Appendix C

Derivation of 1-D wave equation

In this appendix the one-dimensional wave equation for an acoustic medium is derived, starting from the conservation of mass and conservation of momentum (Newton's Second Law).

Derivation

Here we will derive the wave equation for homogeneous media, using the conservation of momentum (Newton's second law) and the conservation of mass. In this derivation, we will follow (Berkhout 1984: appendix C), where we consider a single cube of mass when it is subdued to a seismic disturbance (see figure (C.1)). Such a cube has a volume ΔV with sides $\Delta x, \Delta y$ and Δz.

Conservation of mass gives us:

$$\Delta m(t_0) = \Delta m(t_0 + dt) \tag{C.1}$$

where Δm is the mass of the volume ΔV, and t denotes time. Using the density ρ, the conservation of mass can be written as:

$$\rho(t_0)\Delta V(t_0) = \rho(t_0 + dt)\Delta V(t_0 + dt) \tag{C.2}$$

Making this explicit:

$$\begin{aligned} \rho_0 \Delta V &= (\rho_0 + d\rho)(\Delta V + dV) \\ &= \rho_0 \Delta V + \rho_0 dV + \Delta V d\rho + d\rho dV \end{aligned} \tag{C.3}$$

Ignoring lower-order terms, i.e., $d\rho dV$, it follows that

$$\frac{d\rho}{\rho_0} = -\frac{dV}{\Delta V} \tag{C.4}$$

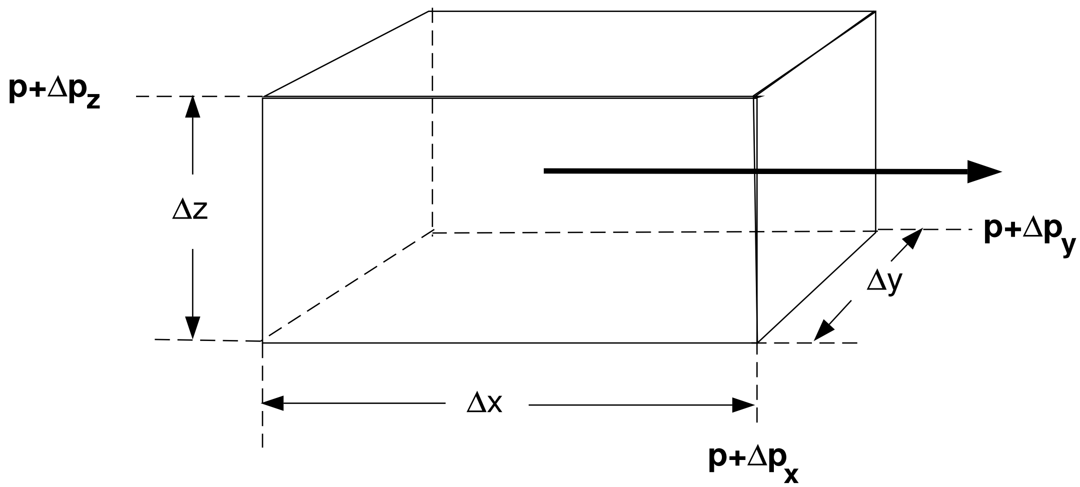

Figure C.1: A cube of mass, used for derivation of the wave equation.

We want to derive an equation with the pressure in it so we assume there is a linear relation between the pressure p and the density:

$$dp = \frac{K}{\rho_0} d\rho \tag{C.5}$$

where K is called the bulk modulus. Then, we can rewrite the above equation as:

$$dp = -K \frac{dV}{\Delta V} \tag{C.6}$$

which formulates Hooke's law. It shows that for a constant mass the pressure is linearly related to the relative volume change. Now we assume that the volume change is only in one direction (1-Dimensional). Then we have:

$$\begin{aligned} \frac{dV}{\Delta V} &= \frac{(\Delta x + dx)\Delta y \Delta z - \Delta x \Delta y \Delta z}{\Delta x \Delta y \Delta z} \\ &= \frac{dx}{\Delta x} \end{aligned} \tag{C.7}$$

Since dx is the difference between the displacements u_x at the sides, we can write:

$$\begin{aligned} dx &= (du_x)_{x+\Delta x} - (du_x)_x \\ &= \frac{\partial (du_x)}{\partial x} \Delta x = \frac{\partial (v_x)}{\partial x} dt \, \Delta x \end{aligned} \tag{C.8}$$

where v_x denotes the particle velocity in the x–direction. Substitute this in Hooke's law (equation C.6):

$$dp = -K\frac{\partial v_x}{\partial x}dt \tag{C.9}$$

or

$$\frac{1}{K}\frac{dp}{dt} = -\frac{\partial v_x}{\partial x} \tag{C.10}$$

The term on the left-hand side can be written as :

$$\frac{1}{K}\frac{dp}{dt} = \frac{1}{K}\left[\frac{\partial p}{\partial t} + v_x\frac{\partial p}{\partial x}\right] \tag{C.11}$$

Ignoring the second term in brackets (low-velocity approximation), we obtain for equation (C.10):

$$\frac{1}{K}\frac{\partial p}{\partial t} = -\frac{\partial v_x}{\partial x} \tag{C.12}$$

This is one basic relation needed for the derivation of the wave equation.

The other relation is obtained via Newton's law applied to the volume ΔV in the direction x, since we consider 1-Dimensional motion:

$$\Delta F_x = \Delta m\frac{dv_x}{dt} \tag{C.13}$$

where F is the force working on the element ΔV. Consider the force in the x–direction:

$$\begin{aligned}\Delta F_x &= -\Delta p_x \Delta S_x \\ &= -\left(\frac{\partial p}{\partial x}\Delta x + \frac{\partial p}{\partial t}dt\right)\Delta S_x \\ &\simeq -\frac{\partial p}{\partial x}\Delta V\end{aligned} \tag{C.14}$$

ignoring the term with dt since it is small, and ΔS_x is the surface in the x–direction, thus $\Delta y \Delta z$. Substituting in Newton's law (equation C.13), we obtain:

$$\begin{aligned}-\Delta V\frac{\partial p}{\partial x} &= \Delta m\frac{dv_x}{dt} \\ &= \rho\Delta V\frac{dv_x}{dt}\end{aligned} \tag{C.15}$$

We can write dv_x/dt as $\partial v_x/\partial t$; for this we use again the low-velocity approximation:

$$\frac{dv_x}{dt} = \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} \approx \frac{\partial v_x}{\partial t} \tag{C.16}$$

We divide by ΔV to give:

$$-\frac{\partial p}{\partial x} = \rho \frac{\partial v_x}{\partial t} \tag{C.17}$$

This equation is called the equation of motion.

We are now going to combine the conservation of mass and the equation of motion. Therefore we let the operator $(\partial/\partial x)$ work on the equation of motion:

$$\begin{aligned} -\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}\right) &= \frac{\partial}{\partial x}\left(\rho \frac{\partial v_x}{\partial t}\right) \\ &= \rho \frac{\partial}{\partial t}\left(\frac{\partial v_x}{\partial x}\right) \end{aligned} \tag{C.18}$$

for constant ρ. Substituting the result of the conservation of mass gives:

$$-\frac{\partial^2 p}{\partial x^2} = \rho \frac{\partial}{\partial t}\left(-\frac{1}{K}\frac{\partial p}{\partial t}\right) \tag{C.19}$$

Rewriting gives us the 1-Dimensional wave equation:

$$\frac{\partial p^2}{\partial x^2} - \frac{\rho}{K}\frac{\partial^2 p}{\partial t^2} = 0 \tag{C.20}$$

or

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0 \tag{C.21}$$

in which c can be seen as the velocity of sound, for which we have: $c = \sqrt{K/\rho}$.