Longitudinal Static Stability

Some definitions

$$C_m = \frac{M}{\frac{1}{2}\rho V^2 S \bar{c}}$$

pitching moment without dimensions
(so without influence of ρ, V and S)
it is a ‘shape’ parameter which
varies with the angle of attack.
Note the chord c in the denominator because of the unit Nm!

$$C_L = \frac{L}{\frac{1}{2}\rho V^2 S}$$

For the wing+aircraft we use the
surface area of the wing *S*!

$$C_{L_H} = \frac{L_H}{\frac{1}{2}\rho V^2 S_H}$$

For the tail we use the surface of the tail: S_H !

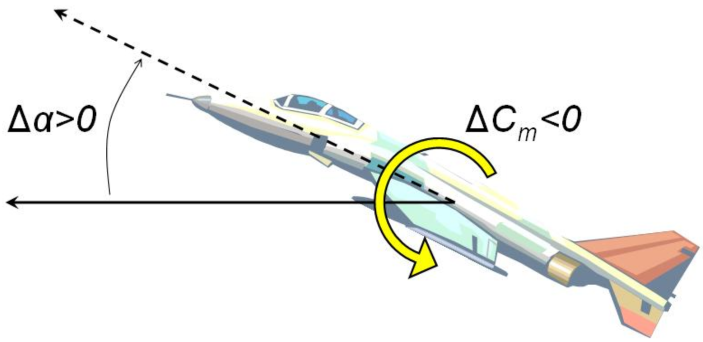
Definition of aerodynamic center of a wing:

> The aerodynamic center (a.c.) is the point around which the moment does not change when the angle of attack changes. We can therefore use Cm_{ac} as a constant moment for all angles of attack.

The aerodynamic center usually lies around a quarter chord from the leading edge.

Criterium for longitudinal static stability (see also Anderson § 7.5):

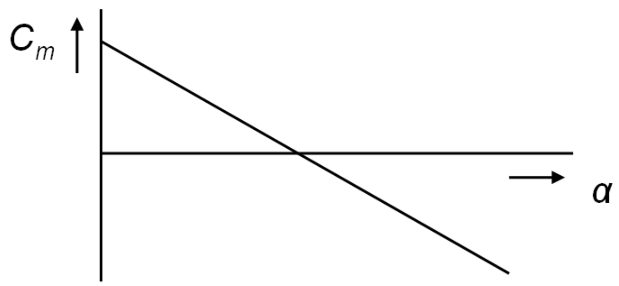
We will look at the consequences of the position of the center of gravity, the wing and the tail for longitudinal static stability.



For stability, we need a *negative* change of the pitching moment if there is a *positive* change of the angle of attack (and vice versa), so:

$$\frac{\Delta C_m}{\Delta\alpha} = \frac{<0}{>0} < 0 \Rightarrow \frac{\Delta C_m}{\Delta\alpha} < 0$$

Graphically this means $C_m(\alpha)$ has to be descending:



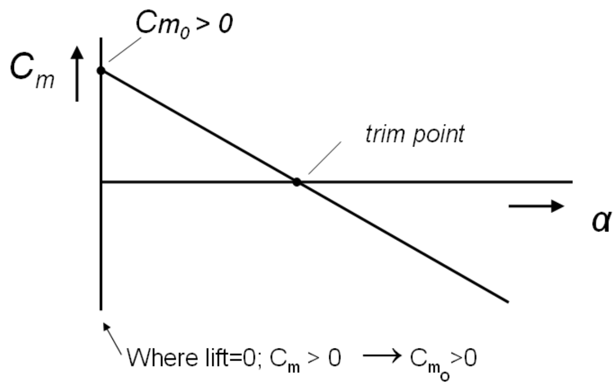
For small changes we write:

$$\frac{dC_m}{d\alpha} < 0$$

We also write this as:

$$C_{m_\alpha} < 0$$

When $C_m(\alpha)$ is descending, the Cm_0 has to be positive to have a trim point where $C_m = 0$ and there is an equilibrium:



So two conditions for stability:

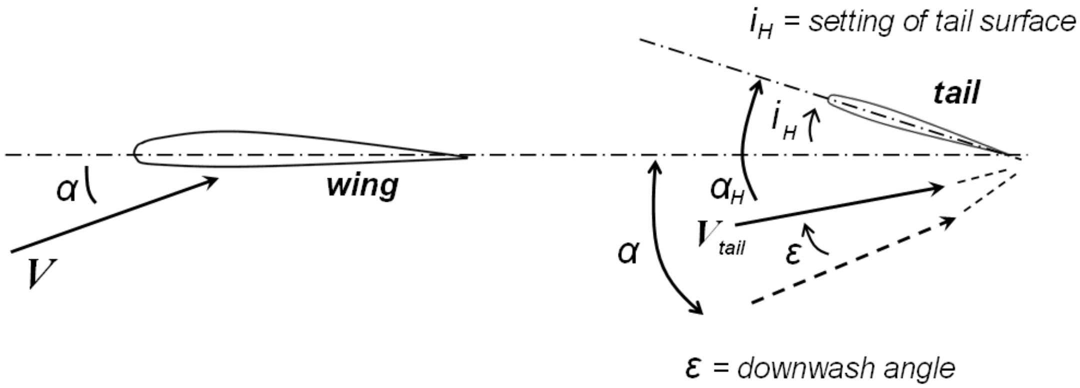
1) $Cm_0 > 0$; if lift = 0; pitching moment has to be positive (nose up)

2) $\frac{dC_m}{d\alpha} < 0$ (or $C_{m_\alpha} < 0$); pitching moment has to become more negative when the angle of attack increases

Condition 1 is easy to check. But what is the consequence of condition 2? For this we have to study what happens when the angle of attack changes. Therefore we have to look at the derivatives to the angle of attack and then use this to predict what the change in pitching moment will be. For this we will first look at the tail and then look at the effect on the whole configuration.

The horizontal tail surface

The angle of attack of the horizontal tail α_H:



The wing has pushed air down and decreased the angle of attack with ε, the downwash angle.

The relation between α and α_H now becomes:

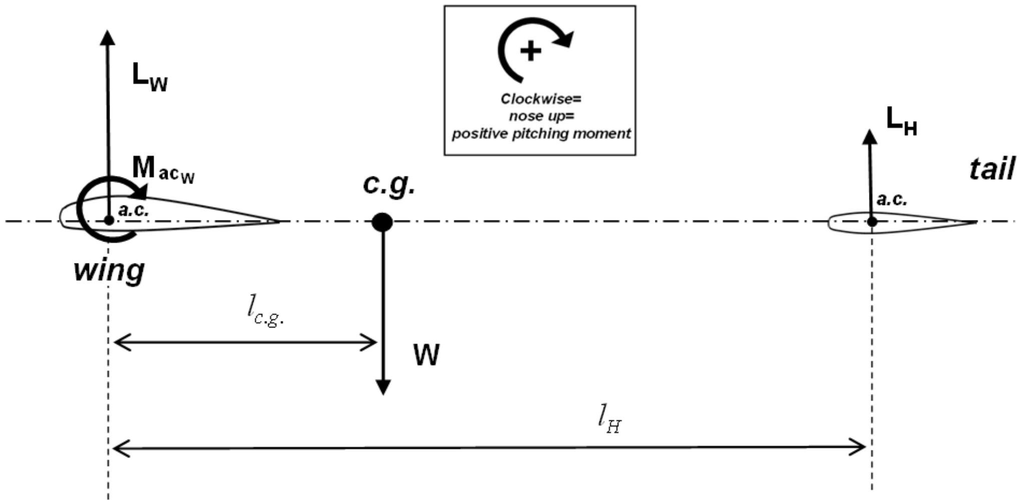
$$\alpha_H = \alpha - \varepsilon + i_H$$

So the change in α_H due to a change in angle of attack α now can be calculated:

$$\frac{\Delta\alpha_H}{\Delta\alpha} \quad = \quad \frac{d\alpha_H}{d\alpha} \quad = \quad \frac{d}{d\alpha}(\alpha - \varepsilon + i_H) \quad = \quad 1 - \frac{d\varepsilon}{d\alpha}$$

The term $\frac{d\varepsilon}{d\alpha}$ basically means: the change in downwash due to the change in angle of attack. Typical values are around 0.10 for tails that do not have a T-configuration.

Calculate change in pitching moment due to a change in angle of attack α: the Cm_α



In this figure L_H is drawn upward. In reality it could just as well be pointed downwards, but the sign convention is that the lift is positive upward, and therefore we draw it like this.

Moment around center of gravity:

Pitching moment:

$$M \quad = \quad \Sigma M \quad = \quad M_{ac_W} + L_W \cdot l_{cg} - L_H (l_H - l_{cg})$$

With:

M_{ac_W} = Moment of wing around aerodynamic center, so constant for all α !

$L_W \cdot l_{cg}$ = Wing (and fuselage) lift force times arm relative to c.g., positive (clockwise)

$-L_H (l_H - l_{cg})$ = Moment of tail lift force rel. to c.g., is negative (counter clockwise)

We can simplify the moment equation by using the total lift force L:

$L = L_W + L_H$

$$M = \quad M_{ac_W} + L_W \cdot l_{cg} - L_H \cdot l_H + L_H \cdot l_{cg}$$

$$= \quad M_{ac_W} + \underbrace{(L_W + L_H)}_{=L} \cdot l_{cg} - L_H \cdot l_H$$

$$= \quad M_{ac_W} + L \cdot l_{cg} - L_H \cdot l_H$$

Now make this pitching moment dimensionless with $\frac{1}{2}\rho V^2 Sc$:

$$\frac{M}{\frac{1}{2}\rho V^2 Sc} = \frac{M_{ac_W}}{\frac{1}{2}\rho V^2 Sc} + \frac{L \cdot l_{cg}}{\frac{1}{2}\rho V^2 Sc} - \frac{L_H \cdot l_H}{\frac{1}{2}\rho V^2 Sc}$$

We can now simplify this enormously by using the definitions in the start (Note how the difference in moments and forces all work out alright). One complication however is that the lift coefficient of the tail surface is defined using the area of the tail surface, so S_H instead of S:

$$L_H = C_{L_H} \cdot \frac{1}{2}\rho V^2 S_H$$

Using all this transforms the moment equation into its dimensionless form:

$$C_m = C_{m\,ac\,W} + C_L \cdot \frac{l_{cg}}{c} - C_{L_H} \cdot \frac{\frac{1}{2}\rho V^2 S_H \cdot l_H}{\frac{1}{2}\rho V^2 S \cdot c}$$

$$C_m = C_{m\,ac\,W} + C_L \cdot \frac{l_{cg}}{c} - C_{L_H} \cdot \frac{S_H \cdot l_H}{S \cdot c}$$

This we now call $V_H = \frac{S_H \cdot l_H}{S \cdot c}$

$$C_m = C_{m\,ac\,W} + C_L \cdot \frac{l_{cg}}{c} - C_{L_H} \cdot V_H$$

The ratio V_H (the tail area times the arm divided by the wing area times the chord) is called the tail volume V_H (even though it is dimensionless).

$$V_H = \frac{S_H \cdot l_H}{S \cdot c}$$

We want to know $\frac{dC_m}{d\alpha}$; so differentiate to α :

$$\frac{dC_m}{d\alpha} = \frac{dC_{m\,ac\,w}}{d\alpha} + \frac{dC_L}{d\alpha} \cdot \frac{l_{cg}}{c} - \frac{dC_{L_H}}{d\alpha} \cdot V_H$$

A number of observations can be made:

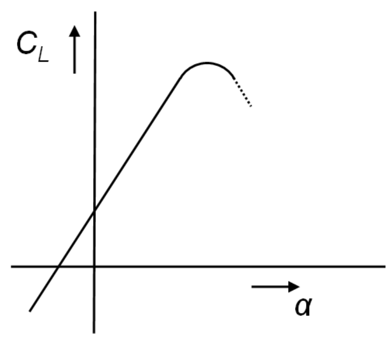
$\frac{dC_{m\,ac\,w}}{d\alpha} = 0$ The moment around the aerodynamic center does not change when the angle of attack α changes. So this term is zero by definition and disappears.

Note how the tail volume V_H is independent of the angle of attack, and so it can be treated as constant.:

What we're left with is this:

$$\frac{dC_m}{d\alpha} = \frac{dC_L}{d\alpha} \cdot \frac{l_{cg}}{c} - \frac{dC_{L_H}}{d\alpha} \cdot V_H$$

The $\frac{dC_L}{d\alpha}$ is simply a characteristic of the aircraft shape: the steepness of the C_L-α curve:



We normally should have similar data for the tail airfoil, however then the the angle of attack of the tail surface α_H is on the x-axis. So we know the $\frac{dC_{L_H}}{d\alpha_H}$ and not the $\frac{dC_{L_H}}{d\alpha}$.

But we have seen:

$$\alpha_H = \alpha - \varepsilon + i_H \qquad \text{and therefore:} \qquad \frac{d\alpha_H}{d\alpha} = 1 - \frac{d\varepsilon}{d\alpha}$$

So:

$$\frac{dC_{L_H}}{d\alpha} = \frac{dC_{L_H}}{d\alpha_H} \cdot \frac{d\alpha_H}{d\alpha} = \frac{dC_{L_H}}{d\alpha_H} \cdot \left(1 - \frac{d\varepsilon}{d\alpha}\right)$$

So we substitute $\frac{dC_{L_H}}{d\alpha}$ with $\frac{dC_{L_H}}{d\alpha_H}\cdot\left(1-\frac{d\varepsilon}{d\alpha}\right)$:

$$\frac{dC_m}{d\alpha} = \frac{dC_L}{d\alpha}\cdot\frac{l_{cg}}{c} - \frac{dC_{L_H}}{d\alpha_H}\cdot V_H\cdot\left(1-\frac{d\varepsilon}{d\alpha}\right)$$

Both $\frac{dC_L}{d\alpha}$ and $\frac{dC_{L_H}}{d\alpha_H}$ are constants for any given shape, and indicate the steepness of the C_L-α curve. They are also written as a and a_t , where the index *t* refers to the tail. When also writing $\frac{dCm}{d\alpha}$ as C_{m_α} we can write the last equation above as follows:

$$C_{m_\alpha} = a\cdot\frac{l_{cg}}{c} - a_t\cdot V_H\cdot\left(1-\frac{d\varepsilon}{d\alpha}\right)$$

And we concluded for static stability that this C_{m_α} should be less than zero, so the aircraft will be stable if:

$$a\cdot\frac{l_{cg}}{c} - a_t\cdot V_H\cdot\left(1-\frac{d\varepsilon}{d\alpha}\right) < 0$$

So for the tail this means:

$$a_t\cdot V_H\cdot\left(1-\frac{d\varepsilon}{d\alpha}\right) > a\cdot\frac{l_{cg}}{c}$$

From this relation we can not only conclude the following:

- A larger tail will contribute to static stability
- A longer distance between tail and wing will contribute to stability
- A center of gravity that is just after the wing or even before the wing contributes to stability (forward cg => more stable, aft c.g. less stable)

From this equation we can, for a given aircraft configuration, calculate what c.g. position is just on the edge of stability. This point is called the ***neutral point*** . If the c.g. is before this point the aircraft will be stable, if the c.g. is after this point the aircraft will be unstable. We can calculate this by solving the borderline case between stability and instability.

So at neutral point: $l_{cg} = l_{np}$ and $C_{m_\alpha} = 0$:

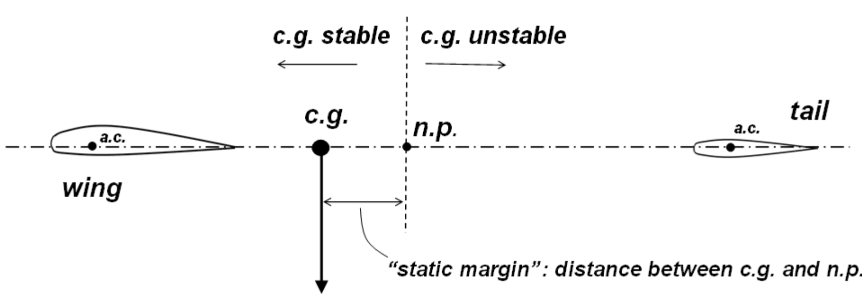
$$a \cdot \frac{l_{np}}{c} \quad - \quad a_t \cdot V_H \cdot \left(1 - \frac{d\varepsilon}{d\alpha}\right) \quad = \quad 0$$

$$a \cdot \frac{l_{np}}{c} \quad = \quad a_t \cdot V_H \cdot \left(1 - \frac{d\varepsilon}{d\alpha}\right)$$

The neutral point then is:

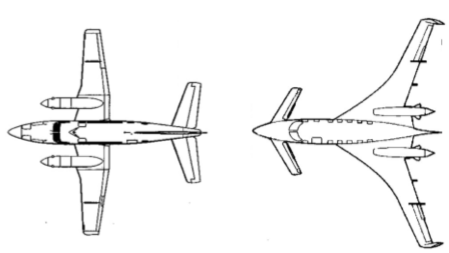
$$\boxed{\frac{l_{np}}{c} \quad = \quad \frac{a_t}{a} \cdot V_H \cdot \left(1 - \frac{d\varepsilon}{d\alpha}\right) \quad \text{with} \quad V_H = \frac{S_H \cdot l_H}{S \cdot c}}$$

The distance between the neutral point and the center of gravity is called the ***static margin***:



Exercise
A similar analysis can be done for a canard plane. Would a forward c.g. there also be a benefit or would everything reverse? We leave that as an exercise for the reader.

Normal configuration (Beech 99)



Canard configuration (Beech Rutan Starship 2000)