Appendix B

Correlation

In this appendix the process of correlation is defined, and its Fourier counterpart is derived.

Definitions and derivations

Here we shall focus on correlation, specifically auto-correlation and cross-correlation, and relate it to the Fourier transformation. As we shall see, correlation can be written as a convolution, and in the frequency domain it is a simple operation.

Let us first define the time reversal of a signal $a(t)$:

$$a_{rev}(t) = a^*(-t) \tag{B.1}$$

where $b(t)$ is allowed to be complex and the asterisk * as superscript denotes complex conjugate. Normally of course we deal with real time signals. However, by allowing these signals to be complex it is easier to see their symmetry properties. When we apply a Fourier transformation to $a(t)$, and take the complex conjugate of each side, we obtain:

$$\begin{aligned}
A^*(f) &= \left[\int_{-\infty}^{\infty} a(t)\exp(-2\pi i f t)dt\right]^* \\
&= \int_{-\infty}^{\infty} a^*(t)\exp(2\pi i f t)dt \\
&= \int_{-\infty}^{\infty} a^*(-t')\exp(-2\pi i f t')dt' \qquad \text{(B.2)} \\
&= \mathcal{F}_t[a^*(-t')] \\
&= \mathcal{F}_t[a_{rev}(t')]
\end{aligned}$$

which is the Fourier transform of $a_{rev}(t)$.

Now, the autocorrelation of $a(t)$ is defined as

$$\phi_{aa}(\tau) = \int_{-\infty}^{\infty} a(t)a^*(t-\tau)\,dt \tag{B.3}$$

Using the concept of the time reverse, equation (B.1), equation (B.3) can be written as

$$\phi_{aa}(\tau) = \int_{-\infty}^{\infty} a(t)a_{rev}(\tau-t)\,dt \tag{B.4}$$

That is, the autocorrelation of a signal is the convolution of the signal with its time reverse. Equation (B.4) may be transformed to the frequency domain. The convolution becomes a multiplication and the result is

$$\begin{aligned}\Phi_{aa}(f) &= A(f)A^*(f)\\ &= |A(f)|^2 \qquad \text{(B.5)}\end{aligned}$$

The function $\Phi_{aa}(f)$ is the power spectrum of $a(t)$ and $|A(f)|$ is the amplitude spectrum of $a(t)$. Both the power spectrum and the amplitude spectrum are real. The power spectrum is the Fourier transform of the autocorrelation function.

The cross-correlation function $\phi_{ab}(t)$ is defined as

$$\phi_{ab}(t) = \int_{-\infty}^{\infty} a(\tau)b^*(\tau-t)\,d\tau \tag{B.6}$$

which can be recognized as the convolution of $a(t)$ with the time reverse of $b(t)$:

$$\phi_{ab}(t) = a(t) * b_{rev}(t) \tag{B.7}$$

In the frequency domain equation (B.7) becomes

$$\Phi_{ab}(f) = A(f)B^*(f) \tag{B.8}$$

The function $\Phi_{ab}(f)$ is known as the cross-spectrum. Note that the cross-correlation function and the cross-spectrum do not, in general, exhibit any symmetry. Also, it can be seen that the correlation of $a(t)$ with $b(t)$ is not necessarily the same as the correlation of $b(t)$ with $a(t)$, that is:

$$\phi_{ab}(t) \neq \phi_{ba}(t) \tag{B.9}$$

and

$$\Phi_{ab}(f) \neq \Phi_{ba}(f) \tag{B.10}$$