Course: WB3413, Dredging Processes 1

Fundamental Theory Required for Sand, Clay and Rock Cutting

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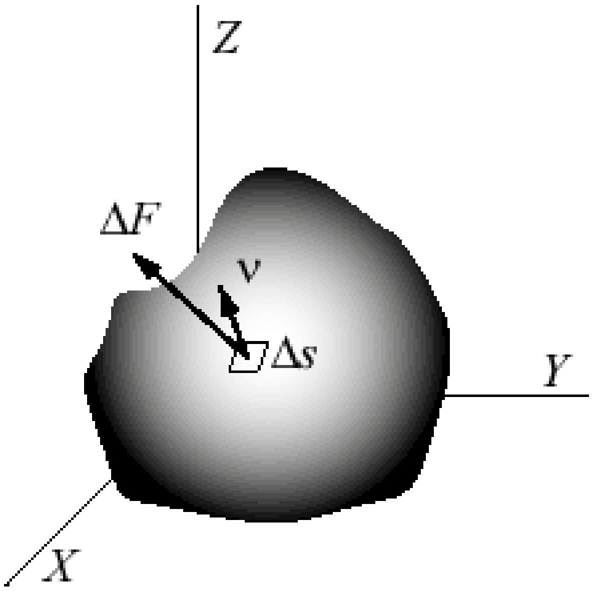
Mechanics of Materials: Stress

The Definition of Stress

The concept of **stress** originated from the study of strength and failure of solids. The stress field is the distribution of internal "tractions" that balance a given set of external tractions and body forces.



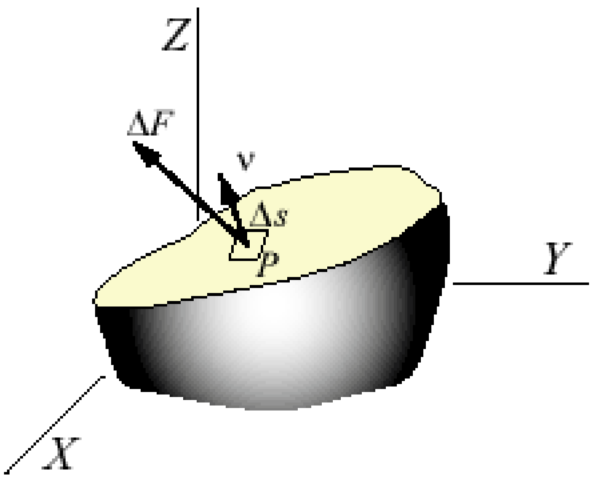
First, we look at the external traction **T** that represents the force per unit area acting at a given location on the body's surface. Traction **T** is a *bound vector*, which means **T** cannot slide along its line of action or translate to another location and keep the same meaning.

In other words, a traction vector cannot be fully described unless both the force and the surface where the force acts on has been specified. Given both ΔF and Δs, the traction **T** can be defined as

$$\mathbf{T} = \lim_{\Delta s \to 0} \frac{\Delta \mathbf{F}}{\Delta s} = \frac{d\mathbf{F}}{ds}$$

The internal traction within a solid, or stress, can be defined in a similar manner. Suppose an arbitrary slice is made across the solid shown in the above figure, leading to the free body diagram shown at right. Surface tractions would appear on the exposed surface, similar in form to the external tractions applied to the body's exterior surface. The stress at point *P* can be defined using the same equation as was used for **T**.

Stress therefore can be interpreted as internal tractions that act on a defined internal datum plane. One cannot measure the stress without first specifying the datum plane.

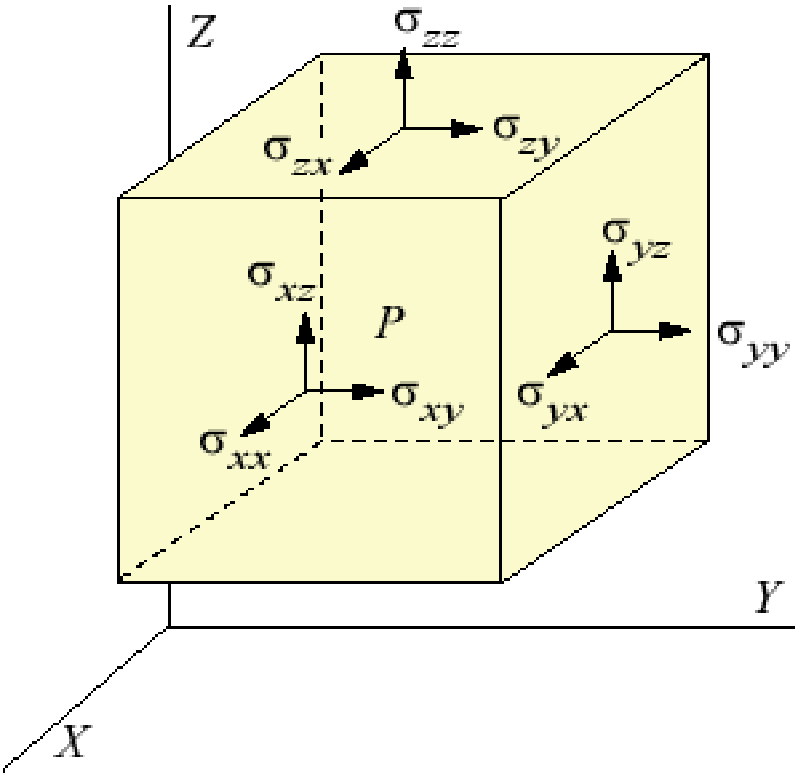


The Stress Tensor (or Stress Matrix)

Surface tractions, or stresses acting on an internal datum plane, are typically decomposed into three mutually orthogonal components. One component is normal to the surface and represents *direct stress*. The other two components are tangential to the surface and represent *shear stresses*.

What is the distinction between normal and tangential tractions, or equivalently, direct and shear stresses? **Direct stresses** tend to change the volume of the material (e.g. hydrostatic pressure) and are resisted by the body's bulk modulus (which depends on the Young's modulus and Poisson ratio). **Shear stresses** tend to deform the material without changing its volume, and are resisted by the body's shear modulus.

Defining a set of internal datum planes aligned with a Cartesian coordinate system allows the stress state at an internal point *P* to be described relative to *x*, *y*, and *z* coordinate directions.



For example, the stress state at point *P* can be represented by an *infinitesimal* cube with three stress components on each of its six sides (one direct and two shear components).

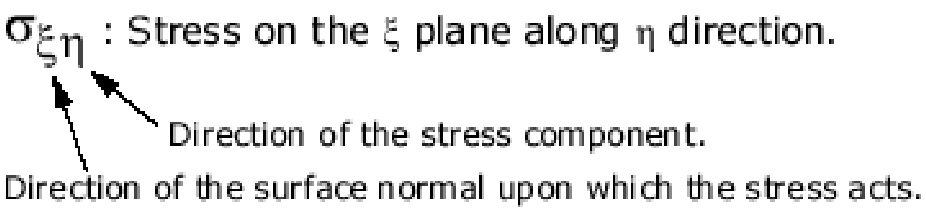
Since each point in the body is under static equilibrium (no net force in the absense of any body forces), only nine stress components from three planes are needed to describe the stress state at a point *P*.

These nine components can be organized into the matrix:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

where shear stresses across the diagonal are identical (i.e. $\sigma_{xy} = \sigma_{yx}$, $\sigma_{yz} = \sigma_{zy}$, and $\sigma_{zx} = \sigma_{xz}$) as a result of static equilibrium (no net moment). This grouping of the nine stress components is known as the **stress tensor** (or stress matrix).

The subscript notation used for the nine stress components have the following meaning:



Note: The stress state is a second order tensor since it is a quantity associated with two directions. As a result, stress components have 2 subscripts.
A surface traction is a first order tensor (i.e. vector) since it a quantity associated with only one direction. Vector components therefore require only 1 subscript.
Mass would be an example of a zero-order tensor (i.e. scalars), which have no relationships with directions (and no subscripts).

Equations of Equilibrium

Consider the static equilibrium of a solid subjected to the body force vector field **b**. Applying Newton's first law of motion results in the following set of differential equations which govern the stress distribution within the solid,

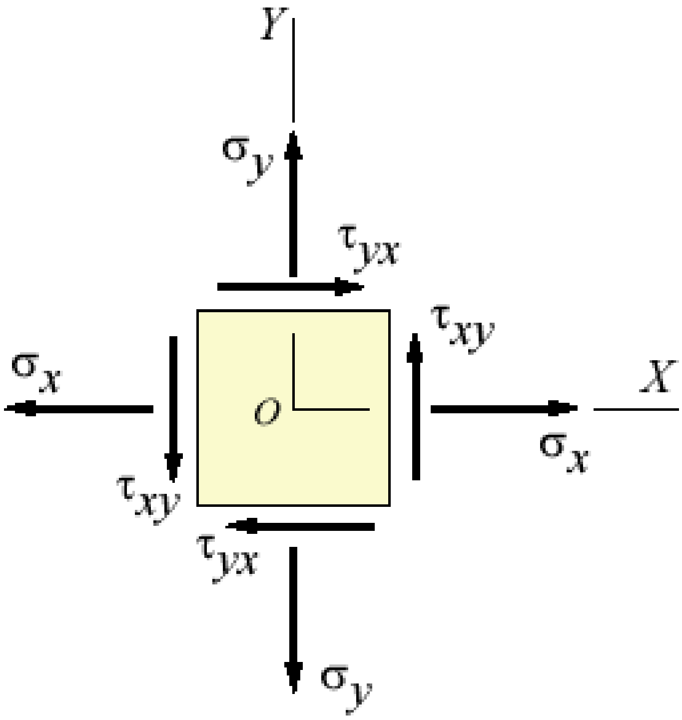
$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{yx}}{\partial y} + \dfrac{\partial \sigma_{zx}}{\partial z} + b_x = 0 \\ \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \sigma_{zy}}{\partial z} + b_y = 0 \\ \dfrac{\partial \sigma_{xz}}{\partial x} + \dfrac{\partial \sigma_{yz}}{\partial y} + \dfrac{\partial \sigma_{zz}}{\partial z} + b_z = 0 \end{cases}$$

In the case of two dimensional stress, the above equations reduce to,

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} + b_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + b_y = 0 \end{cases}$$

Plane Stress and Coordinate Transformations

Plane State of Stress



A class of common engineering problems involving stresses in a thin plate or on the free surface of a structural element, such as the surfaces of thin-walled pressure vessels under external or internal pressure, the free surfaces of shafts in torsion and beams under transverse load, have one principal stress that is much smaller than the other two. By assuming that this small principal stress is zero, the three-dimensional stress state can be reduced to two dimensions. Since the remaining two principal stresses lie in a plane, these simplified 2D problems are called **plane stress** problems.

Assume that the negligible principal stress is oriented in the *z*-direction. To reduce the 3D stress matrix to the 2D plane stress matrix, remove all components with *z* subscripts to get,

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$$

where $\tau_{xy} = \tau_{yx}$ for static equilibrium. The sign convention for positive stress components in plane stress is illustrated in the above figure on the 2D element.

Coordinate Transformations

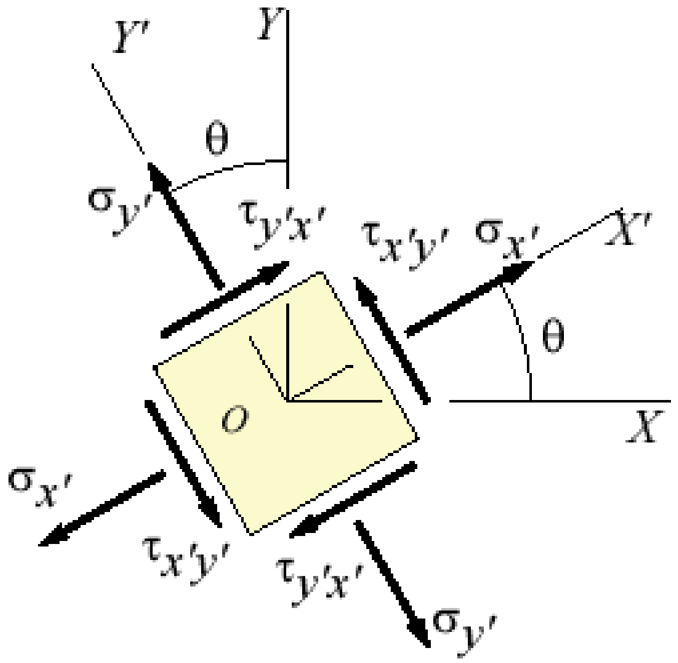
The coordinate directions chosen to analyze a structure are usually based on the shape of the structure. As a result, the direct and shear stress components are associated with these directions. For example, to analyze a bar one almost always directs one of the coordinate directions along the bar's axis.

Nonetheless, stresses in directions that do not line up with the original coordinate set are also important. For example, the failure plane of a brittle shaft under torsion is often at a 45° angle with respect to the shaft's axis. Stress transformation formulas are required to analyze these stresses.

The transformation of stresses with respect to the $\{x,y,z\}$ coordinates to the stresses with respect to $\{x',y',z'\}$ is performed via the equations,

$$\begin{cases} \sigma_{x'} = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \sigma_{y'} = \dfrac{\sigma_x + \sigma_y}{2} - \dfrac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \\ \qquad = \sigma_x + \sigma_y - \sigma_{x'} \\ \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \end{cases}$$

where θ is the rotation angle between the two coordinate sets (positive in the counterclockwise direction). This angle along with the stresses for the {*x'*,*y'*,*z'*} coordinates are shown in the figure below,



Principal Stress for the Case of Plane Stress

Principal Directions, Principal Stress

The normal stresses ($\sigma_{x'}$ and $\sigma_{y'}$) and the shear stress ($\tau_{x'y'}$) vary smoothly with respect to the rotation angle θ, in accordance with the coordinate transformation equations. There exist a couple of particular angles where the stresses take on special values.

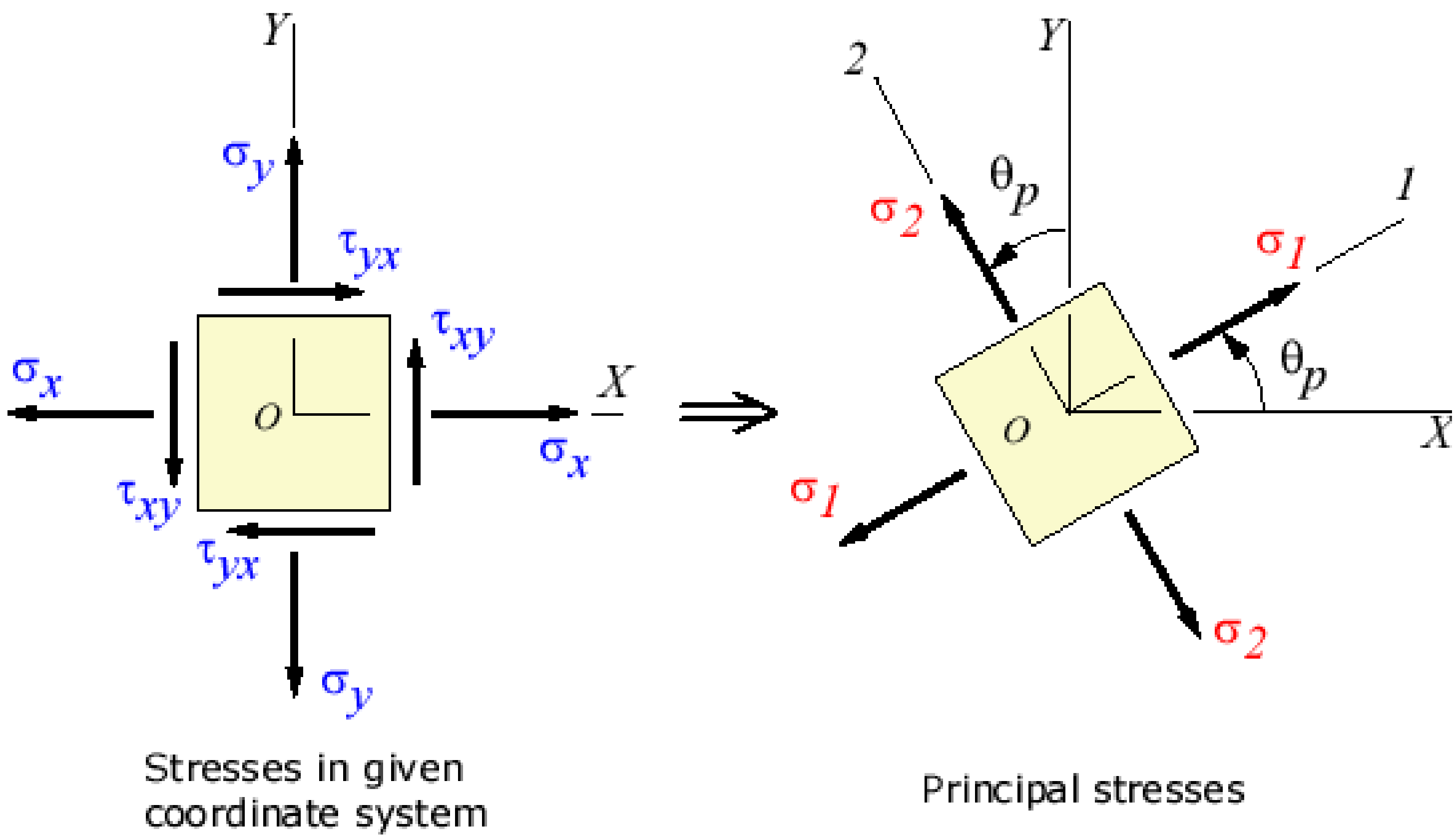
First, there exists an angle θ_p where the shear stress $\tau_{x'y'}$ becomes zero. That angle is found by setting $\tau_{x'y'}$ to zero in the above shear transformation equation and solving for θ (set equal to θ_p). The result is,

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

The angle θ_p defines the *principal directions* where the only stresses are normal stresses. These stresses are called *principal stresses* and are found from the original stresses (expressed in the *x,y,z* directions) via,

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

The transformation to the principal directions can be illustrated as:



Maximum Shear Stress Direction

Another important angle, θ_s, is where the maximum shear stress occurs. This is found by finding the maximum of the shear stress transformation equation, and solving for θ. The result is,

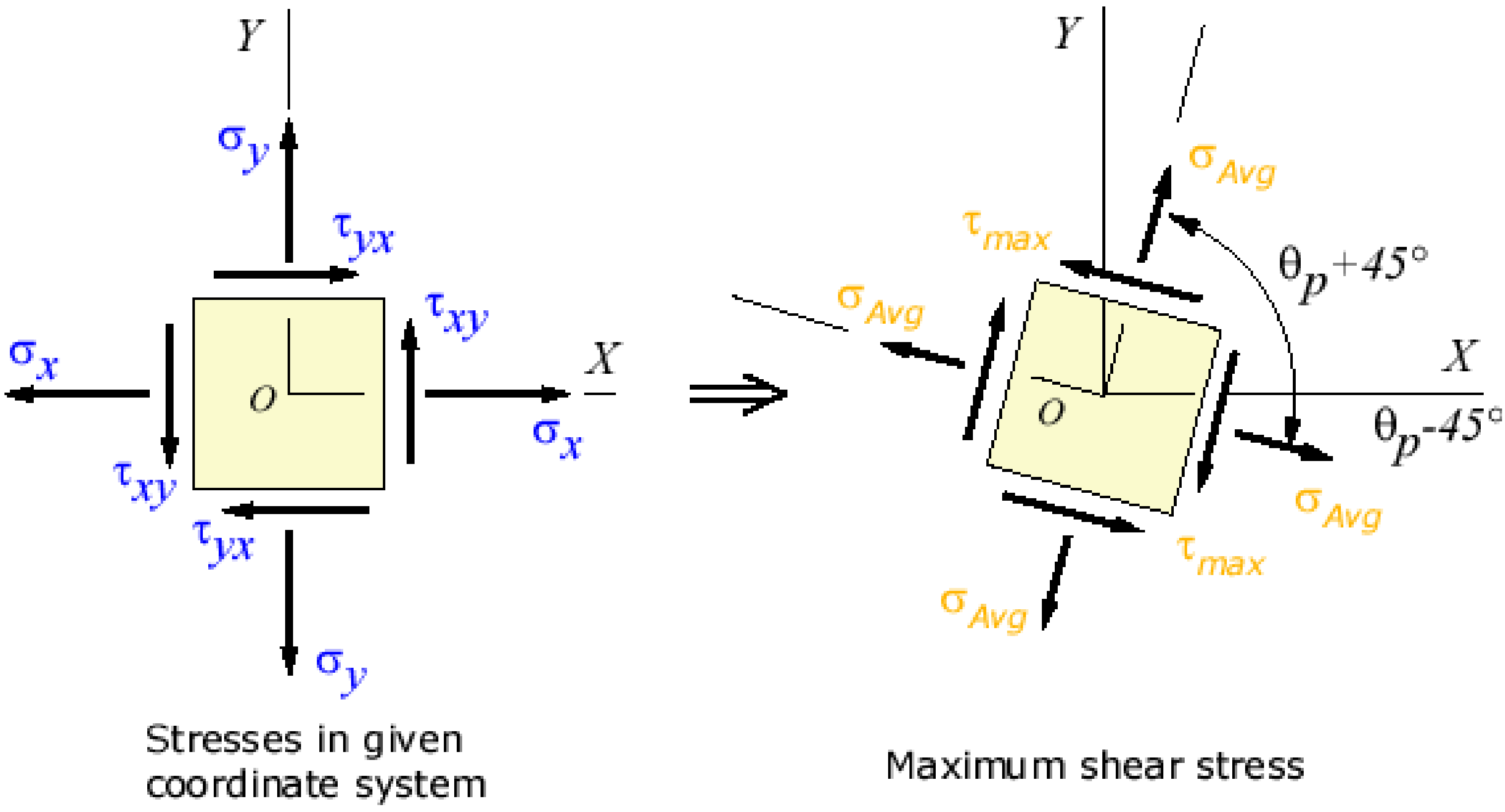
$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

$$\Rightarrow \theta_s = \theta_p \pm 45^\circ$$

The maximum shear stress is equal to one-half the difference between the two principal stresses,

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} = \frac{\sigma_1 - \sigma_2}{2}$$

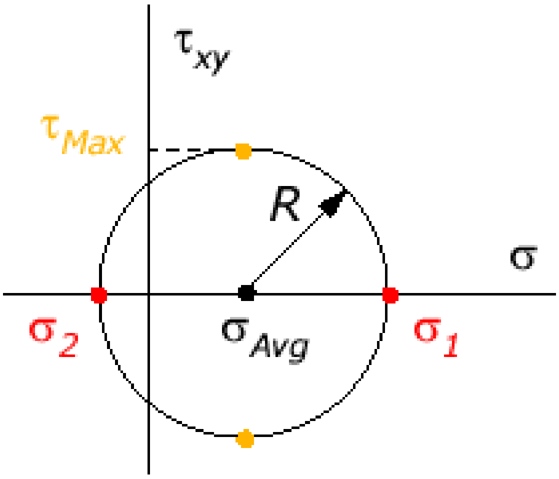
The transformation to the maximum shear stress direction can be illustrated as:



Mohr's Circle for Plane Stress

Mohr's Circle

Introduced by Otto Mohr in 1882, Mohr's Circle illustrates principal stresses and stress transformations via a graphical format,



The two principal stresses are shown in **red**, and the maximum shear stress is shown in **orange**. Recall that the normal stesses equal the principal stresses when the stress element is aligned with the principal directions, and the shear stress equals the maximum shear stress when the stress element is rotated 45° away from the principal directions.

As the stress element is rotated away from the principal (or maximum shear) directions, the normal and shear stress components will always lie on Mohr's Circle.

Mohr's Circle was the leading tool used to visualize relationships between normal and shear stresses, and to estimate the maximum stresses, before hand-held calculators became popular. Even today, Mohr's Circle is still widely used by engineers all over the world.

Derivation of Mohr's Circle

To establish Mohr's Circle, we first recall the stress transformation formulas for plane stress at a given location,

$$\begin{cases} \sigma_{x'} - \dfrac{\sigma_x + \sigma_y}{2} = \dfrac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \\ \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \end{cases}$$

Using a basic trigonometric relation ($\cos^2 2\theta + \sin^2 2\theta = 1$) to combine the two above equations we have,

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}{}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2$$

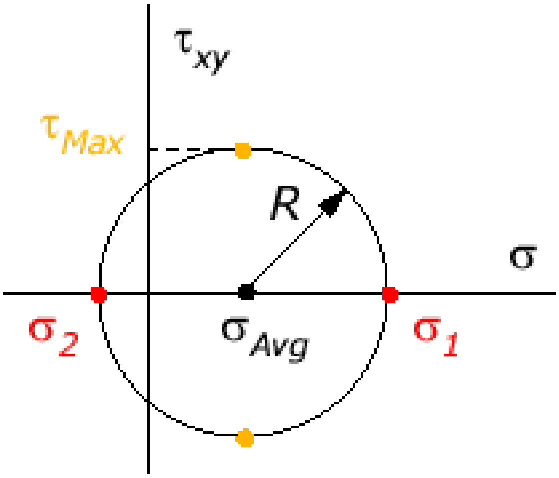
This is the equation of a circle, plotted on a graph where the abscissa is the normal stress and the ordinate is the shear stress. This is easier to see if we interpret σ_x and σ_y as being the two principal stresses, and τ_{xy} as being the maximum shear stress. Then we can define the average stress, σ_{avg}, and a "radius" R (which is just equal to the maximum shear stress),

$$\sigma_{\text{Avg}} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

The circle equation above now takes on a more familiar form,

$$\left(\sigma_{x'} - \sigma_{\text{Avg}}\right)^2 + \tau_{x'y'}{}^2 = R^2$$

The circle is centered at the average stress value, and has a radius R equal to the maximum shear stress, as shown in the figure below,



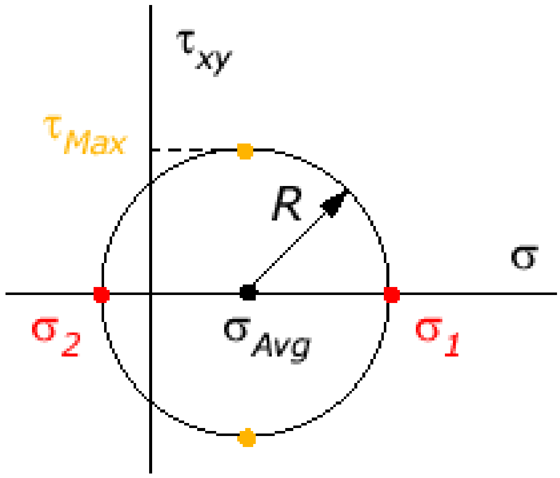
Related Topics

The procedure of drawing a Mohr's Circle from a given stress state is discussed in the Mohr's Circle usage page.

The Mohr's Circle for plane strain can also be obtained from similar procedures.

Mohr's Circle Usage in Plane Stress

Principal Stresses from Mohr's Circle



A chief benefit of Mohr's circle is that the principal stresses σ_1 and σ_2 and the maximum shear stress τ_{max} are obtained immediately after drawing the circle,

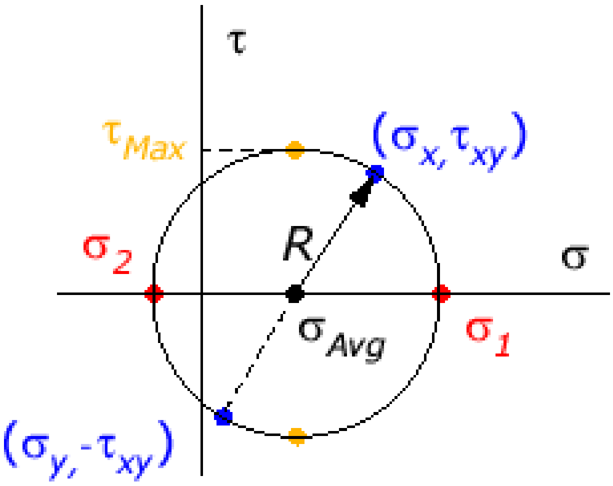
$$\begin{cases} \sigma_{1,2} = \sigma_{\text{Avg}} \pm R \\ \tau_{\text{Max}} = R \end{cases}$$

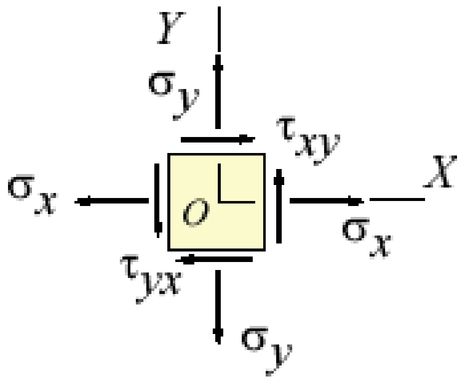
where,

$$\sigma_{\text{Avg}} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

Principal Directions from Mohr's Circle

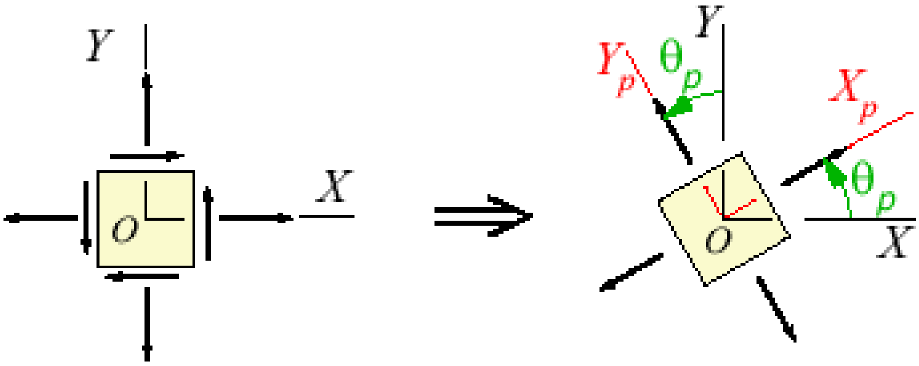
Mohr's Circle can be used to find the directions of the principal axes. To show this, first suppose that the normal and shear stresses, σ_x, σ_y, and τ_{xy}, are obtained at a given point *O* in the body. They are expressed relative to the coordinates *XY*, as shown in the stress element at right below.

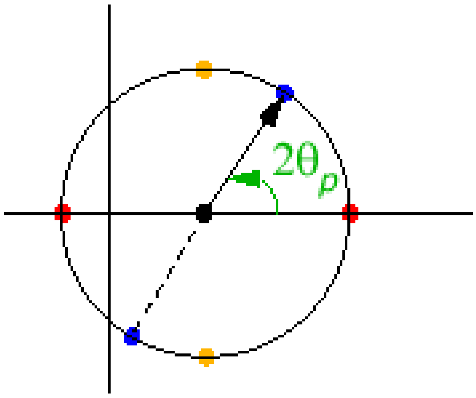




The Mohr's Circle for this general stress state is shown at left above. Note that it's centered at σ_{avg} and has a radius *R*, and that the two points $\{\sigma_x, \tau_{xy}\}$ and $\{\sigma_y, -\tau_{xy}\}$ lie on opposites sides of the circle. The line connecting σ_x and σ_y will be defined as L_{xy}.

The **angle** between the current axes (*X* and *Y*) and the **principal axes** is defined as θ_p, and is equal to one half the angle between the line L_{xy} and the σ-axis as shown in the schematic below,





A set of six Mohr's Circles representing most stress state possibilities are presented on the examples page.

Also, principal directions can be computed by the principal stress calculator.

Rotation Angle on Mohr's Circle

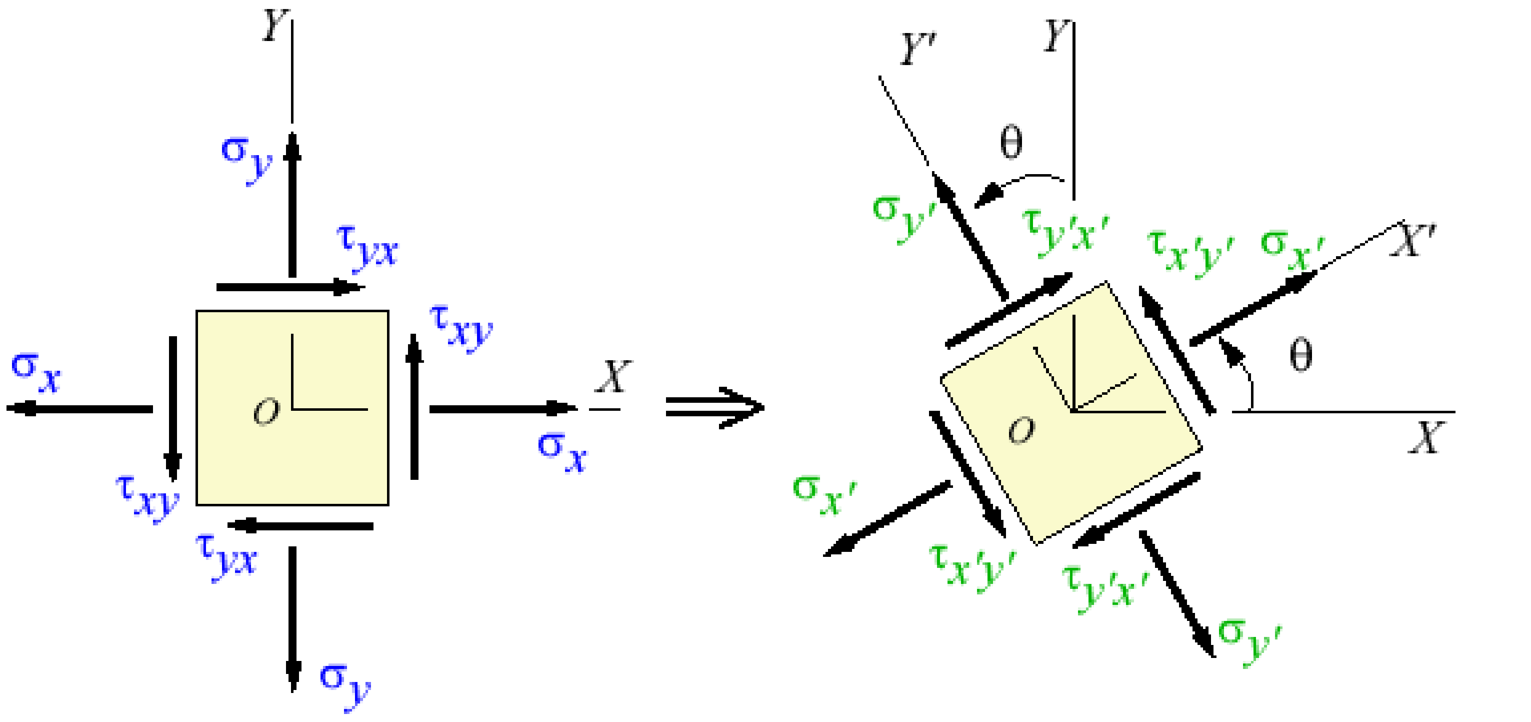
Note that the coordinate rotation angle θ_p is defined positive when starting at the *XY* coordinates and proceeding to the X_pY_p coordinates. In contrast, on the Mohr's Circle θ_p is defined positive starting on the principal stress line (i.e. the σ-axis) and proceeding to the *XY* stress line (i.e. line L_{xy}). The angle θ_p has the opposite sense between the two figures, because on one it starts on the *XY* coordinates, and on the other it starts on the principal coordinates.

Some books avoid this dichotomy between θ_p on Mohr's Circle and θ_p on the stress element by locating (σ_x, $-\tau_{xy}$) instead of (σ_x, τ_{xy}) on Mohr's Circle. This will switch the polarity of θ_p on the circle. Whatever method you choose, the bottom line is that an *opposite* sign is needed either in the interpretation or in the plotting to make Mohr's Circle physically meaningful.

Stress Transform by Mohr's Circle

Mohr's Circle can be used to transform stresses from one coordinate set to another, similar that that described on the plane stress page.

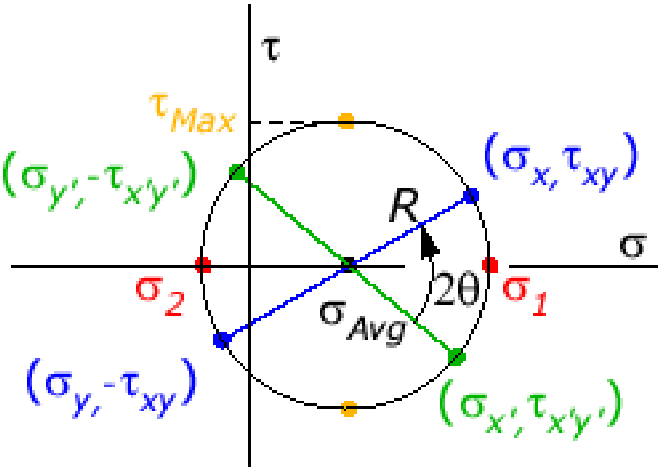
Suppose that the normal and shear stresses, σ_x, σ_y, and τ_{xy}, are obtained at a point *O* in the body, expressed with respect to the coordinates *XY*. We wish to find the stresses expressed in the new coordinate set *X'Y'*, rotated an angle θ from *XY*, as shown below:



Stresses at given coordinate system   Stresses transformed to another coordinate

To do this we proceed as follows:

- Draw Mohr's circle for the **given stress state** (σ_x, σ_y, and τ_{xy}; shown below).
- Draw the line L_{xy} across the circle from (σ_x, τ_{xy}) to (σ_y, $-\tau_{xy}$).
- Rotate the line L_{xy} by 2*θ (twice as much as the angle between *XY* and *X'Y'*) and in the *opposite* direction of θ.
- The **stresses in the new coordinates** ($\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$) are then read off the circle.

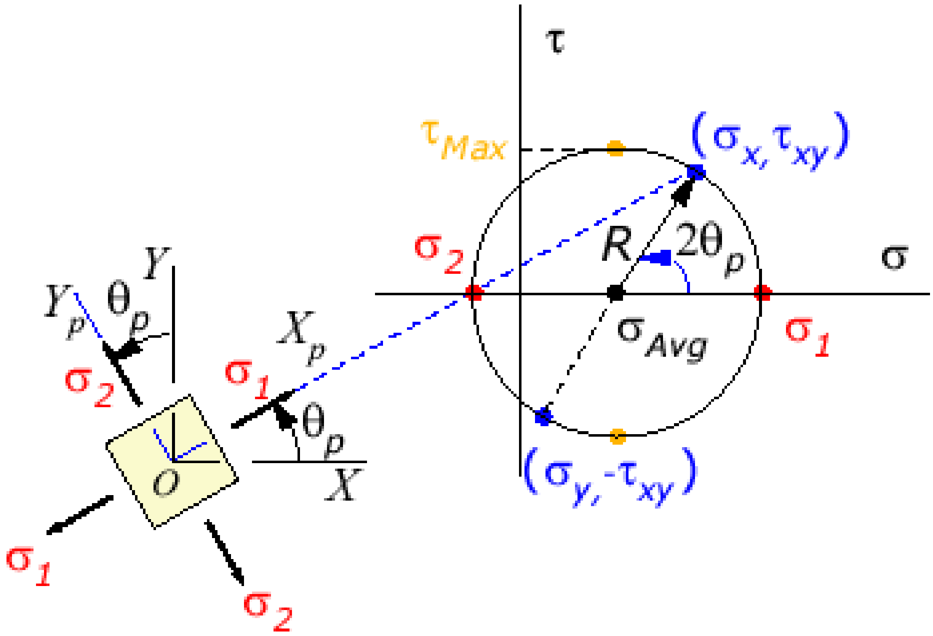


Stress transforms can be performed using eFunda's stress transform calculator.

Examples of Mohr's Circles in Plane Stress

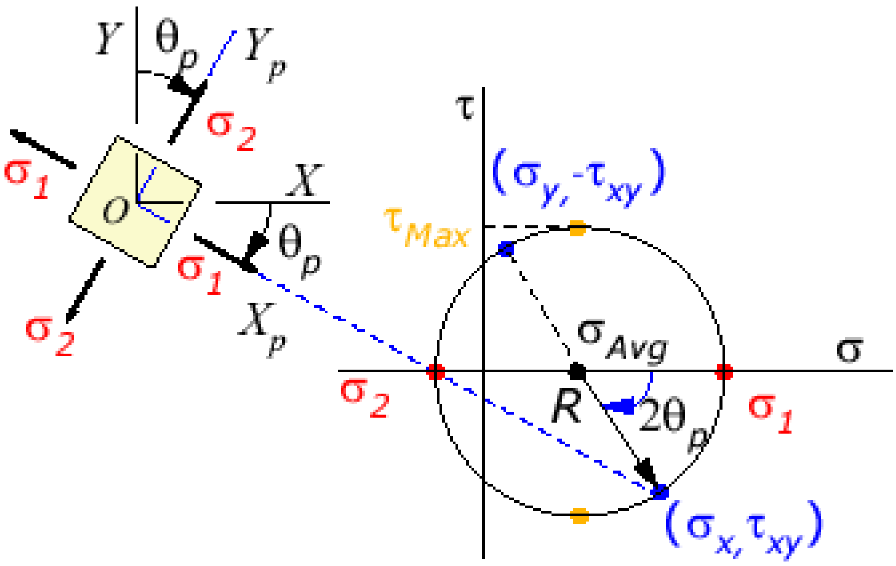
Case 1: $\tau_{xy} > 0$ and $\sigma_x > \sigma_y$

The principal axes are counterclockwise to the current axes (because $\tau_{xy} > 0$) and no more than 45° away (because $\sigma_x > \sigma_y$).



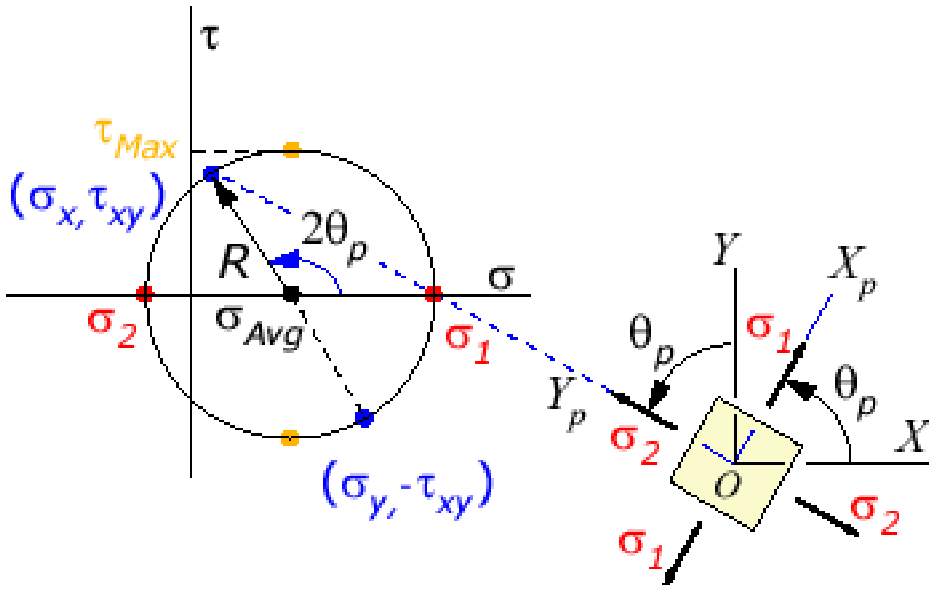
Case 2: $\tau_{xy} < 0$ and $\sigma_x > \sigma_y$

The principal axes are clockwise to the current axes (because $\tau_{xy} < 0$) and no more than 45° away (because $\sigma_x > \sigma_y$).



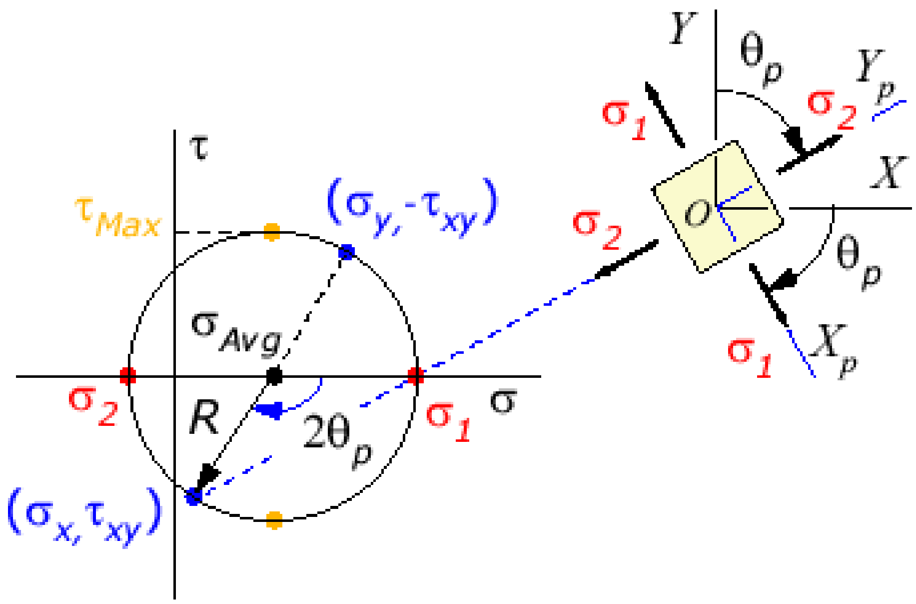
Case 3: $\tau_{xy} > 0$ and $\sigma_x < \sigma_y$

The principal axes are counterclockwise to the current axes (because $\tau_{xy} > 0$) and between 45° and 90° away (because $\sigma_x < \sigma_y$).



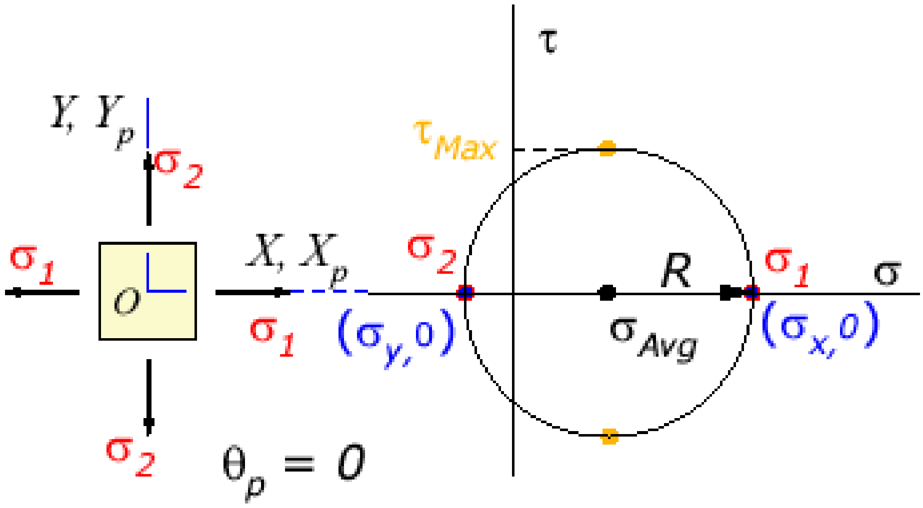
Case 4: $\tau_{xy} < 0$ and $\sigma_x < \sigma_y$

The principal axes are clockwise to the current axes (because $\tau_{xy} < 0$) and between 45° and 90° away (because $\sigma_x < \sigma_y$).



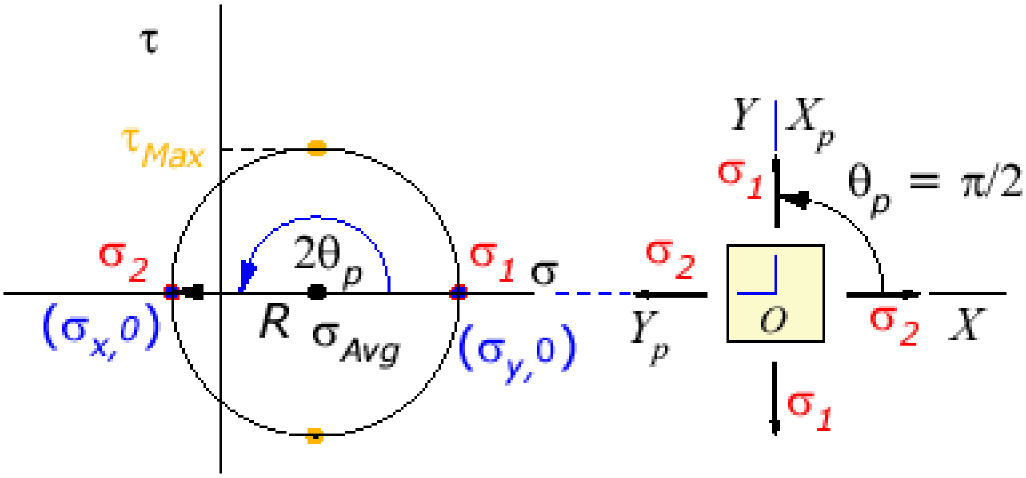
Case 5: $\tau_{xy} = 0$ and $\sigma_x > \sigma_y$

The principal axes are aligned with the current axes (because $\sigma_x > \sigma_y$ and $\tau_{xy} = 0$).



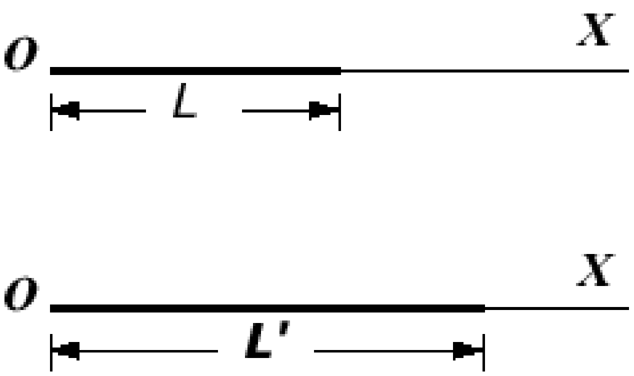
Case 6: $\tau_{xy} = 0$ **and** $\sigma_x < \sigma_y$

The principal axes are exactly 90° from the current axes (because $\sigma_x < \sigma_y$ and $\tau_{xy} = 0$).



Mechanics of Materials: Strain

Global 1D Strain

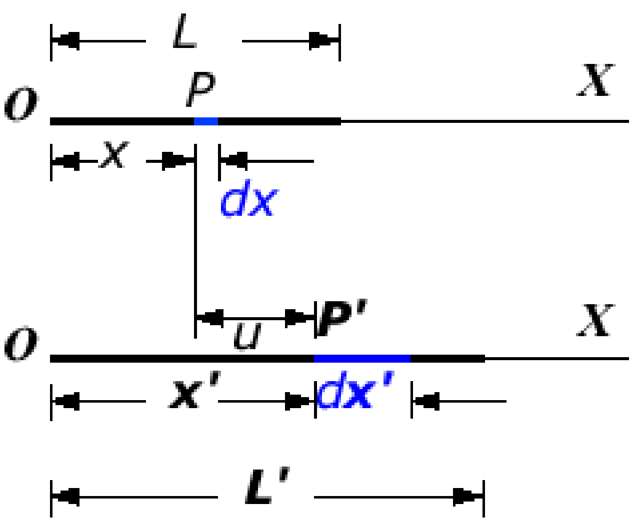


Consider a rod with initial length *L* which is stretched to a length ***L'***. The strain measure ε, a dimensionless ratio, is defined as the ratio of elongation with respect to the original length,

$$\varepsilon = \frac{L' - L}{L}$$

Infinitesimal 1D Strain

The above strain measure is defined in a global sense. The strain at each point may vary dramatically if the bar's elastic modulus or cross-sectional area changes. To track down the strain at each point, further refinement in the definition is needed.



Consider an arbitrary point in the bar *P*, which has a position vector *x*, and its infinitesimal neighbor *dx*. Point *P* shifts to ***P'***, which has a position vector ***x'***, after the stretch. In the meantime, the small "step" *dx* is stretched to ***dx'***.

The strain at point *p* can be defined the same as in the global strain measure,

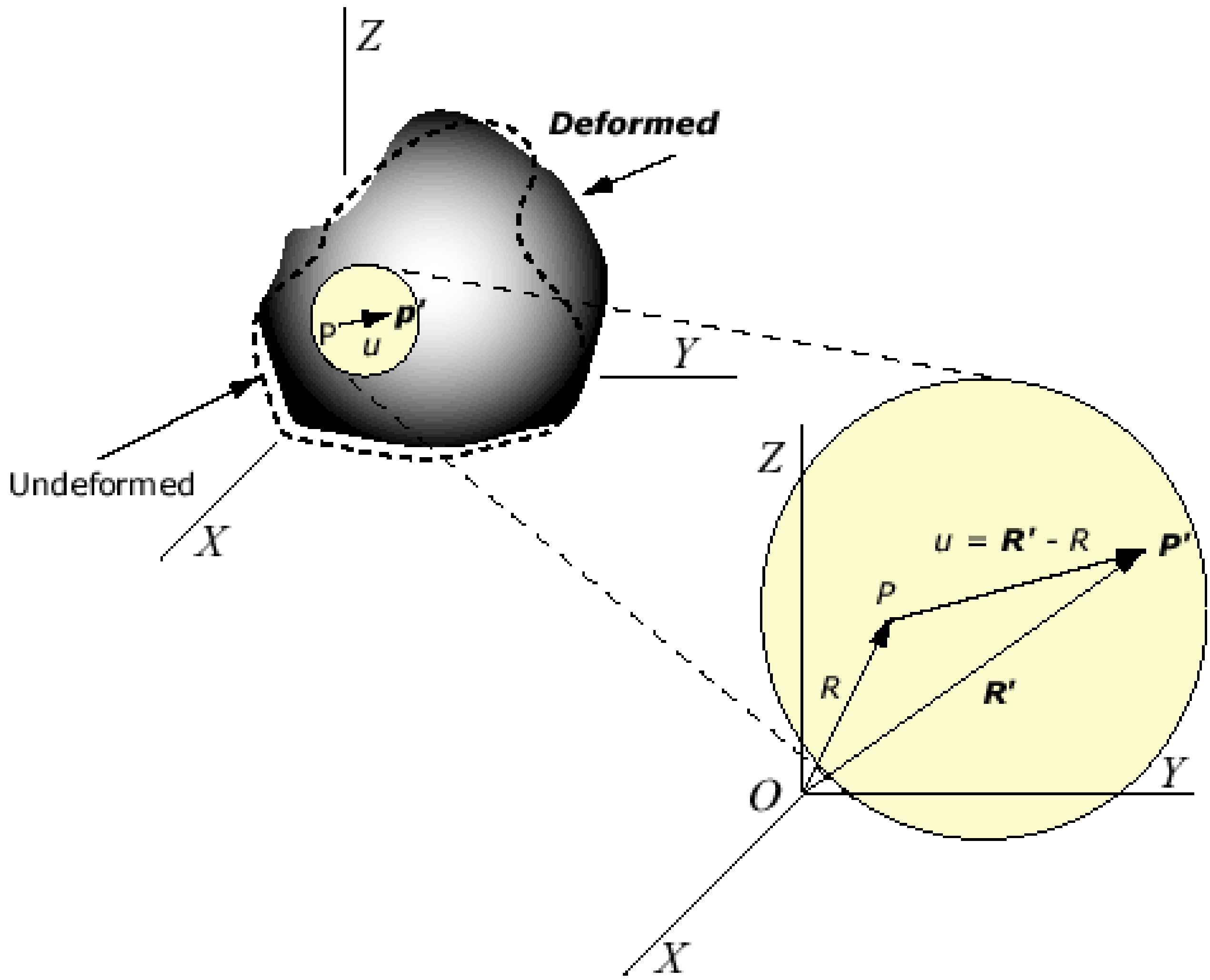
$$\varepsilon = \frac{dx' - dx}{dx}$$

Since the displacement $u = x' - x$, the strain can hence be rewritten as,

$$\varepsilon = \frac{dx' - dx}{dx} = \frac{du}{dx}$$

General Definition of 3D Strain

As in the one dimensional strain derivation, suppose that point *P* in a body shifts to point ***P*** after deformation.



The infinitesimal strain-displacement relationships can be summarized as,

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

where **u** is the displacement vector, *x* is coordinate, and the two indices *i* and *j* can range over the three coordinates {1, 2, 3} in three dimensional space.

Expanding the above equation for each coordinate direction gives,

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = \varepsilon_{zy}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \varepsilon_{xz}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \qquad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \varepsilon_{yx}$$

where *u*, *v*, and *w* are the displacements in the *x*, *y*, and *z* directions respectively (i.e. they are the components of **u**).

3D Strain Matrix

There are a total of 6 strain measures. These 6 measures can be organized into a matrix (similar in form to the 3D stress matrix), shown here,

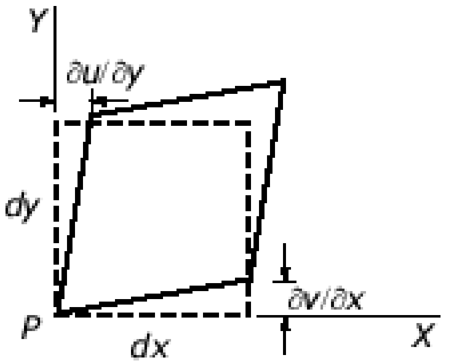
$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

Engineering Shear Strain

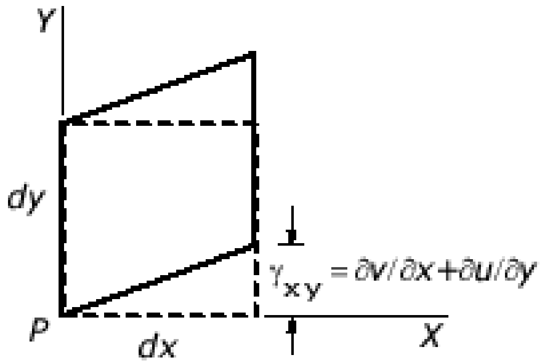
Focus on the strain ε_{xy} for a moment. The expression inside the parentheses can be rewritten as,

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

where $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$. Called the **engineering shear strain**, γ_{xy} is a total measure of shear strain in the *x*-*y* plane. In contrast, the shear strain ε_{xy} is the average of the shear strain on the *x* face along the *y* direction, and on the *y* face along the *x* direction.



Shear strain tensor is the ***average*** of two strains, i.e.,
$\varepsilon_{xy} = (\partial v/\partial x+\partial u/\partial x)/2 = \varepsilon_{yx}$



Engineer shear strain is the ***total*** shear strain, i.e.,
$\gamma_{xy} = \partial v/\partial x+\partial u/\partial x$

Engineering shear strain is commonly used in engineering reference books. However, please beware of the difference between shear strain and engineering shear strain, so as to avoid errors in mathematical manipulations.

Compatibility Conditions

In the strain-displacement relationships, there are six strain measures but only three independent displacements. That is, there are 6 unknowns for only 3 independent variables. As a result there exist 3 constraint, or compatibility, equations.

These compatibility conditions for infinitesimal strain reffered to rectangular Cartesian coordinates are,

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \qquad \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} \qquad \frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \varepsilon_{yz}}{\partial x} - \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = 2\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x} \qquad \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial z}\right)$$

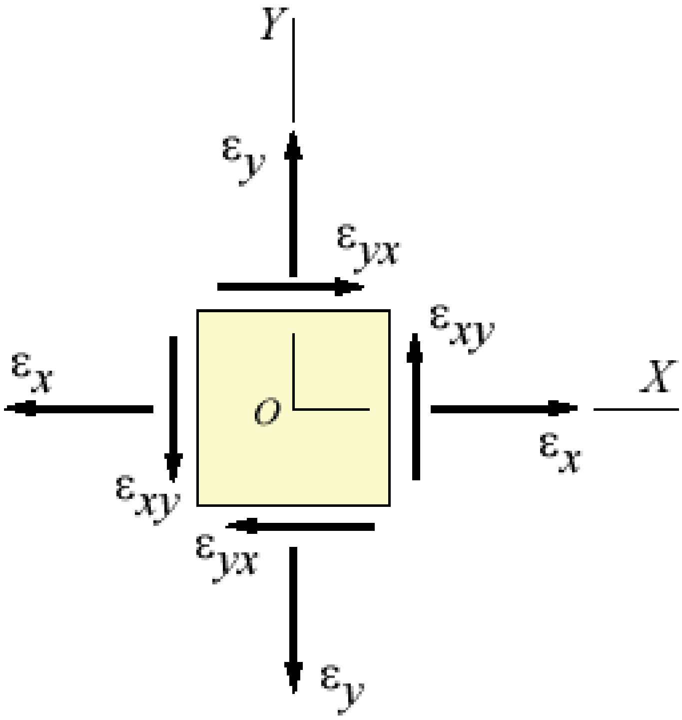
In two dimensional problems (e.g. plane strain), all *z* terms are set to zero. The compatibility equations reduce to,

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}$$

Note that some references use engineering shear strain ($\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$) when referencing compatibility equations.

Plane Strain and Coordinate Transformations

Plane State of Strain



Some common engineering problems such as a dam subjected to water loading, a tunnel under external pressure, a pipe under internal pressure, and a cylindrical roller bearing compressed by force in a diametral plane, have significant strain only in a plane; that is, the strain in one direction is much less than the strain in the two other orthogonal directions. If small enough, the smallest strain can be ignored and the part is said to experience **plane strain**.

Assume that the negligible strain is oriented in the *z*-direction. To reduce the 3D strain matrix to the 2D plane stress matrix, remove all components with *z* subscripts to get,

$$\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_y \end{bmatrix}$$

where $\varepsilon_{xy} = \varepsilon_{yx}$ by definition.

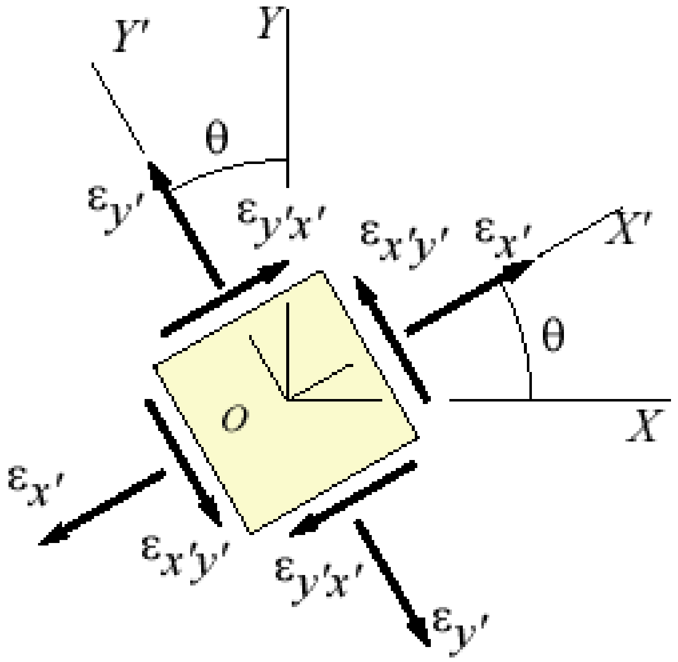
The sign convention here is consistent with the sign convention used in plane stress analysis.

Coordinate Transformation

The transformation of strains with respect to the $\{x,y,z\}$ coordinates to the strains with respect to $\{x',y',z'\}$ is performed via the equations,

$$\begin{cases} \varepsilon_{x'} = \dfrac{\varepsilon_x + \varepsilon_y}{2} + \dfrac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \varepsilon_{xy}\sin 2\theta \\ \varepsilon_{y'} = \dfrac{\varepsilon_x + \varepsilon_y}{2} - \dfrac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta - \varepsilon_{xy}\sin 2\theta \\ \quad = \varepsilon_x + \varepsilon_y - \varepsilon_{x'} \\ \varepsilon_{x'y'} = -\dfrac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \varepsilon_{xy}\cos 2\theta \end{cases}$$

The rotation between the two coordinate sets is shown here,



where θ is defined positive in the counterclockwise direction.

Principal Strain for the Case of Plane Strain

Principal Directions, Principal Strain

The normal strains ($\varepsilon_{x'}$ and $\varepsilon_{y'}$) and the shear strain ($\varepsilon_{x'y'}$) vary smoothly with respect to the rotation angle θ, in accordance with the transformation equations given above. There exist a couple of particular angles where the strains take on special values.

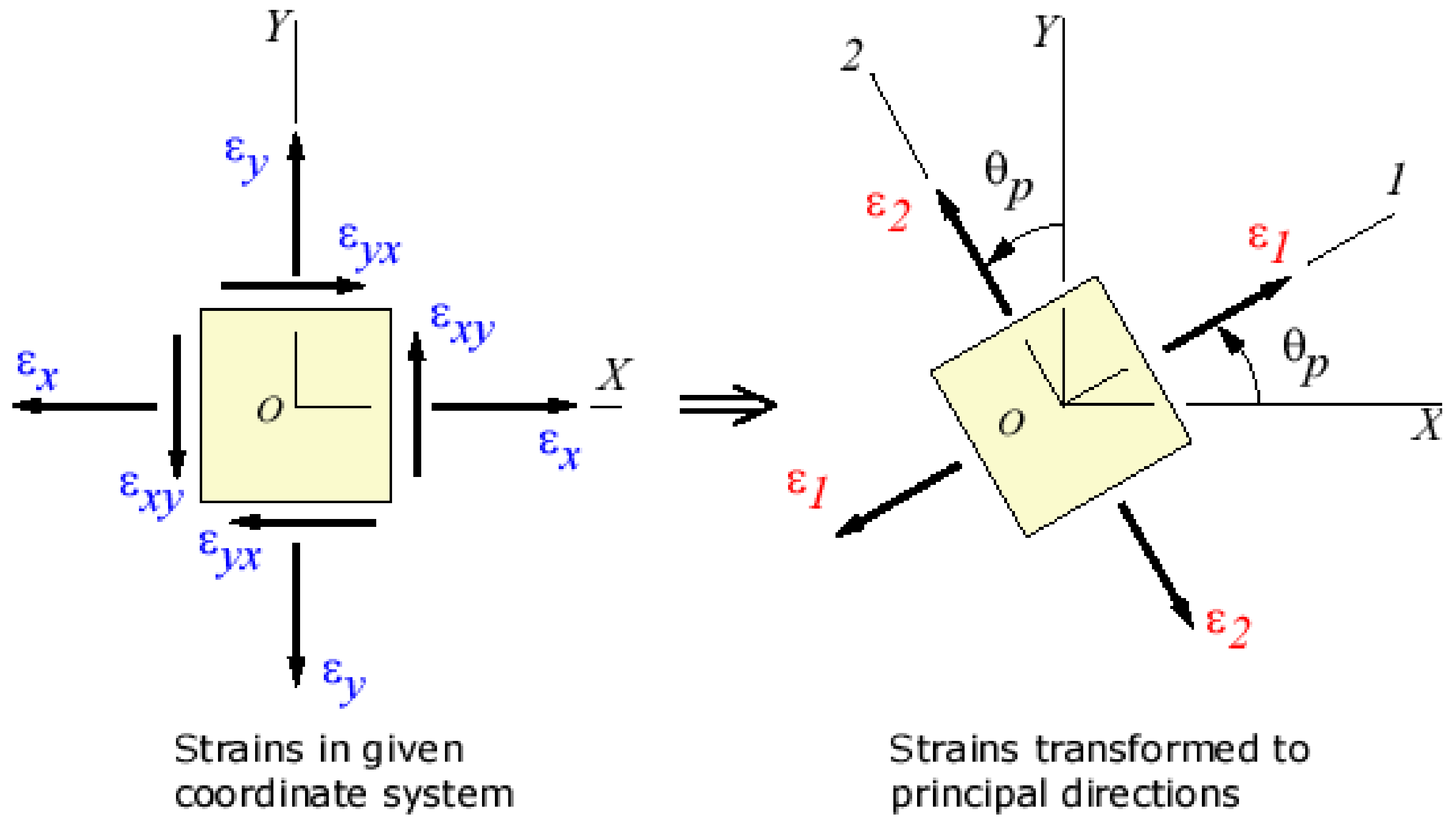
First, there exists an angle θ_p where the shear strain $\varepsilon_{x'y'}$ vanishes. That angle is given by,

$$\tan 2\theta_p = \frac{2\varepsilon_{xy}}{\varepsilon_x - \varepsilon_y}$$

This angle defines the *principal directions*. The associated *principal strains* are given by,

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \varepsilon_{xy}{}^2}$$

The transformation to the principal directions with their principal strains can be illustrated as:



Maximum Shear Strain Direction

Another important angle, θ_s, is where the maximum shear strain occurs and is given by,

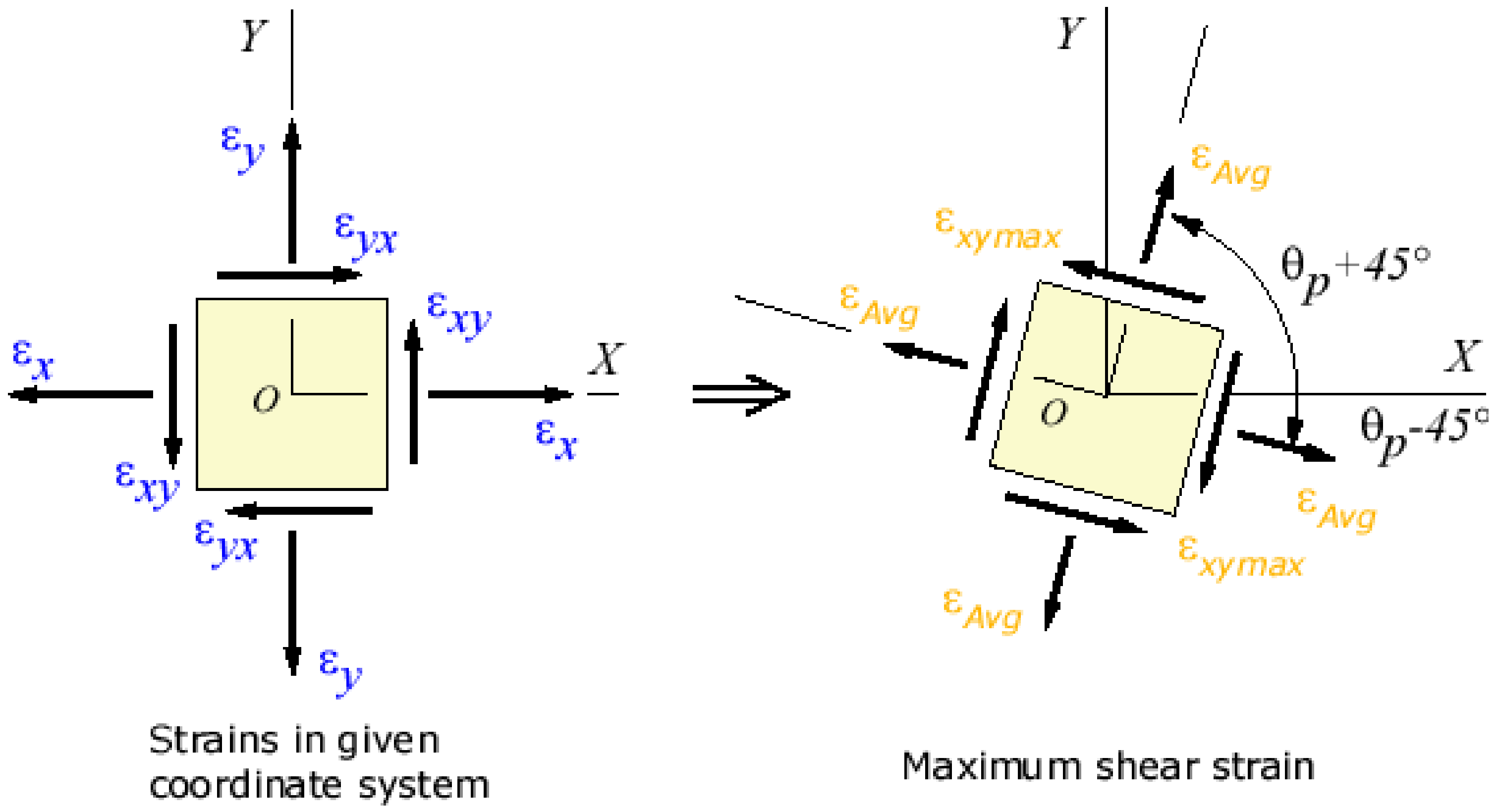
$$\tan 2\theta_s = -\frac{\varepsilon_x - \varepsilon_y}{2\varepsilon_{xy}}$$

$$\Rightarrow \theta_s = \theta_p \pm 45^\circ$$

The maximum shear strain is found to be one-half the difference between the two principal strains,

$$\varepsilon_{\max} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \varepsilon_{xy}{}^2} = \frac{\varepsilon_1 - \varepsilon_2}{2}$$

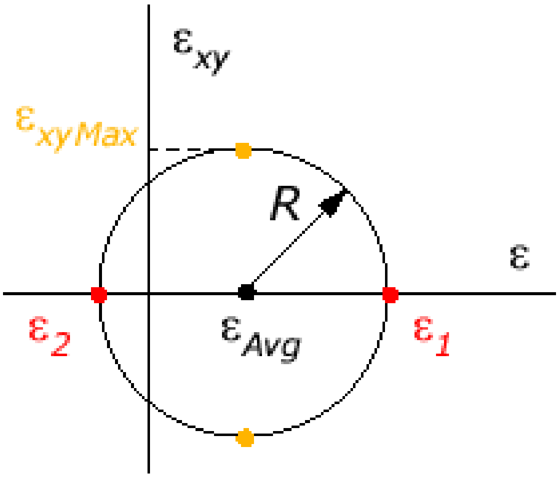
The transformation to the maximum shear strain direction can be illustrated as:



Mohr's Circle for Plane Strain

Mohr's Circle

Strains at a point in the body can be illustrated by Mohr's Circle. The idea and procedures are exactly the same as for Mohr's Circle for plane stress.



The two principal strains are shown in **red**, and the maximum shear strain is shown in **orange**. Recall that the normal strains are equal to the principal strains when the element is aligned with the principal directions, and the shear strain is equal to the maximum shear strain when the element is rotated 45° away from the principal directions.

As the element is rotated away from the principal (or maximum strain) directions, the normal and shear strain components will always lie on Mohr's Circle.

Derivation of Mohr's Circle

To establish the Mohr's circle, we first recall the strain transformation formulas for plane strain,

$$\begin{cases} \varepsilon_{x'} - \dfrac{\varepsilon_x + \varepsilon_y}{2} = \dfrac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta + \varepsilon_{xy} \sin 2\theta \\ \varepsilon_{x'y'} = -\dfrac{\varepsilon_x - \varepsilon_y}{2} \sin 2\theta + \varepsilon_{xy} \cos 2\theta \end{cases}$$

Using a basic trigonometric relation ($\cos^2 2\theta + \sin^2 2\theta = 1$) to combine the above two formulas we have,

$$\left(\varepsilon_{x'} - \frac{\varepsilon_x + \varepsilon_y}{2}\right)^2 + \varepsilon_{x'y'}{}^2 = \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \varepsilon_{xy}{}^2$$

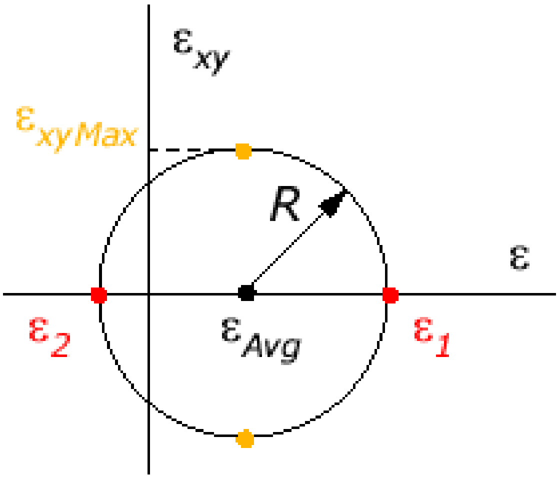
This equation is an equation for a circle. To make this more apparent, we can rewrite it as,

$$\left(\varepsilon_{x'} - \varepsilon_{\mathrm{Avg}}\right)^2 + \varepsilon_{x'y'}{}^2 = R^2$$

where,

$$\varepsilon_{\mathrm{Avg}} = \frac{\varepsilon_x + \varepsilon_y}{2} \qquad R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \varepsilon_{xy}{}^2}$$

The circle is centered at the average strain value ε_{Avg}, and has a radius *R* equal to the maximum shear strain, as shown in the figure below,



Related Topics

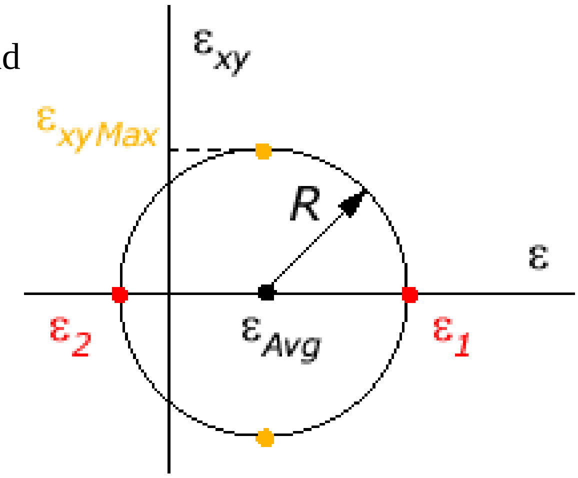
The procedure of drawing Mohr's Circle from a given strain state is discussed in the Mohr's Circle usage and examples pages.

The Mohr's Circle for plane stress can also be obtained from similar procedures.

Mohr's Circle Usage in Plane Strain

Principal Strains from Mohr's Circle

A chief benefit of Mohr's circle is that the principal strains ε_1 and ε_2 and the maximum shear strain ε_{xyMax} are obtained immediately after drawing the circle,



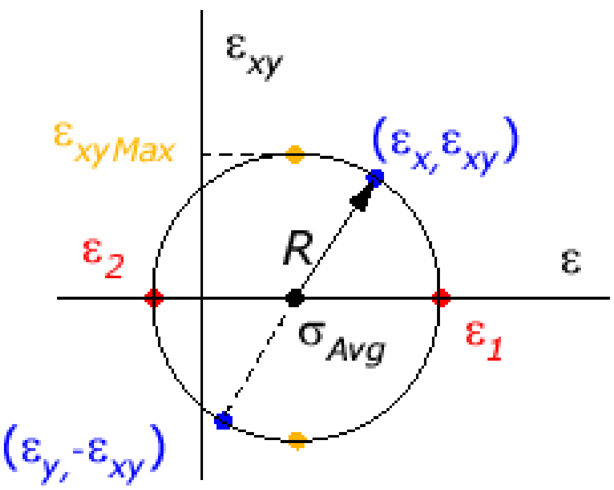
$$\begin{cases} \varepsilon_{1,2} = \varepsilon_{Avg} \pm R \\ \varepsilon_{xy_{Max}} = R \end{cases}$$

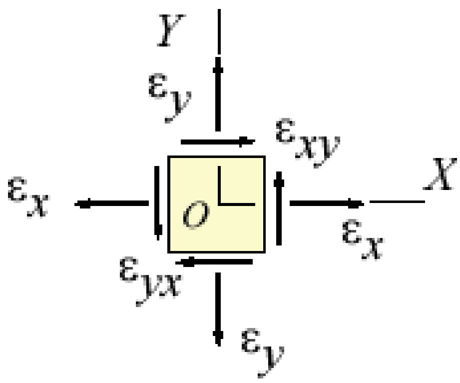
where,

$$\varepsilon_{Avg} = \frac{\varepsilon_x + \varepsilon_y}{2} \qquad R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \varepsilon_{xy}{}^2}$$

Principal Directions from Mohr's Circle

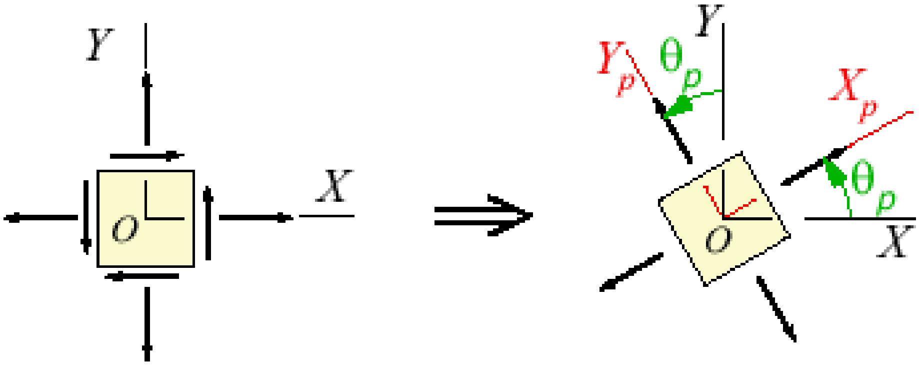
Mohr's Circle can be used to find the directions of the principal axes. To show this, first suppose that the normal and shear strains, ε_x, ε_y, and ε_{xy}, are obtained at a given point *O* in the body. They are expressed relative to the coordinates *XY*, as shown in the strain element at right below.

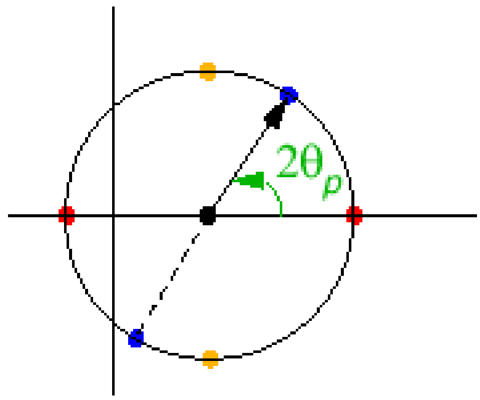




The Mohr's Circle for this general strain state is shown at left above. Note that it's centered at ε_{Avg} and has a radius *R*, and that the two points (ε_x, ε_{xy}) and (ε_y, -ε_{xy}) lie on opposites sides of the circle. The line connecting ε_x and ε_y will be defined as L_{xy}.

The **angle** between the current axes (*X* and *Y*) and the **principal axes** is defined as θ_p, and is equal to one half the angle between the line L_{xy} and the ε-axis as shown in the schematic below,





A set of six Mohr's Circles representing most strain state possibilities are presented on the examples page.

Also, principal directions can be computed by the principal strain calculator.

Rotation Angle on Mohr's Circle

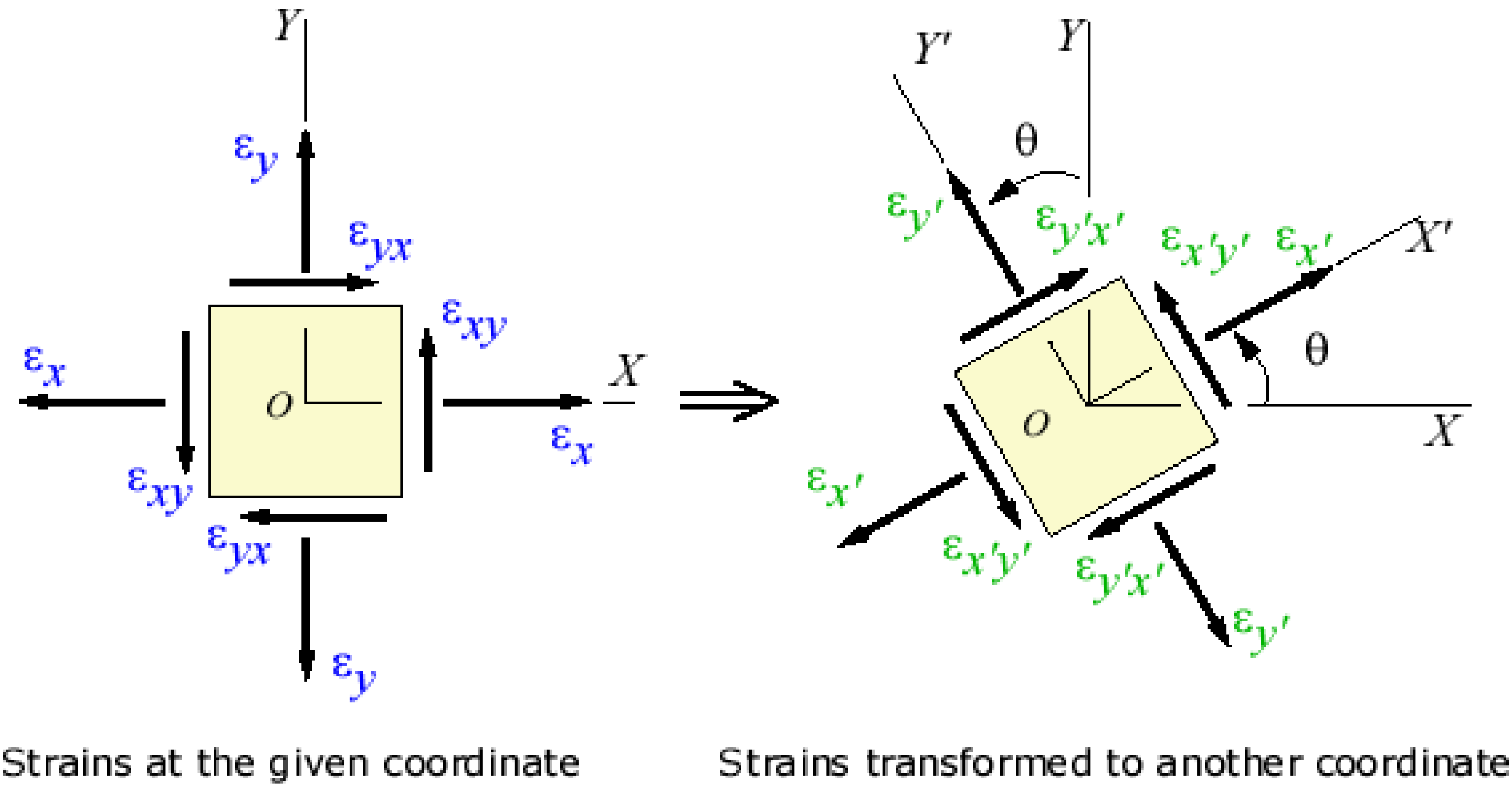
Note that the coordinate rotation angle θ_p is defined positive when starting at the *XY* coordinates and proceeding to the X_pY_p coordinates. In contrast, on the Mohr's Circle θ_p is defined positive starting on the principal strain line (i.e. the ε-axis) and proceeding to the *XY* strain line (i.e. line L_{xy}). The angle θ_p has the opposite sense between the two figures, because on one it starts on the *XY* coordinates, and on the other it starts on the principal coordinates.

Some books avoid the sign difference between θ_p on Mohr's Circle and θ_p on the stress element by locating (ε_x, -ε_{xy}) instead of (ε_x, ε_{xy}) on Mohr's Circle. This will switch the polarity of θ_p on the circle. Whatever method you choose, the bottom line is that an *opposite* sign is needed either in the interpretation or in the plotting to make Mohr's Circle physically meaningful.

Strain Transform by Mohr's Circle

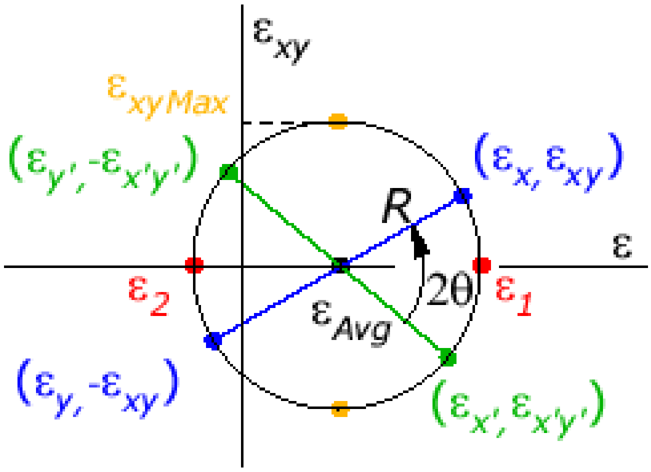
Mohr's Circle can be used to transform strains from one coordinate set to another, similar that that described on the plane strain page.

Suppose that the normal and shear strains, ε_x, ε_y, and ε_{xy}, are obtained at a point *O* in the body, expressed with respect to the coordinates *XY*. We wish to find the strains expressed in the new coordinate set *X'Y'*, rotated an angle θ from *XY*, as shown below:



To do this we proceed as follows:

- Draw Mohr's circle for the **given strain state** (ε_x, ε_y, and ε_{xy}; shown below).
- Draw the line L_{xy} across the circle from (ε_x, ε_{xy}) to (ε_y, -ε_{xy}).
- Rotate the line L_{xy} by 2*θ (twice as much as the angle between *XY* and *X'Y'*) and in the *opposite* direction of θ.
- The **strains in the new coordinates** ($\varepsilon_{x'}$, $\varepsilon_{y'}$, and $\varepsilon_{x'y'}$) are then read off the circle.

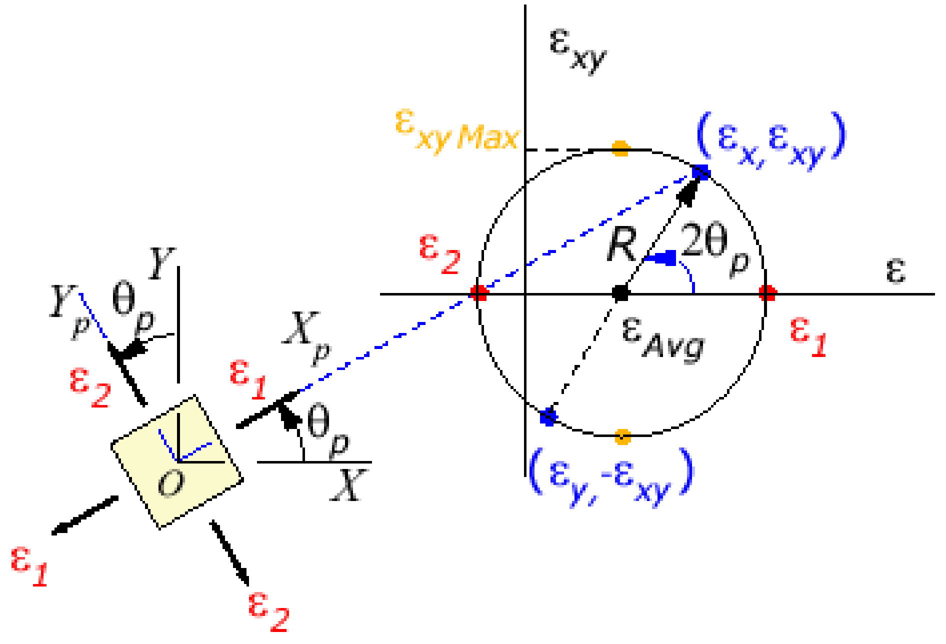


Strain transforms can be performed using eFunda's strain transform calculator.

Examples of Mohr's Circles in Plane Strain

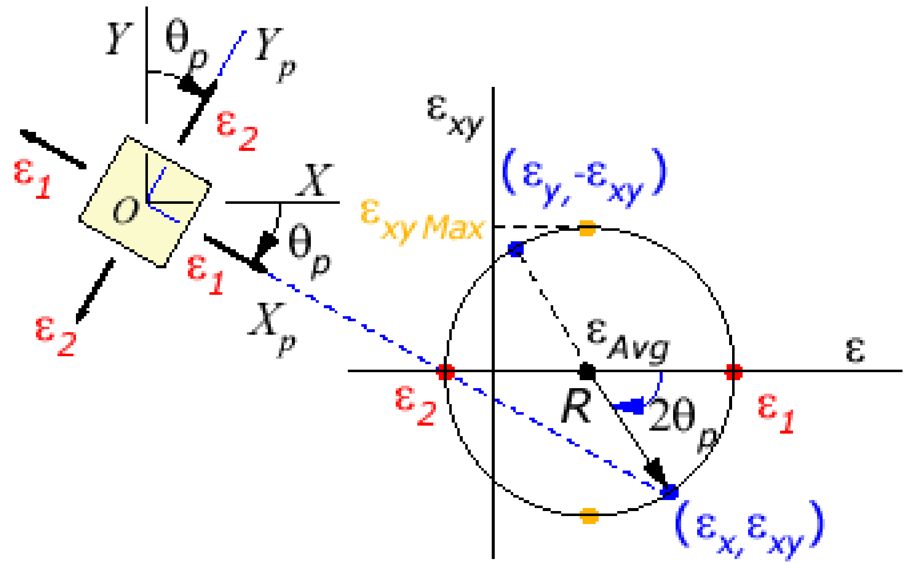
Case 1: $\varepsilon_{xy} > 0$ **and** $\varepsilon_x > \varepsilon_y$

The principal axes are counterclockwise to the current axes (because $\varepsilon_{xy} > 0$) and no more than 45° away (because $\varepsilon_x > \varepsilon_y$).



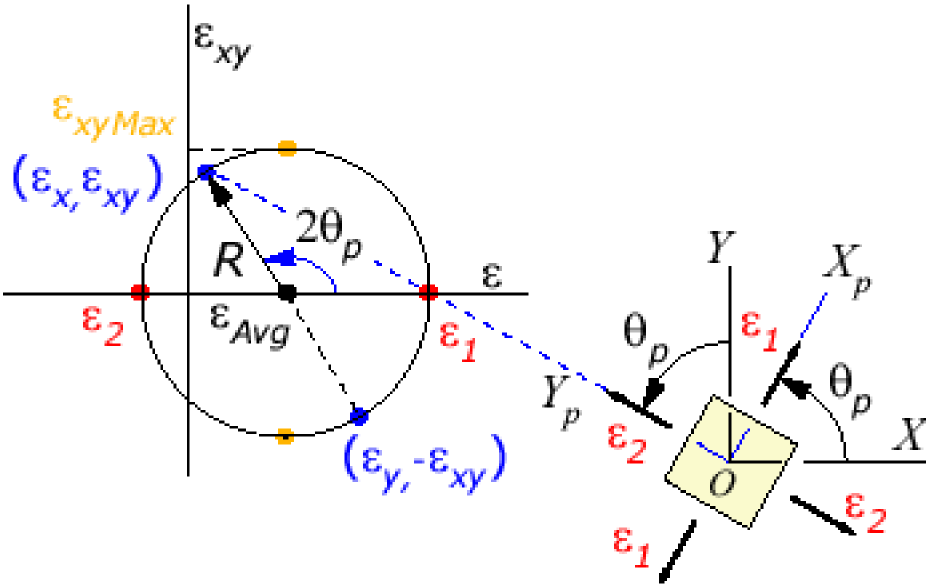
Case 2: $\varepsilon_{xy} < 0$ **and** $\varepsilon_x > \varepsilon_y$

The principal axes are clockwise to the current axes (because $\varepsilon_{xy} < 0$) and no more than 45° away (because $\varepsilon_x > \varepsilon_y$).



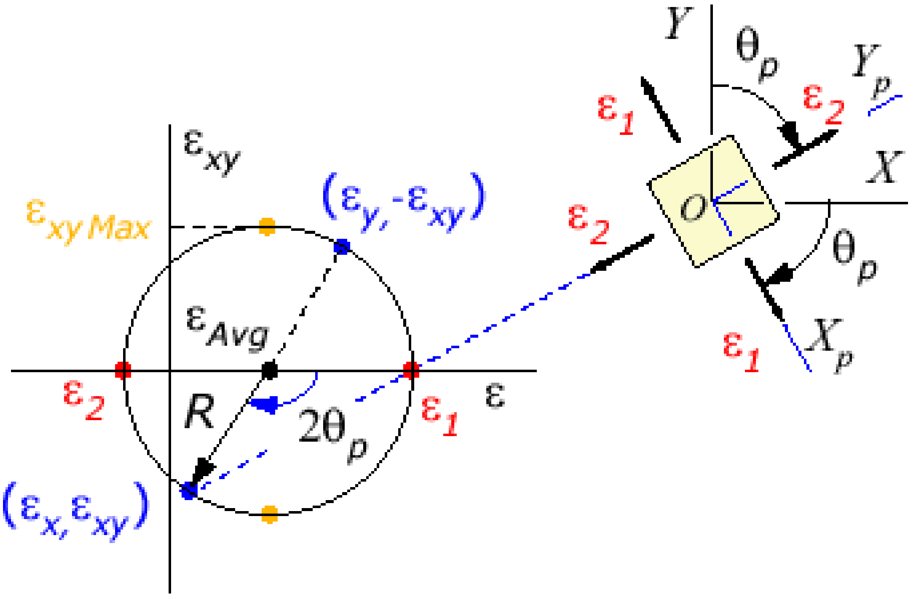
Case 3: $\varepsilon_{xy} > 0$ **and** $\varepsilon_x < \varepsilon_y$

The principal axes are counterclockwise to the current axes (because $\varepsilon_{xy} > 0$) and between 45° and 90° away (because $\varepsilon_x < \varepsilon_y$).



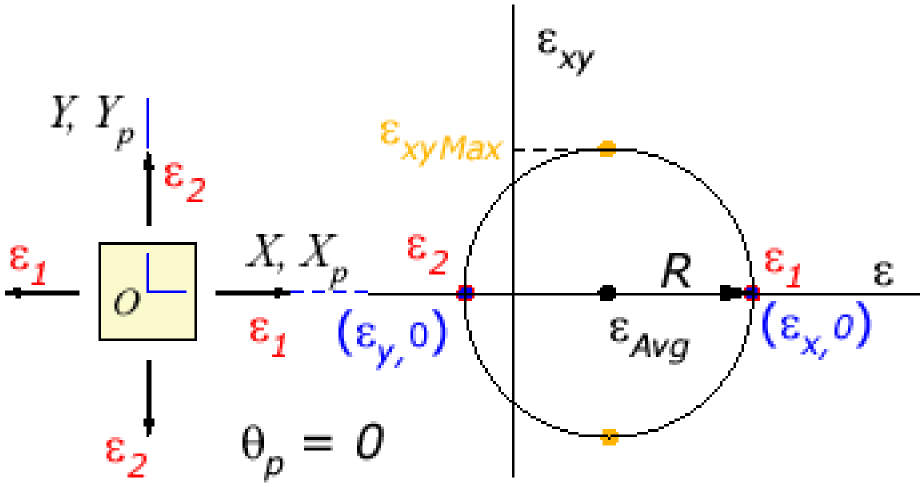
Case 4: $\varepsilon_{xy} < 0$ **and** $\varepsilon_x < \varepsilon_y$

The principal axes are clockwise to the current axes (because $\varepsilon_{xy} < 0$) and between 45° and 90° away (because $\varepsilon_x < \varepsilon_y$).



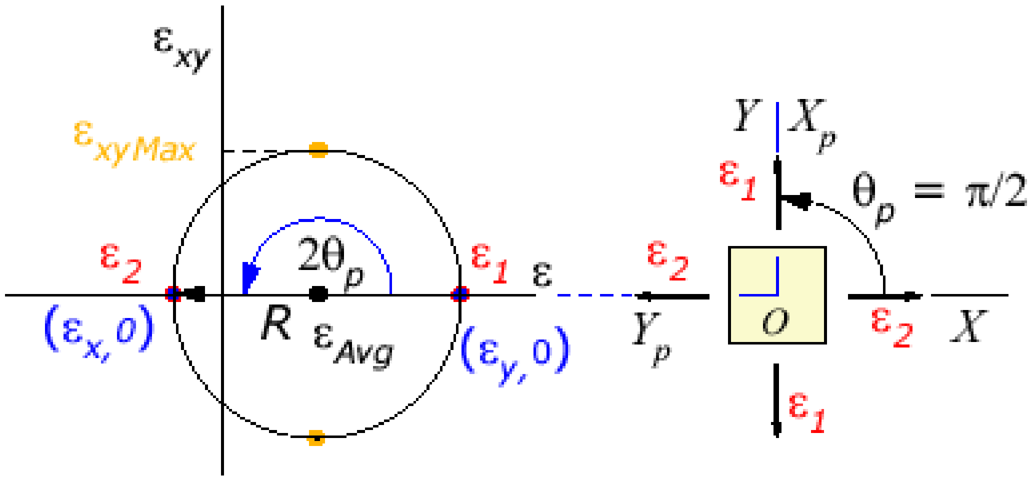
Case 5: $\varepsilon_{xy} = 0$ **and** $\varepsilon_x > \varepsilon_y$

The principal axes are aligned with the current axes (because $\varepsilon_x > \varepsilon_y$ and $\varepsilon_{xy} = 0$).



Case 6: $\varepsilon_{xy} = 0$ and $\varepsilon_x < \varepsilon_y$

The principal axes are exactly 90° from the current axes (because $\varepsilon_x < \varepsilon_y$ and $\varepsilon_{xy} = 0$).



Mechanics of Materials: Hooke's Law

One-dimensional Hooke's Law

Robert Hooke, who in 1676 stated,

> The power (*sic*.) of any springy body is in the same proportion with the extension.

announced the birth of elasticity. Hooke's statement expressed mathematically is,

$$F = k \cdot u$$

where *F* is the applied force (and not the power, as Hooke mistakenly suggested), *u* is the deformation of the elastic body subjected to the force *F*, and *k* is the spring constant (i.e. the ratio of previous two parameters).

Generalized Hooke's Law (Anisotropic Form)

Cauchy generalized Hooke's law to three dimensional elastic bodies and stated that the 6 components of stress are linearly related to the 6 components of strain.

The stress-strain relationship written in matrix form, where the 6 components of stress and strain are organized into column vectors, is,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}, \quad \varepsilon = \mathbf{S} \cdot \sigma$$

or,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix}, \quad \sigma = \mathbf{C} \cdot \varepsilon$$

where **C** is the **stiffness matrix**, **S** is the **compliance matrix**, and $\mathbf{S} = \mathbf{C}^{-1}$.

In general, stress-strain relationships such as these are known as **constitutive relations**.

In general, there are 36 stiffness matrix components. However, it can be shown that conservative materials possess a strain energy density function and as a result, the stiffness and compliance matrices are symmetric. Therefore, only 21 stiffness components are actually independent in Hooke's law. The vast majority of engineering materials are conservative.

Please note that the **stiffness** matrix is traditionally represented by the symbol **C**, while **S** is reserved for the **compliance** matrix. This convention may seem backwards, but perception is not always reality. For instance, Americans hardly ever use their feet to play (American) football.

Hooke's Law for Orthotropic Materials

Orthotropic Definition

Some engineering materials, including certain piezoelectric materials (e.g. Rochelle salt) and 2-ply fiber-reinforced composites, are **orthotropic**.

By definition, an orthotropic material has at least 2 orthogonal planes of symmetry, where material properties are independent of direction within each plane. Such materials require 9 independent variables (i.e. elastic constants) in their constitutive matrices.

In contrast, a material without any planes of symmetry is fully anisotropic and requires 21 elastic constants, whereas a material with an infinite number of symmetry planes (i.e. every plane is a plane of symmetry) is isotropic, and requires only 2 elastic constants.

Hooke's Law in Compliance Form

By convention, the 9 elastic constants in orthotropic constitutive equations are comprised of 3 Young's modulii E_x, E_y, E_z, the 3 Poisson's ratios ν_{yz}, ν_{zx}, ν_{xy}, and the 3 shear modulii G_{yz}, G_{zx}, G_{xy}.

The **compliance matrix** takes the form,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{yz}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{zx}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{xy}} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}$$

where $\frac{\nu_{yz}}{E_y} = \frac{\nu_{zy}}{E_z}$, $\frac{\nu_{zx}}{E_z} = \frac{\nu_{xz}}{E_x}$, $\frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y}$.

Note that, in orthotropic materials, there is no interaction between the normal stresses σ_x, σ_y, σ_z and the shear strains ε_{yz}, ε_{zx}, ε_{xy}

The factor 2 multiplying the shear modulii in the compliance matrix results from the difference between shear strain and engineering shear strain, where $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$, etc.

Hooke's Law in Stiffness Form

The **stiffness matrix** for orthotropic materials, found from the inverse of the compliance matrix, is given by,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1-\nu_{yz}\nu_{zy}}{E_y E_z \Delta} & \frac{\nu_{yx}+\nu_{zx}\nu_{yz}}{E_y E_z \Delta} & \frac{\nu_{zx}+\nu_{yx}\nu_{zy}}{E_y E_z \Delta} & 0 & 0 & 0 \\ \frac{\nu_{xy}+\nu_{xz}\nu_{zx}}{E_z E_x \Delta} & \frac{1-\nu_{zx}\nu_{xz}}{E_z E_x \Delta} & \frac{\nu_{zy}+\nu_{zx}\nu_{xy}}{E_z E_x \Delta} & 0 & 0 & 0 \\ \frac{\nu_{xz}+\nu_{xy}\nu_{yz}}{E_x E_y \Delta} & \frac{\nu_{zy}+\nu_{xz}\nu_{yz}}{E_x E_y \Delta} & \frac{1-\nu_{xy}\nu_{yx}}{E_x E_y \Delta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2G_{yz} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2G_{zx} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2G_{xy} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix}$$

where,

$$\Delta = \frac{1-\nu_{xy}\nu_{yx}-\nu_{yz}\nu_{zy}-\nu_{zx}\nu_{xz}-\nu_{xy}\nu_{yz}\nu_{zx}}{E_x E_y E_z}$$

The fact that the stiffness matrix is symmetric requires that the following statements hold,

$$\begin{cases} \frac{\nu_{yx}+\nu_{zx}\nu_{yz}}{E_y E_z \Delta} = \frac{\nu_{xy}+\nu_{xz}\nu_{zx}}{E_z E_x \Delta} \\ \frac{\nu_{zy}+\nu_{zx}\nu_{xy}}{E_z E_x \Delta} = \frac{\nu_{zy}+\nu_{xz}\nu_{yz}}{E_x E_y \Delta} \\ \frac{\nu_{zx}+\nu_{yx}\nu_{zy}}{E_y E_z \Delta} = \frac{\nu_{xz}+\nu_{xy}\nu_{yz}}{E_x E_y \Delta} \end{cases}$$

The factor of 2 multiplying the shear modulii in the stiffness matrix results from the difference between shear strain and engineering shear strain, where $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$, etc.

Hooke's Law for Transversely Isotropic Materials

Transverse Isotropic Definition

A special class of orthotropic materials are those that have the same properties in one plane (e.g. the *x-y* plane) and different properties in the direction normal to this plane (e.g. the *z*-axis). Such materials are called **transverse isotropic**, and they are described by 5 independent elastic constants, instead of 9 for fully orthotropic.

Examples of transversely isotropic materials include some piezoelectric materials (e.g. PZT-4, barium titanate) and fiber-reinforced composites where all fibers are in parallel.

Hooke's Law in Compliance Form

By convention, the 5 elastic constants in transverse isotropic constitutive equations are the Young's modulus and poisson ratio in the *x-y* symmetry plane, E_p and ν_p, the Young's modulus and poisson ratio in the *z*-direction, E_{pz} and ν_{pz}, and the shear modulus in the *z*-direction G_{zp}.

The **compliance matrix** takes the form,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_p} & -\frac{\nu_p}{E_p} & -\frac{\nu_{zp}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_p}{E_p} & \frac{1}{E_p} & -\frac{\nu_{zp}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{pz}}{E_p} & -\frac{\nu_{pz}}{E_p} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{zp}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{zp}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1+\nu_p}{E_p} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}$$

where $\frac{\nu_{pz}}{E_p} = \frac{\nu_{zp}}{E_z}$.

The factor 2 multiplying the shear modulii in the compliance matrix results from the difference between shear strain and engineering shear strain, where $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$, etc.

Hooke's Law in Stiffness Form

The **stiffness matrix** for transverse isotropic materials, found from the inverse of the compliance matrix, is given by,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1-\nu_{pz}\nu_{zp}}{E_p E_z \Delta} & \frac{\nu_p+\nu_{zp}\nu_{pz}}{E_p E_z \Delta} & \frac{\nu_{zp}+\nu_p\nu_{zp}}{E_p E_z \Delta} & 0 & 0 & 0 \\ \frac{\nu_p+\nu_{pz}\nu_{zp}}{E_z E_p \Delta} & \frac{1-\nu_{zp}\nu_{pz}}{E_z E_p \Delta} & \frac{\nu_{zp}+\nu_{zp}\nu_p}{E_z E_p \Delta} & 0 & 0 & 0 \\ \frac{\nu_{pz}+\nu_p\nu_{pz}}{E_p^2 \Delta} & \frac{\nu_{zp}+\nu_{pz}^2}{E_p^2 \Delta} & \frac{1-\nu_p^2}{E_p^2 \Delta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2G_{zp} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2G_{zp} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{E_p}{1+\nu_p} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix}$$

where,

$$\Delta = \frac{1-\nu_p^2 - 2\nu_{pz}\nu_{zp} - \nu_p\nu_{pz}\nu_{zp}}{E_p^2 E_z}$$

The fact that the stiffness matrix is symmetric requires that the following statements hold,

$$\begin{cases} \frac{\nu_p+\nu_{zp}\nu_{pz}}{E_p E_z \Delta} = \frac{\nu_p+\nu_{pz}\nu_{zp}}{E_z E_p \Delta} \\ \frac{\nu_{zp}+\nu_{zp}\nu_p}{E_z E_p \Delta} = \frac{\nu_{zp}+\nu_{pz}^2}{E_p^2 \Delta} \\ \frac{\nu_{zp}+\nu_p\nu_{zp}}{E_p E_z \Delta} = \frac{\nu_{pz}+\nu_p\nu_{pz}}{E_p^2 \Delta} \end{cases}$$

The factor of 2 multiplying the shear modulii in the stiffness matrix results from the difference between shear strain and engineering shear strain, where $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$, etc.

Hooke's Law for Isotropic Materials

Isotropic Definition

Most metallic alloys and thermoset polymers are considered **isotropic**, where by definition the material properties are independent of direction. Such materials have only 2 independent variables (i.e. elastic constants) in their stiffness and compliance matrices, as opposed to the 21 elastic constants in the general anisotropic case.

The two elastic constants are usually expressed as the Young's modulus *E* and the Poisson's ratio ν. However, the alternative elastic constants *K* (bulk modulus) and/or *G* (shear modulus) can also be used. For isotropic materials, *G* and *K* can be found from *E* and ν by a set of equations, and vice-versa.

Hooke's Law in Compliance Form

Hooke's law for isotropic materials in **compliance matrix** form is given by,

$$\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\\ \varepsilon_{xy}\end{bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1+\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1+\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1+\nu\end{bmatrix}\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{zx}\\ \sigma_{xy}\end{bmatrix}$$

Hooke's Law in Stiffness Form

The **stiffness matrix** is equal to the inverse of the compliance matrix, and is given by,

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{zx}\\ \sigma_{xy}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & 1-2\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1-2\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1-2\nu\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\\ \varepsilon_{xy}\end{bmatrix}$$

Visit the elastic constant calculator to see the interplay amongst the 4 elastic constants (*E*, ν, *G*, *K*).

Hooke's Law for Plane Stress

Hooke's Law for Plane Stress

For the simplification of plane stress, where the stresses in the *z* direction are considered to be negligible, $\sigma_{zz} = \sigma_{yz} = \sigma_{xz} = 0$, the stress-strain compliance relationship for an isotropic material becomes,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ 0 \\ 0 \\ 0 \\ \sigma_{xy} \end{bmatrix}$$

The three zero'd stress entries in the stress vector indicate that we can ignore their associated columns in the compliance matrix (i.e. columns 3, 4, and 5). If we also ignore the rows associated with the strain components with *z*-subscripts, the **compliance matrix** reduces to a simple 3x3 matrix,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$

The **stiffness matrix** for plane stress is found by inverting the plane stress compliance matrix, and is given by,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix}$$

Note that the stiffness matrix for plane stress is **NOT** found by removing columns and rows from the general isotropic stiffness matrix.

Plane Stress Hooke's Law via Engineering Strain

Some reference books incorporate the shear modulus G and the engineering shear strain γ_{xy}, related to the shear strain ε_{xy} via,

$$\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$$

The stress-strain **compliance matrix** using G and γ_{xy} are,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

The **stiffness matrix** is,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

The shear modulus G is related to E and ν via,

$$G = \frac{E}{2(1+\nu)}$$

Hooke's Law for Plane Strain

Hooke's Law for Plane Strain

For the case of plane strain, where the strains in the *z* direction are considered to be negligible, $\varepsilon_{zz} = \varepsilon_{yz} = \varepsilon_{xz} = 0$, the stress-strain stiffness relationship for an isotropic material becomes,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 0 \\ 0 \\ 0 \\ \varepsilon_{xy} \end{bmatrix}$$

The three zero'd strain entries in the strain vector indicate that we can ignore their associated columns in the stiffness matrix (i.e. columns 3, 4, and 5). If we also ignore the rows associated with the stress components with *z*-subscripts, the **stiffness matrix** reduces to a simple 3x3 matrix,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

The **compliance matrix** for plane stress is found by inverting the plane stress stiffness matrix, and is given by,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}$$

Note that the compliance matrix for plane stress is **NOT** found by removing columns and rows from the general isotropic compliance matrix.

Plane Strain Hooke's Law via Engineering Strain

The stress-strain **stiffness matrix** expressed using the shear modulus **G** and the engineering shear strain $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$ is,

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} & \frac{\nu E}{(1+\nu)(1-2\nu)} & 0 \\ \frac{\nu E}{(1+\nu)(1-2\nu)} & \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

The **compliance matrix** is,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1-\nu^2}{E} & -\frac{\nu(1+\nu)}{E} & 0 \\ -\frac{\nu(1+\nu)}{E} & \frac{1-\nu^2}{E} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

The shear modulus G is related to E and ν via,

$$G = \frac{E}{2(1+\nu)}$$

Finding Young's Modulus and Poisson's Ratio

Youngs Modulus from Uniaxial Tension

When a specimen made from an isotropic material is subjected to uniaxial tension, say in the *x* direction, σ_{xx} is the only non-zero stress. The strains in the specimen are obtained by,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The modulus of elasticity in tension, also known as **Young's modulus *E***, is the ratio of stress to strain on the loading plane along the loading direction,

$$E = \frac{\sigma_{xx}}{\varepsilon_{xx}}$$

Common sense (and the 2nd Law of Thermodynamics) indicates that a material under uniaxial tension must elongate in length. Therefore the Young's modulus *E* is required to be non-negative for all materials,

$$E > 0$$

Poisson's Ratio from Uniaxial Tension

A rod-like specimen subjected to uniaxial tension will exhibit some shrinkage in the lateral direction for most materials. The ratio of lateral strain and axial strain is defined as **Poisson's ratio** ν,

$$\nu = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}}$$

The Poisson ratio for most metals falls between 0.25 to 0.35. Rubber has a Poisson ratio close to 0.5 and is therefore almost incompressible. Theoretical materials with a Poisson ratio of **exactly 0.5** are truly **incompressible**, since the sum of all their strains leads to a zero volume change. Cork, on the other hand, has a Poisson ratio close to zero. This makes cork function well as a bottle stopper, since an axially-loaded cork will not swell laterally to resist bottle insertion.

The Poisson's ratio is bounded by two theoretical limits: it must be greater than -1, and less than or equal to 0.5,

$$-1 < \nu \le \frac{1}{2}$$

The proof for this stems from the fact that *E*, *G*, and *K* are all positive and mutually dependent. However, it is rare to encounter engineering materials with negative Poisson ratios. Most materials will fall in the range,

$$0 \le \nu \le \frac{1}{2}$$

Finding the Shear Modulus and the Bulk Modulus

Shear Modulus from Pure Shear

When a specimen made from an isotropic material is subjected to pure shear, for instance, a cylindrical bar under torsion in the *xy* sense, σ_{xy} is the only non-zero stress. The strains in the specimen are obtained by,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sigma_{xy} \end{bmatrix}$$

The **shear modulus *G***, is defined as the ratio of shear stress to engineering shear strain on the loading plane,

$$G = \frac{\sigma_{xy}}{\varepsilon_{xy} + \varepsilon_{yx}} = \frac{\sigma_{xy}}{2\varepsilon_{xy}} = \frac{\sigma_{xy}}{\gamma_{xy}}$$
$$= \frac{E}{2(1+\nu)}$$

where $\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$.

The shear modulus *G* is also known as the rigidity modulus, and is equivalent to the 2nd Lamé constant μ mentioned in books on continuum theory.

Common sense and the 2nd Law of Thermodynamics require that a positive shear stress leads to a positive shear strain. Therefore, the shear modulus *G* is required to be nonnegative for all materials,

$$G > 0$$

Since both *G* and the elastic modulus *E* are required to be positive, the quantity in the denominator of *G* must also be positive. This requirement places a **lower bound restriction on the range for Poisson's ratio**,

$$\nu > -1$$

Bulk Modulus from Hydrostatic Pressure

When an isotropic material specimen is subjected to hydrostatic pressure σ, all shear stress will be zero and the normal stress will be uniform, $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma$. The strains in the specimen are given by,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma \\ \sigma \\ \sigma \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In response to the hydrostatic load, the specimen will change its volume. Its resistance to do so is quantified as the **bulk modulus *K***, also known as the modulus of compression. Technically, *K* is defined as the ratio of hydrostatic pressure to the relative volume change (which is related to the direct strains),

$$K = \frac{\sigma}{\Delta V / V} = \frac{\sigma}{\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}}$$

$$= \frac{E}{3(1-2\nu)}$$

Common sense and the 2nd Law of Thermodynamics require that a positive hydrostatic load leads to a positive volume change. Therefore, the bulk modulus *K* is required to be nonnegative for all materials,

$$K > 0$$

Since both *K* and the elastic modulus *E* are required to be positive, the following requirement is placed on the **upper bound of Poisson's ratio** by the denominator of *K*,

$$\nu < 1/2$$

Relation Between Relative Volume Change and Strain

For simplicity, consider a rectangular block of material with dimensions a_0, b_0, and c_0. Its volume V_0 is given by,

$$V_0 = a_0 b_0 c_0$$

When the block is loaded by stress, its volume will change since each dimension now includes a direct strain measure. To calculate the volume when loaded V_f, we multiply the new dimensions of the block,

$$\begin{aligned} V_f = a_f b_f c_f &= \left[a_0\left(1+\varepsilon_{xx}\right)\right]\left[b_0\left(1+\varepsilon_{yy}\right)\right]\left[c_0\left(1+\varepsilon_{zz}\right)\right] \\ &= V_0\left(1+\varepsilon_{xx}\right)\left(1+\varepsilon_{yy}\right)\left(1+\varepsilon_{zz}\right) \\ &= V_0\left(1+\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}+\varepsilon_{yy}\varepsilon_{zz}+\varepsilon_{zz}\varepsilon_{xx}+\varepsilon_{xx}\varepsilon_{yy}+\varepsilon_{xx}\varepsilon_{yy}\varepsilon_{zz}\right) \\ &\approx V_0\left(1+\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}\right) \end{aligned}$$

Products of strain measures will be much smaller than individual strain measures when the overall strain in the block is small (i.e. linear strain theory). Therefore, we were able to drop the strain products in the equation above.

The relative change in volume is found by dividing the volume difference by the initial volume,

$$\frac{\Delta V}{V_0} = \frac{V_f - V_0}{V_0} \approx \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

Hence, the relative volume change (for small strains) is equal to the sum of the 3 direct strains.

Failure Criteria

Stress-Based Criteria

The purpose of **failure criteria** is to predict or estimate the failure/yield of machine parts and structural members.

A considerable number of theories have been proposed. However, only the most common and well-tested theories applicable to isotropic materials are discussed here. These theories, dependent on the nature of the material in question (i.e. brittle or ductile), are listed in the following table:

Material Type	Failure Theories
Ductile	Maximum shear stress criterion, von Mises criterion
Brittle	Maximum normal stress criterion, Mohr's theory

All four criteria are presented in terms of principal stresses. Therefore, all stresses should be transformed to the principal stresses before applying these failure criteria.

Note:

1. Whether a material is *brittle* or *ductile* could be a subjective guess, and often depends on temperature, strain levels, and other environmental conditions. However, a *5% elongation* criterion at break is a reasonable dividing line. Materials with a larger elongation can be considered ductile and those with a lower value brittle.
   Another distinction is a brittle material's compression strength is usually significantly larger than its tensile strength.

2. All popular failure criteria rely on only a handful of basic tests (such as uniaxial tensile and/or compression strength), even though most machine parts and structural members are typically subjected to multi-axial loading. This disparity is usually driven by cost, since complete multi-axial failure testing requires extensive, complicated, and expensive tests.

Non Stress-Based Criteria

The success of all machine parts and structural members are not necessarily determined by their strength. Whether a part succeeds or fails may depend on other factors, such as stiffness, vibrational characteristics, fatigue resistance, and/or creep resistance.

For example, the automobile industry has endeavored many years to increase the rigidity of passenger cages and install additional safety equipment. The bicycle industry continues to decrease the weight and increase the stiffness of bicycles to enhance their performance.

In civil engineering, a patio deck only needs to be strong enough to carry the weight of several people. However, a design based on the "strong enough" precept will often result a bouncy deck that most people will find objectionable. Rather, the *stiffness* of the deck determines the success of the design.

Many factors, in addition to stress, may contribute to the design requirements of a part. Together, these requirements are intended to increase the sense of security, safety, and quality of service of the part.

Failure Criteria for Ductile Materials

Maximum Shear Stress Criterion

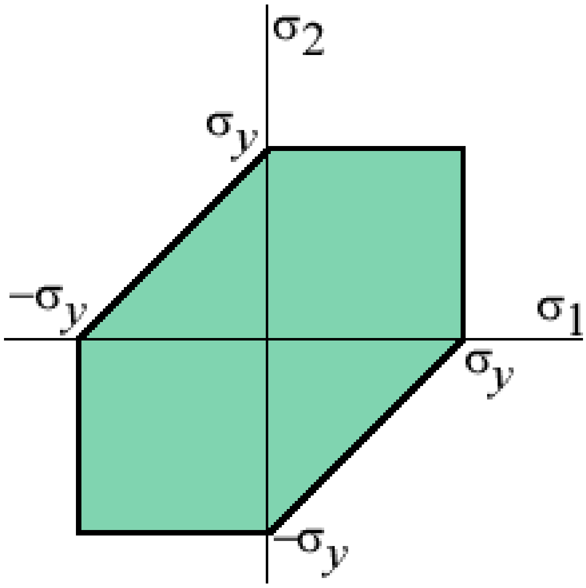
The maximum shear stress criterion, also known as Tresca's or Guest's criterion, is often used to predict the yielding of ductile materials.

Yield in ductile materials is usually caused by the *slippage* of crystal planes along the maximum shear stress surface. Therefore, a given point in the body is considered safe as long as the maximum shear stress at that point is under the yield shear stress σ_y obtained from a uniaxial tensile test.

With respect to 2D stress, the maximum shear stress is related to the difference in the two principal stresses (see Mohr's Circle). Therefore, the criterion requires the principal stress difference, along with the principal stresses themselves, to be less than the yield shear stress,

$$|\sigma_1| \leq \sigma_y, \quad |\sigma_2| \leq \sigma_y, \text{ and } |\sigma_1 - \sigma_2| \leq \sigma_y$$

Graphically, the maximum shear stress criterion requires that the two principal stresses be within the green zone indicated below,



Von Mises Criterion

The von Mises Criterion (1913), also known as the maximum distortion energy criterion, octahedral shear stress theory, or Maxwell-Huber-Hencky-von Mises theory, is often used to estimate the yield of ductile materials.

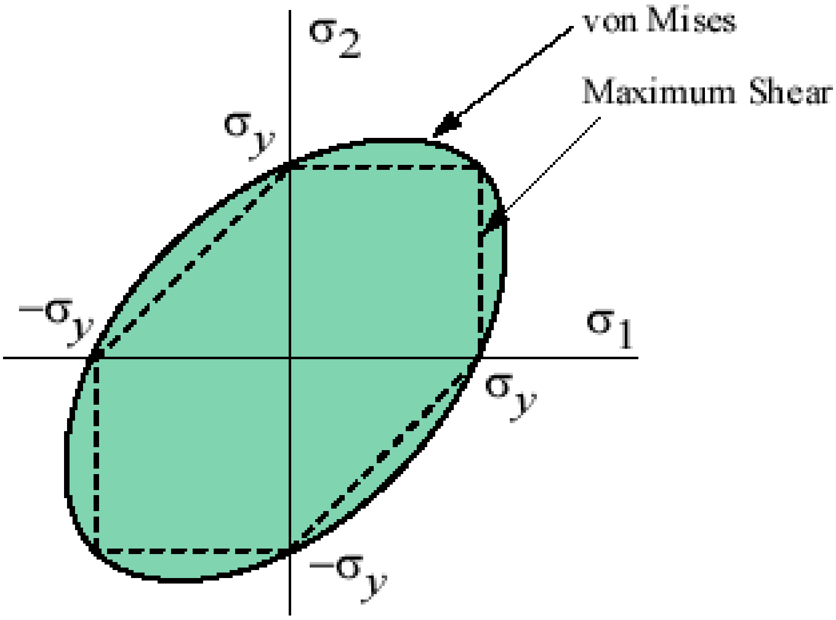
The von Mises criterion states that failure occurs when the energy of distortion reaches the same energy for yield/failure in uniaxial tension. Mathematically, this is expressed as,

$$\frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]\leq\sigma_y{}^2$$

In the cases of plane stress, $\sigma_3 = 0$. The von Mises criterion reduces to,

$$\sigma_1^2-\sigma_1\sigma_2+\sigma_2^2\leq\sigma_y^2$$

This equation represents a principal stress ellipse as illustrated in the following figure,



Also shown on the figure is the maximum shear stress criterion (dashed line). This theory is more conservative than the von Mises criterion since it lies inside the von Mises ellipse.

In addition to bounding the principal stresses to prevent ductile failure, the von Mises criterion also gives a reasonable estimation of fatigue failure, especially in cases of repeated tensile and tensile-shear loading.

Failure Criteria for Brittle Materials

Maximum Normal Stress Criterion

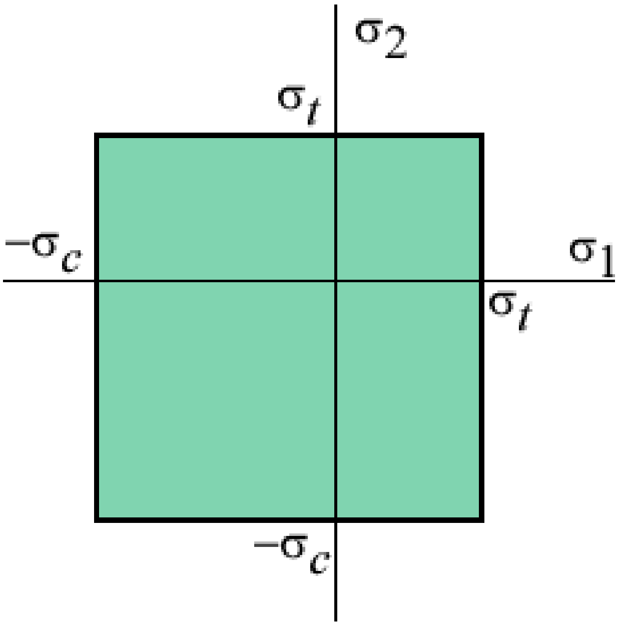
The maximum stress criterion, also known as the normal stress, Coulomb, or Rankine criterion, is often used to predict the failure of brittle materials.

The maximum stress criterion states that failure occurs when the maximum (normal) principal stress reaches either the *uniaxial* tension strength σ_t, or the *uniaxial* compression strength σ_c,

$$-\sigma_c < \{\sigma_1, \sigma_2\} < \sigma_t$$

where σ_1 and σ_2 are the principal stresses for 2D stress.

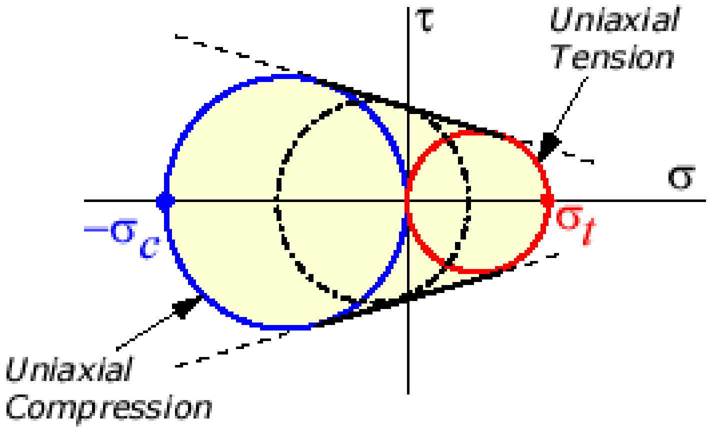
Graphically, the maximum stress criterion requires that the two principal stresses lie within the green zone depicted below,



Mohr's Theory

The Mohr Theory of Failure, also known as the Coulomb-Mohr criterion or internal-friction theory, is based on the famous Mohr's Circle. Mohr's theory is often used in predicting the failure of brittle materials, and is applied to cases of 2D stress.

Mohr's theory suggests that failure occurs when Mohr's Circle at a point in the body exceeds the envelope created by the two Mohr's circles for uniaxial tensile strength and uniaxial compression strength. This envelope is shown in the figure below,



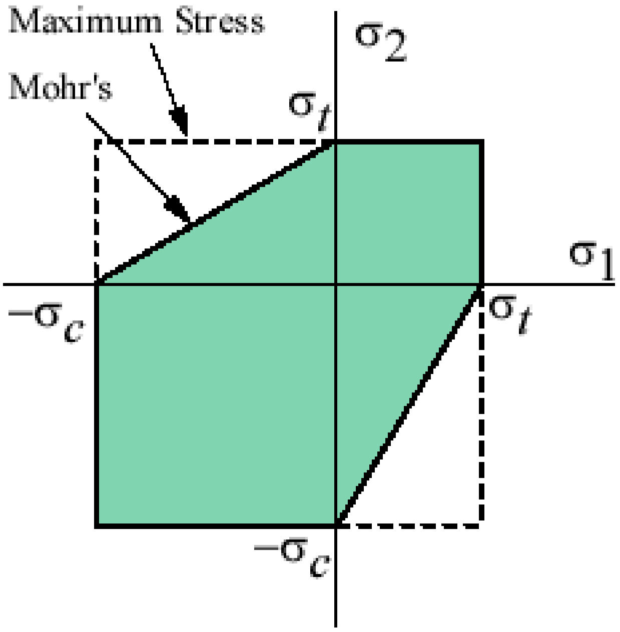
The left circle is for uniaxial compression at the limiting compression stress σ_c of the material. Likewise, the right circle is for uniaxial tension at the limiting tension stress σ_t.

The middle Mohr's Circle on the figure (dash-dot-dash line) represents the maximum allowable stress for an intermediate stress state.

All intermediate stress states fall into one of the four categories in the following table. Each case defines the maximum allowable values for the two principal stresses to avoid failure.

Case	Principal Stresses		Criterion requirements
1	Both in tension	$\sigma_1 > 0, \sigma_2 > 0$	$\sigma_1 < \sigma_t, \sigma_2 < \sigma_t$
2	Both in compression	$\sigma_1 < 0, \sigma_2 < 0$	$\sigma_1 > -\sigma_c, \sigma_2 > -\sigma_c$
3	σ_1 in tension, σ_2 in compression	$\sigma_1 > 0, \sigma_2 < 0$	$\frac{\sigma_1}{\sigma_t} + \frac{\sigma_2}{-\sigma_c} < 1$
4	σ_1 in compression, σ_2 in tension	$\sigma_1 < 0, \sigma_2 > 0$	$\frac{\sigma_1}{-\sigma_c} + \frac{\sigma_2}{\sigma_t} < 1$

Graphically, Mohr's theory requires that the two principal stresses lie within the green zone depicted below,



Also shown on the figure is the maximum stress criterion (dashed line). This theory is less conservative than Mohr's theory since it lies outside Mohr's boundary.

Techniques for Failure Prevention and Diagnosis

There exist a set of basic techniques for preventing failure in the design stage, and for diagnosing failure in manufacturing and later stages.

In the Design Stage

It is quite commonplace today for design engineers to verify design stresses with finite element (FEA) packages. This is fine and good when FEA is applied appropriately. However, the popularity of finite element analysis can condition engineers to look just for red spots in simulation output, without really understanding the essence or *funda* at play.

By following basic rules of thumb, such danger points can often be anticipated and avoided without total reliance on computer simulation.

Loading Points	Maximum stresses are often located at loading points, supports, joints, or maximum deflection points.
Stress Concentrations	Stress concentrations are usually located near corners, holes, crack tips, boundaries, between layers, and where cross-section areas change rapidly. Sound design avoids rapid changes in material or geometrical properties. For example, when a large hole is removed from a structure, a reinforcement composed of generally no less than the material removed should be added around the opening.
Safety Factors	The addition of safety factors to designs allow engineers to reduce sensitivity to manufacturing defects and to compensate for stress prediction limitations.

In Post-Manufacturing Stages

Despite the best efforts of design and manufacturing engineers, unanticipated failure may occur in parts after design and manufacturing. In order for projects to succeed, these failures must be diagnosed and resolved quickly and effectively. Often, the failure is caused by a singular factor, rather than an involved collection of factors.

Such failures may be caught early in initial quality assurance testing, or later after the part is delivered to the customer.

Induced Stress Concentrations	Stress concentrations may be induced by inadequate manufacturing processes. For example, initial surface imperfections can result from sloppy machining processes. Manufacturing defects such as size mismatches and improper fastener application can lead to residual stresses and even cracks, both strong stress concentrations.
Damage and Exposure	Damages during service life can lead a part to failure. Damages such as cracks, debonding, and delamination can result from unanticipated resonant vibrations and impacts that exceed the design loads. Reduction in strength can result from exposure to UV lights and chemical corrosion.
Fatigue and Creep	Fatigue or creep can lead a part to failure. For example, unanticipated fatigue can result from repeated mechanical or thermal loading.

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13 december 2001 13:01

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Meet sinds ...	1 mei 1999
Totaal aantal pageviews tot nu toe	8058
Drukste dag tot nu toe	27 september 2001
Pageviews	67

Prognose voor vandaag

Gemiddeld komt 41 procent van het dagelijkse bezoek vóór 13:01. Op grond van het bezoekersaantal van 10 van vandaag tot nu toe kan uw site vandaag op 24 pageviews (+/- 4) uitkomen.

Laatste 10 bezoekers

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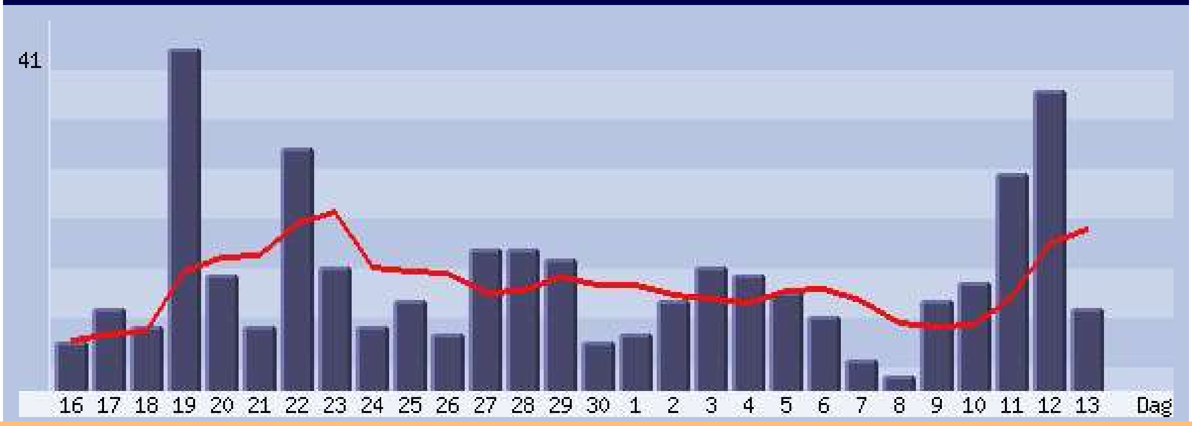
2.	13 december	03:05	Nottingham, Verenigd Koninkrijk (djanogly.notts.)
3.	13 december	07:11	ALLTEL, Verenigde Staten
4.	13 december	08:25	TU Delft, Delft, Nederland
5.	13 december	08:27	TU Delft, Delft, Nederland
6.	13 december	08:32	TU Delft, Delft, Nederland
7.	13 december	10:09	TU Delft, Delft, Nederland
8.	13 december	10:58	TU Delft, Delft, Nederland
9.	13 december	12:26	Chinanet, China
10.	13 december	13:00	TU Delft, Delft, Nederland

Pageviews per dag



Pageviews per dag	
16 november 2001	6
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