## Appendix A

## Discrete Fourier Transform and Sampling Theorem.

In this appendix the discrete Fourier Transform is derived, starting from the Continuous Fourier Transform. As part of the derivation, the sampling theorem or Nyquist criterion is obtained.

## Derivation

The continuous integrals are nearly always used in deriving any mathematical results, but, in performing transforms on data, the integrals are always replaced by summations. The continuous signal a(t) becomes the discrete signal, or time series,  $a_k$ , in which k is an integer, and the sampling has taken place at regular intervals  $k\Delta t$ . Thus the discrete signal corresponds exactly to the continuous signal at times

$$t = k\Delta t. \tag{A.1}$$

Consider the inverse Fourier Transform (1.2) at the discrete times  $k\Delta t$ :

$$a_k = \int_{-\infty}^{\infty} A(f) \exp(2\pi i f k \Delta t) df \qquad k = ..., -2, -1, 0, 1, 2, ...$$
(A.2)

where  $a_k$  stands for the fact that time is now discrete so:

$$a_k = a(t)$$
, when  $t = k\Delta t$   $k = ..., -2, -1, 0, 1, 2, ...$  (A.3)

An important aspect of the integrand is that the exponential function  $\exp(2\pi i f k \Delta t)$  is *periodic* with a period of  $1/\Delta t$ , i.e.,

$$f = F$$
 :  $\exp(2\pi i F k \Delta t)$ 

$$f = F + 1/\Delta t : \exp(2\pi i (F + 1/\Delta t) k \Delta t)$$
$$: = \exp(2\pi i F k \Delta t + 2\pi i k)$$
$$: = \exp(2\pi i F k \Delta t)$$

since  $\exp(2\pi i k) = 1$ , and therefore the exponential at f = F is identical to the exponential at  $f = F + 1/\Delta t$ . Therefore, for all k, the integral above may be replaced by an infinite sum of pieces of the integral with period  $1/\Delta t$ :

$$a_{k} = \left(\dots + \int_{-\frac{5}{2\Delta t}}^{-\frac{3}{2\Delta t}} + \int_{-\frac{3}{2\Delta t}}^{-\frac{1}{2\Delta t}} + \int_{+\frac{1}{2\Delta t}}^{+\frac{1}{2\Delta t}} + \int_{+\frac{1}{2\Delta t}}^{+\frac{3}{2\Delta t}} + \dots \right) A(f) \exp(2\pi i f k \Delta t) df$$
$$= \sum_{m=-\infty}^{\infty} \int_{\frac{m}{\Delta t} - \frac{1}{2\Delta t}}^{\frac{m}{\Delta t} + \frac{1}{2\Delta t}} A(f) \exp(2\pi i f k \Delta t) df$$
(A.4)

In order to get the bounds of the integral from  $-1/(2\Delta t)$  to  $+1/(2\Delta t)$ , we change to the variable  $f' = f - m/\Delta t$  to yield:

$$a_k = \sum_{m=-\infty}^{\infty} \int_{-\frac{1}{2\Delta t}}^{\frac{1}{2\Delta t}} A(f' + \frac{m}{\Delta t}) \exp(2\pi i \{f' + \frac{m}{\Delta t}\} k \Delta t) df'$$
(A.5)

Changing the order of the integration and summation, and noting that the exponential becomes periodic (so  $\exp(2\pi i m k) = 1$ ), this becomes

$$a_k = \int_{-\frac{1}{2\Delta t}}^{\frac{1}{2\Delta t}} \left[ \sum_{m=-\infty}^{\infty} A(f' + \frac{m}{\Delta t}) \right] \exp(2\pi i f' k \Delta t) df'$$
(A.6)

The Fourier transform of the discrete time series is thus

$$a_k = \int_{-\frac{1}{2\Delta t}}^{\frac{1}{2\Delta t}} A_D(f') \exp(2\pi i f' k \Delta t) df' \qquad k = \dots, -2, -1, 0, 1, 2, \dots$$
(A.7)

provided

$$A_D(f') = \sum_{m=-\infty}^{\infty} A(f' + \frac{m}{\Delta t})$$
(A.8)

So this is an infinite series of shifted spectra as shown in figure 1.3(b) in the main text. The discretisation of the time signal forces the Fourier transform to become periodic. In the discrete case we get the same spectrum as the continuous case if we only take the period from  $-1/(2\Delta t)$  to  $+1/(2\Delta t)$ , and else be zero; the signal must be *band-limited*. So

this means means that the discrete signal must be zero for frequencies  $|f| \ge f_N = 1/(2\Delta t)$ . The frequency  $f_N$  is known as the Nyquist frequency.

Let us now look at the other integral of the continuous Fourier-transform pair, i.e. (1.1). We evaluate the integral by discretisation, so then we obtain for  $A_D(f)$ :

$$A_D(f) = \Delta t \sum_{k=-\infty}^{\infty} a_k \exp(-2\pi i f k \Delta t)$$
(A.9)

In practice the number of samples is always finite since we measure only for a certain time. Say we have N samples. Then we obtain the pair:

$$A_D(f) = \Delta t \sum_{k=0}^{N-1} a_k \exp(-2\pi i f k \Delta t)$$
(A.10)

$$a_k = \int_{-\frac{1}{2\Delta t}}^{\frac{1}{2\Delta t}} A_D(f) \exp(2\pi i f k \Delta t) df \qquad k = 0, 1, 2, ..., N - 1$$
(A.11)

This is the transform pair for continuous frequency and discrete time. Notice that the integral runs from  $-1/2\Delta t$  to  $+1/2\Delta t$ , i.e. one period where one spectrum of  $A_D(f)$  is present.

As said above, the values for frequencies above the Nyquist frequency must be set to zero. Equivalently, we can say that if there is no information in the continuous time signal a(t) at frequencies above  $f_N$ , the maximum sampling interval  $\Delta t$  is

$$\Delta t_{max} = \frac{1}{2f_N} \tag{A.12}$$

This is the sampling theorem.

In practice the number samples in a time series is always finite. We wish to find the discrete Fourier transform of a finite length sequence. We approach the problem by dividing the definite integral (A.7) into the sum of N pieces of equal frequency interval  $\Delta f$ . Because  $A_D(f)$  is periodic, with period  $1/\Delta t$ , we may first rewrite the integral with different limits, but with the same frequency interval:

$$a_k = \int_0^{\frac{1}{\Delta t}} A_D(f) \exp(2\pi i f k \Delta t) df \qquad k = 0, 1, 2, ..., N - 1$$
(A.13)

Writing the integral as a summation, we obtain

$$a_k = \Delta f \sum_{n=0}^{N-1} A_n \exp(2\pi i n \Delta f k \Delta t) \qquad k = 0, 1, 2, ..., N-1$$
(A.14)

where

$$A_n = A_D(f),$$
 when  $f = n\Delta f.$  (A.15)

We now notice that the series  $a_k$  is periodic with period N:

$$a_{k+N} = \Delta f \sum_{n=0}^{N-1} A_n \exp(2\pi i n \Delta f \{k+N\} \Delta t)$$

$$= \Delta f \sum_{n=0}^{N-1} A_n \exp(2\pi i n \Delta f k \Delta t + 2\pi i n \Delta f N \Delta t)$$

$$= \Delta f \sum_{n=0}^{N-1} A_n \exp(2\pi i n \Delta f k \Delta t)$$

$$= a_k \qquad (A.16)$$

since  $N\Delta f = 1/\Delta t$  and so  $\exp(2\pi i n) = 1$ . Thus we arrive at the following discrete Fourier transform pair for a finite-length time series

$$A_n = \Delta t \sum_{k=0}^{N-1} a_k \exp(-2\pi i n k/N) \qquad n = 0, 1, 2, ..., N-1$$
(A.17)

$$a_k = \Delta f \sum_{n=0}^{N-1} A_n \exp(2\pi i n k/N) \qquad k = 0, 1, 2, ..., N-1$$
(A.18)

These two equations are the final discrete-time and discrete-frequency Fourier transform pair.

There needs to be some caution with applying these transforms. We have used both for the frequencies and the times N samples as if we had a choice. But in the frequency domain, we have negative and positive frequencies, while in the time domain we only have samples for positive times. Therefore, when transforming to the frequency domain, we must have enough space allocated for the negative frequencies. So we must always add zeroes to the time series, as many as there are (non-zero) samples.