = f_b(x) \tag{12}$$

one has

$$\delta u(x,t_a) = \delta f_a(x) = 0; \qquad \delta u(x,t_b) = \delta f_b(x) = 0, \tag{13}$$

because f_a and f_b are prescribed functions and can, therefore, not experience any variation. Equations (11–13) together lead to the immediate conclusion

$$\int_{x_a}^{x_b} \frac{\partial F}{\partial u_t} \delta u \Big|_{t_a}^{t_b} dx = 0. \tag{14}$$

The essential boundary conditions (2) thus ensure that the contribution of this integral to the variation vanishes.

The first boundary integral in (7) with respect to the variable t,

$$\int_{t_a}^{t_b} \frac{\partial F}{\partial u_{xx}} \delta u_x \Big|_{x_a}^{x_b} dt, \tag{15}$$

is developed as

$$\int_{t_a}^{t_b} \left(\left.\frac{\partial F}{\partial u_{xx}}\right|_{x=x_b} \delta u_x(x_b,t) - \left.\frac{\partial F}{\partial u_{xx}}\right|_{x=x_a} \delta u_x(x_a,t) \right) dt. \tag{16}$$

The boundary conditions (2) are not providing any information on the variations

$$\delta u_x(x_b,t) \quad \text{and} \quad \delta u_x(x_a,t) \tag{17}$$

now. The only possibility for integral (16) to identically vanish is

$$\left.\frac{\partial F}{\partial u_{xx}}\right|_{x=x_b} = 0 \quad \text{and} \quad \left.\frac{\partial F}{\partial u_{xx}}\right|_{x=x_a} = 0, \tag{18}$$

which is a set of *natural* boundary conditions. Imposing these to the solution of (9) ensures that (16) vanishes and thus

$$\int_{t_a}^{t_b} \frac{\partial F}{\partial u_{xx}} \delta u_x \Big|_{x_a}^{x_b} dt = 0. \tag{19}$$

The same procedure is carried out for the second boundary integral in (7) with respect to the variable t,

$$\int_{t_a}^{t_b} \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_{xx}} \right) \delta u \Big|_{x_a}^{x_b} dt, \tag{20}$$

leading to

$$\int_{t_a}^{t_b} \left[\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_{xx}} \right) \bigg|_{x=x_b} \delta u(x_b, t) - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_{xx}} \right) \bigg|_{x=x_a} \delta u(x_a, t) \right] dt. \tag{21}$$

As in (16), the boundary conditions (2) do not provide any information on the variations

$$\delta u(x_b, t) \quad \text{and} \quad \delta u(x_a, t). \tag{22}$$

Indeed, boundary conditions (2) are not imposing restrictions to the evolution of the solution $u(x_a, t)$ and $u(x_b, t)$ with respect to variable t. The value of u is prescribed for $t = t_a$ and $t = t_b$, but not along the integration domain in t. The only possibility for (21) to identically vanish is

$$\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_{xx}} \right) \bigg|_{x=x_b} = 0 \quad \text{and} \quad \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_{xx}} \right) \bigg|_{x=x_a} = 0, \tag{23}$$

which is one more set of natural boundary conditions. Imposing these to the solution of (9) ensures that (21) vanishes and thus

$$\int_{t_a}^{t_b} \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_{xx}} \right) \delta u \bigg|_{x_a}^{x_b} dt = 0. \tag{24}$$

Summarising, the essential boundary conditions (2) together with the natural boundary conditions (18) and (23) ensure that the boundary integrals (14), (19) and (24) vanish. Together with the Euler-Lagrange equation (9), it is ensured that the variation δI expressed in equation (7) vanishes for the solution of the variational problem.

Conclusion

The Euler-Lagrange equation for the variational problem (1) is

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial t} \left(\frac{\partial F}{\partial u_t} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial F}{\partial u_{xx}} \right) = 0, \tag{25}$$

with the essential boundary conditions (2)

$$u(x, t_a) = f_a(x); \qquad u(x, t_b) = f_b(x), \tag{26}$$

and the natural boundary conditions (18) and (23)

$$\frac{\partial F}{\partial u_{xx}} \bigg|_{x=x_a} = \frac{\partial F}{\partial u_{xx}} \bigg|_{x=x_b} = \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_{xx}} \right) \bigg|_{x=x_a} = \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_{xx}} \right) \bigg|_{x=x_b} = 0 \tag{27}$$