

Chapter 1

Fourier analysis

In this chapter we review some basic results from signal analysis and processing. We shall not go into detail and assume the reader has some basic background in signal analysis and processing. As basis for signal analysis, we use the Fourier transform. We start with the continuous Fourier transformation. But in applications on the computer we deal with a discrete Fourier transformation, which introduces the special effect known as aliasing. We use the Fourier transformation for processes such as convolution, correlation and filtering. Some special attention is given to deconvolution, the inverse process of convolution, since it is needed in later chapters of these lecture notes.

1.1 Continuous Fourier Transform.

The Fourier transformation is a special case of an integral transformation: the transformation decomposes the signal in weighed basis functions. In our case these basis functions are the cosine and sine (remember $\exp(i\phi) = \cos(\phi) + i \sin(\phi)$). The result will be the weight functions of each basis function.

When we have a function which is a function of the independent variable t , then we can transform this independent variable to the independent variable frequency f via:

$$A(f) = \int_{-\infty}^{+\infty} a(t) \exp(-2\pi i f t) dt \quad (1.1)$$

In order to go back to the independent variable t , we define the inverse transform as:

$$a(t) = \int_{-\infty}^{+\infty} A(f) \exp(2\pi i f t) df \quad (1.2)$$

Notice that for the function in the time domain, we use lower-case letters, while for the frequency-domain expression the corresponding uppercase letters are used. $A(f)$ is called the *spectrum* of $a(t)$.

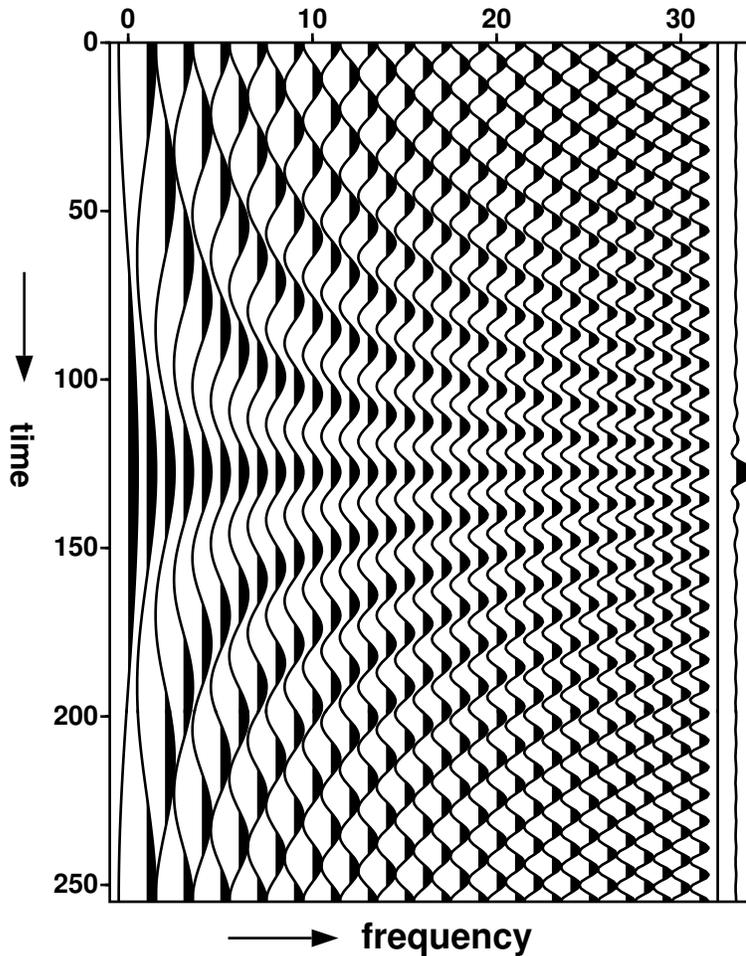


Figure 1.1: 32 cosine's with increasing frequencies; when added together, the rightmost trace is obtained.

A total signal can be built up via using cosines of different frequencies. If we would add many cosines together, we can make some specific signals. Let us consider Figure (1.1). We see 32 cosines with increasing frequencies. When we add the first 32 traces together, we obtain the trace as plotted on the right of the Figure: it has only one peak.

In this figure we used cosines with constant amplitudes, so the cosines were not shifted and the weights were just 1. We can shift the cosines, and we can vary the weights of the different frequency components, to obtain a certain signal. Actually, one can synthesize any signal by using shifted and weighted cosines. This is the Fourier Transform. As an example of this, consider Figure (1.2). On the leftmost trace, we see a time signal. When we look at the different components of this signal, we obtain the other traces. On the

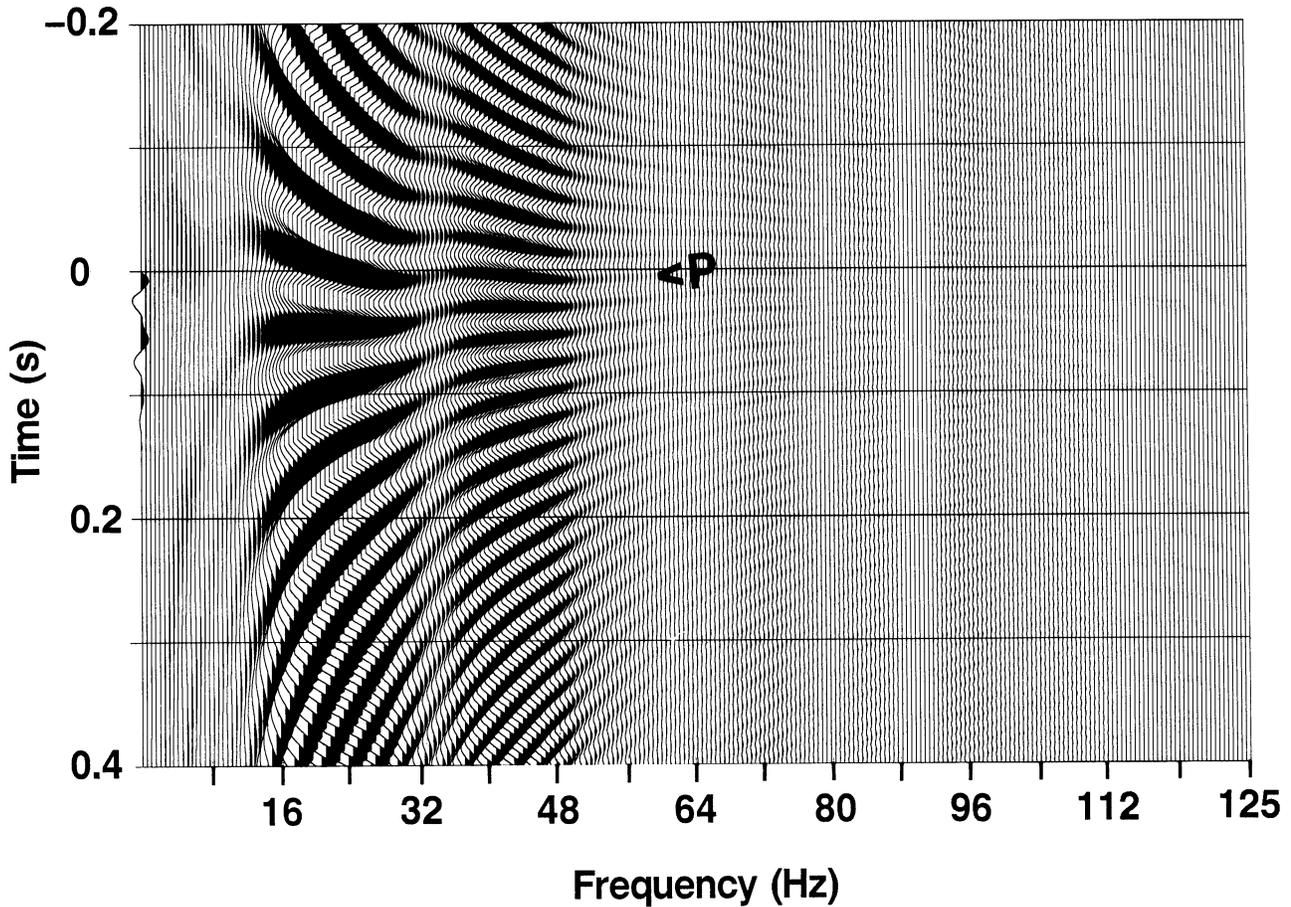


Figure 1.2: A time signal (leftmost trace) decomposed into shifted, weighted cosines (From: Yilmaz, 1987).

horizontal axis the frequency is given. First, it can be seen the weights of the frequency components is different, with the largest amplitudes around 24 Hz. Next, it can be seen that the different cosines are slightly time-shifted compared to its neighbour.

The amplitude of the components are obtained as the *amplitude* spectrum of the Fourier transformation of the signal. The shift of each cosine, is obtained via the *phase* spectrum of the Fourier transformation of the signal. The Fourier transform of a signal gives, in general, complex values for the frequency components. The amplitude gives the amplitude spectrum, and the phase of the complex value gives the phase spectrum.

1.2 Discrete Fourier Transform and Sampling Theorem.

The above continuous integrals are nearly always used in deriving any mathematical results, but, in performing transforms on data, the integrals are always replaced by summations. The continuous signal becomes a discrete signal. As is shown in appendix A, discretisation of the continuous Fourier integral *makes the spectrum periodic*:

$$A_{\text{Discrete}}(f) = \sum_{m=-\infty}^{\infty} A_{\text{Continuous}}\left(f + \frac{m}{\Delta t}\right) \quad (1.3)$$

So this is an infinite series of shifted spectra as shown in Figure (1.3)(b). The discretisation of the time signal forces the Fourier transform to become periodic. In the discrete case we get the same spectrum as the continuous case if we only take the interval from $-1/(2\Delta t)$ to $+1/(2\Delta t)$, and else be zero; the signal must be *band-limited*. So this means means that the discrete signal must be zero for frequencies $|f| \geq f_N = 1/(2\Delta t)$. The frequency f_N is known as the Nyquist frequency. Equivalently, we can say that if there is no information in the continuous time signal $a(t)$ at frequencies above f_N , the maximum sampling interval Δt is

$$\Delta t_{\text{max}} = \frac{1}{2f_N} \quad (1.4)$$

This is the sampling theorem. If we choose Δt too large, we undersample the signal and we get aliasing as shown in Figure 1.4. The original signal appears to have a lower frequency.

Another basic relation originates from the discretisation of the inverse Fourier transformation. The frequencies become discrete and therefore the *time* signal becomes periodic. The interval $1/\Delta t$ is divided up into N samples at Δf sampling so that we obtain the relation:

$$N\Delta t\Delta f = 1 \quad (1.5)$$

This relation can be used when we want increase the number of samples, for instance. In that case, if the time sampling remains the same, the frequency sampling decreases! This can be useful for interpolating data.

Finally, we obtain the pair:

$$A_n = \Delta t \sum_{k=0}^{N-1} a_k \exp(-2\pi ink/N) \quad n = 0, 1, 2, \dots, N-1 \quad (1.6)$$

$$a_k = \Delta f \sum_{n=0}^{N-1} A_n \exp(2\pi ink/N) \quad k = 0, 1, 2, \dots, N-1 \quad (1.7)$$

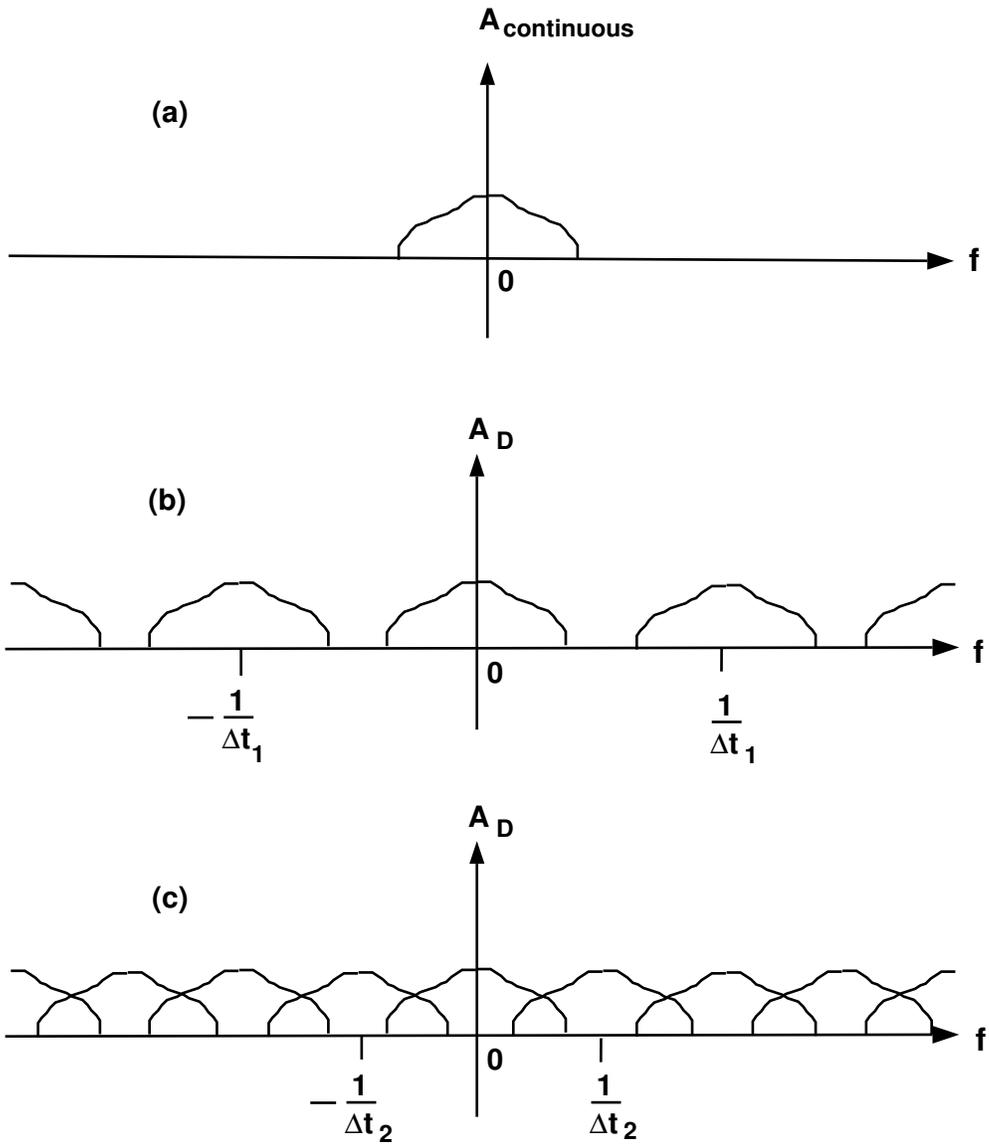


Figure 1.3: Effect of time-discretisation in frequency domain: (a) continuous spectrum; (b) properly time-sampled spectra giving rise to periodicity (period $1/\Delta t_1$); (c) too coarse time sampling Δt_2 such that spectra overlap (= aliasing in time domain).

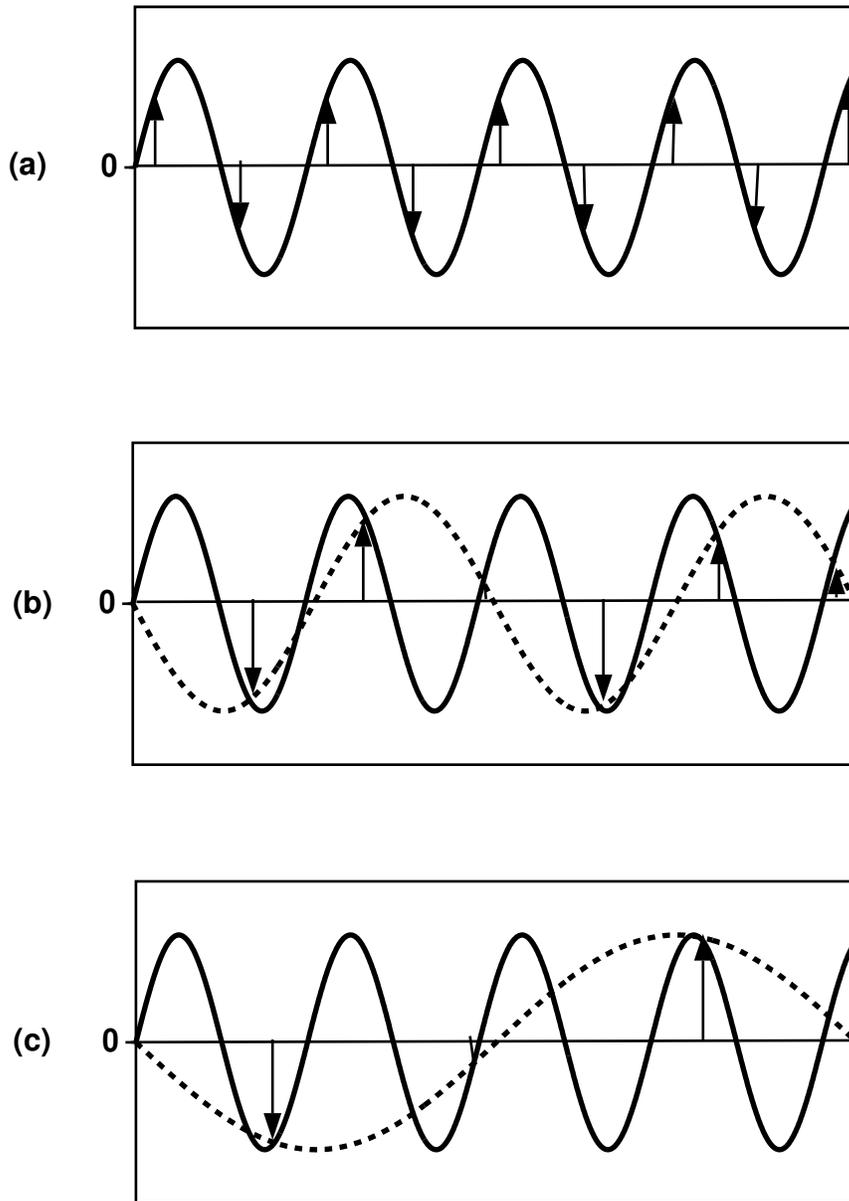


Figure 1.4: Effect of discretisation in time: (a) properly sampled signal; (b) just under-sampled signal; (c) fully undersampled signal.

in which a_k and A_n are now the discrete-time and discrete-frequency values of the continuous signals $a(t)$ and $A_{\text{Continuous}}(f)$. These two equations are the final discrete-time and discrete-frequency Fourier transform pair.

1.3 LTI Systems and Convolution

In this section a signal is fed into a linear time-invariant system. To that purpose a signal $s(t)$ can be written as:

$$s(t) = \int_{-\infty}^{+\infty} s(\tau)\delta(t - \tau)d\tau \quad (1.8)$$

Let us feed the integral to the system by building up the signal from the δ -pulse responses, as shown in Figure 1.5.

- On top, $\delta(t)$ is fed into the system giving $h(t)$ as output.
- Next, a time-shifted pulse $\delta(t - \tau)$ is fed into the system: because the system is time-invariant, the response will be $h(t - \tau)$.
- Next, a scaled pulse $s(\tau)\delta(t - \tau)$ is fed into the system: because the system is linear, the response is $s(\tau)h(t - \tau)$. This is valid for each τ , so for all values of the integrand.
- Then, finally, scaling the input each by $d\tau$, we can feed the whole integral into the system: because the system is linear, the total response $x(t)$ to this signal will be:

$$x(t) = \int_{-\infty}^{+\infty} s(\tau)h(t - \tau)d\tau \quad (1.9)$$

This equation is a *convolution*: the output $x(t)$ is the convolution of the input $s(t)$ with the impulse response $h(t)$. A physical system is causal and assuming the input signal starts at $t = 0$, the responses $s(t)$ and $h(t)$ are zero for times smaller than zero. Substituting this in the above, the equation becomes:

$$x(t) = \int_0^t s(\tau)h(t - \tau)d\tau \quad (1.10)$$

The convenient shorthand notation for the convolution integral is

$$x(t) = s(t) * h(t) \quad (1.11)$$

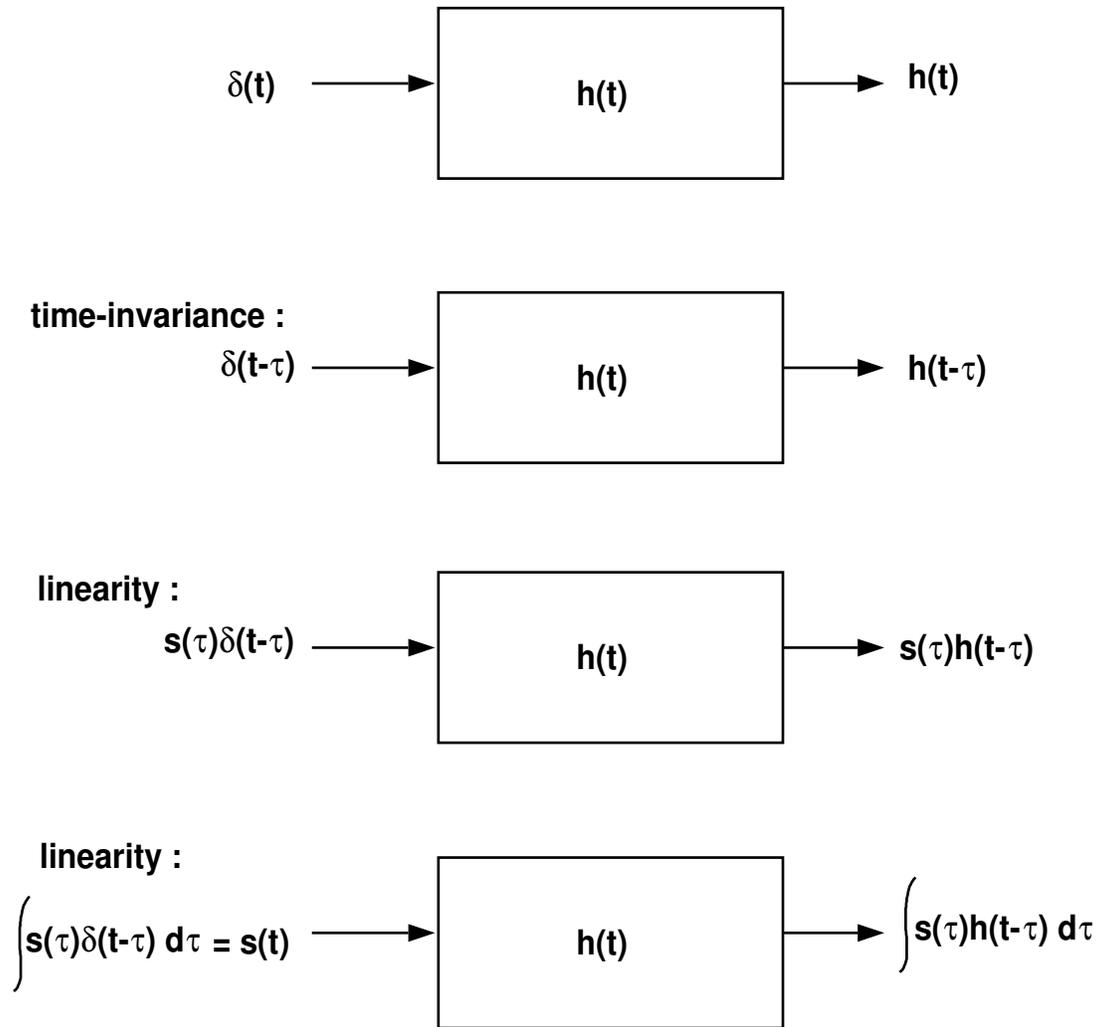


Figure 1.5: Convolution built up from scaled, time-shifted δ -pulses.

1.4 Convolution Theorem

The convolution theorem states that convolution in one Fourier domain is equivalent to multiplication in the other Fourier domain. Thus, the result of convolving of two time signals is equivalent, in the frequency domain, to multiplying their Fourier transforms. Equally, convolution of two (complex) signals in the frequency domain is equivalent to multiplication of their inverse Fourier transforms in the time domain. Of course, this result applies to all Fourier-transformable functions, including functions of space.

The theorem may be stated mathematically as follows

$$\mathcal{F}_t \left(\int_{-\infty}^{+\infty} h(t')g(t-t')dt' \right) = \mathcal{F}_t[h(t) * g(t)] = H(f)G(f) \quad (1.12)$$

in which \mathcal{F}_t means "Fourier transform of".

1.5 Filters

A filter is a system that has an input and an output. The linear time-invariant systems considered previously can also be treated as filters. Filters usually have a purpose to do something to a signal: the input signal needs to be shaped or formed, depending on the application.

Fourier analysis can give very much insight in how a signal is built up : one cannot only recognize certain features arriving at certain times, such as a reflection in reflection seismics, but one can also recognize certain resonances in systems: many electronic circuits have their own resonances, and they can be analyzed by Fourier analysis. One can do more than just analyze signals: one can also *remove* features from a signal. Thus, removal cannot only be done in the time domain, but also in the frequency domain. This is called *filtering*.

An example of filtering is given in the next figure (Fig. 1.6). Let us now consider a signal as given in Figure (1.6). The signal is built up of a part which is slowly varying (low-frequent), and a part which is rapidly varying (high-frequent). Say, we are interested in the slowly varying part, so the rapidly varying (high-frequent) part needs to be removed. This removal cannot be done in the time domain since the two parts are not separated. From the previous sections it may be obvious that we can establish a separation via the frequency domain. For that reason, we transform the signal to the frequency domain. This is shown on the upper right figure. Two peaks can be seen, each associated with the parts which were described above. Now the signal is fed into a filter, which is a window function. This means simply that the spectrum of the input signal is multiplied with the transfer function of the system, which is a window function. When the multiplication is performed, the figure as given in the right-bottom figure is obtained: only the low-frequency part is retained. When the signal is transformed back to the time domain, the left-bottom figure is obtained : we have got rid of the high-frequency part using the window-function in the frequency domain as filter.

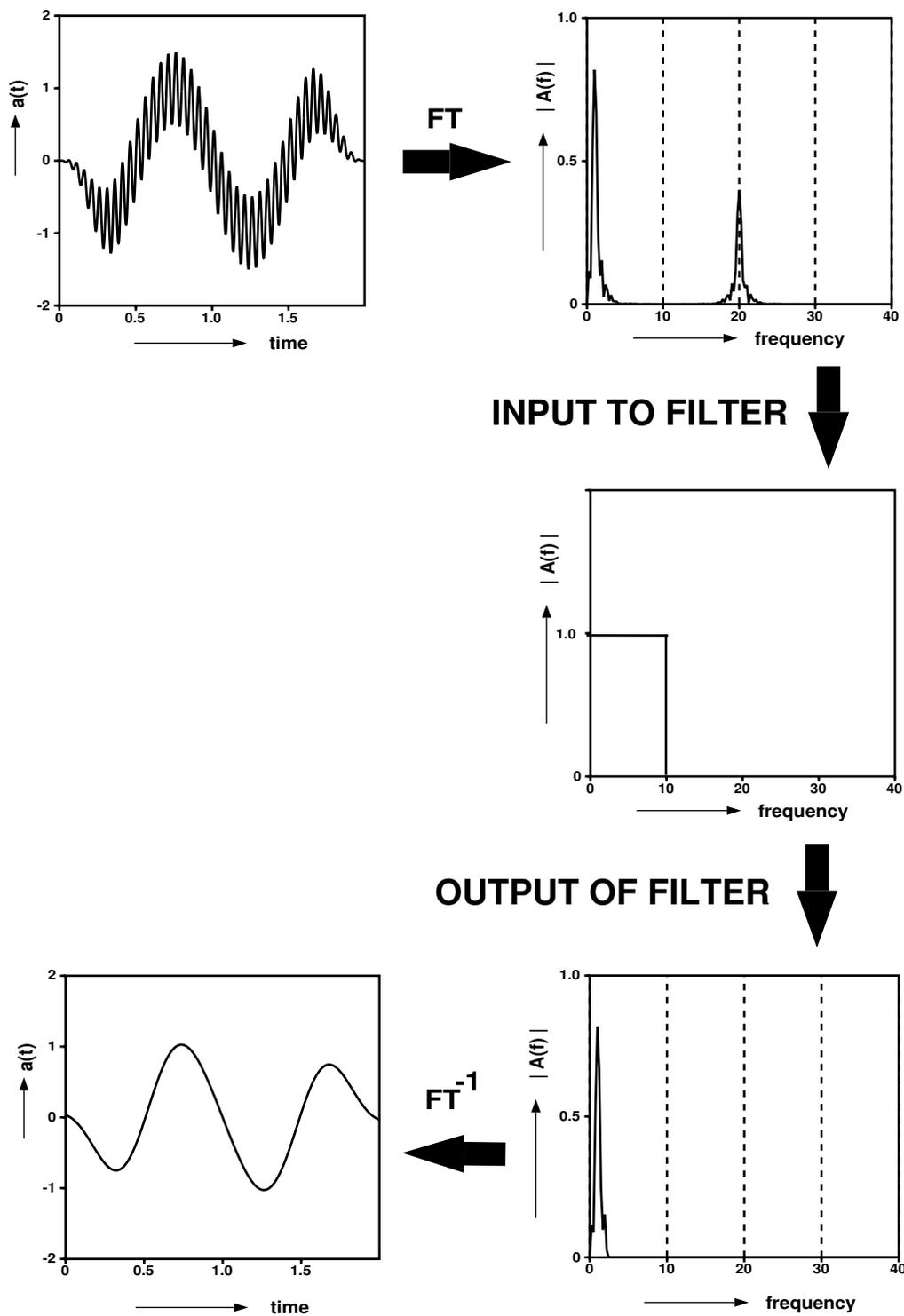


Figure 1.6: Filtering of two-sine signal using window in frequency domain. Upper left: two superposed sine waves. Upper right: amplitude spectrum. Lower right: amplitude spectrum of filtered signal. Lower left: filtered signal in time-domain.

The procedure as given in the above is called *filtering*. Filtering in this case is nothing else than windowing in the frequency domain.

1.6 Correlation

In the same way as convolution, we can easily derive that a correlation in the time domain is equivalent to a multiplication *with the complex conjugate* in the frequency domain. The derivation of this, is given in appendix B. We can recognize two types of correlations, namely an autocorrelation and a cross-correlation. For the autocorrelation, the Fourier transform is given by:

$$\mathcal{F}_t \left(\int_{-\infty}^{+\infty} a(\tau) a^*(\tau - t) d\tau \right) = A(f) A^*(f) = |A(f)|^2 \quad (1.13)$$

Note here that the *phase* of the spectrum is absent, and therefore called zero phase. In the time domain, it can be shown mathematically that this autocorrelation-signal is symmetric around $t = 0$.

In the same way, it is shown in appendix B that the Fourier transform of a cross-correlation is given by:

$$\mathcal{F}_t \left(\int_{-\infty}^{+\infty} a(\tau) b^*(\tau - t) d\tau \right) = A(f) B^*(f) \quad (1.14)$$

1.7 Deconvolution

Deconvolution concerns itself with neutralizing a part of a signal which is convolutional. We consider that the output signal $x(t)$ consists of the convolution of an input signal $s(t)$ and the impulse response $g(t)$ of an LTI system, i.e.,

$$x(t) = s(t) * g(t). \quad (1.15)$$

Often, we are interested in the impulse response response of the system. Ideally, we would like the input signal to have a flat amplitude spectrum of 1, with no phase, which corresponds to a delta-function in time. In practice, this will never be the case. Generally, the input signal $s(t)$ has a certain shape and amplitude. So therefore we want to find a filter $f(t)$ that converts the signal $s(t)$ into a δ -function:

$$f(t) * s(t) = \delta(t). \quad (1.16)$$

By applying the filter $f(t)$ to the output signal $x(t)$, we neutralize the effect of the input signal since

$$f(t) * x(t) = f(t) * s(t) * g(t)$$

$$\begin{aligned}
&= \delta(t) * g(t) \\
&= g(t).
\end{aligned}
\tag{1.17}$$

Neutralizing the effect of the input signal from a output signal is called a *deconvolution* process.

Let us assume we have a signal $s(t)$ with a known spectrum, $S(f)$. Then the convolution (1.15) becomes a multiplication in the frequency domain:

$$X(f) = S(f)G(f), \tag{1.18}$$

in which $X(f)$ is the spectrum of the output signal, and $G(f)$ is the spectrum of the system response. Now if we want to neutralize the input signal, then we have to divide each side by $S(f)$, or equivalently apply the inverse operator $F(f) = 1/S(f)$ to each side, obtaining:

$$\frac{X(f)}{S(f)} = G(f). \tag{1.19}$$

Of course, this states the problem too simple: the signal $x(t)$ always contains some noise. When the signal $x(t)$ is taken as the convolution above together with some noise term, i.e., $X(f) = S(f)G(f) + N(f)$ in which $N(f)$ denotes the noise term, then the deconvolution in the frequency domain becomes:

$$\frac{X(f)}{S(f)} = G(f) + \frac{N(f)}{S(f)}. \tag{1.20}$$

The next problem is that due to this division, the noise is blown up outside the bandwidth of signal $S(f)$, i.e., there where the amplitude of $S(f)$ is (very) small. This effect is shown in Figure (1.7).

There are two ways to tackle this problem. The first one is that we stabilize the division. This is done by not applying a filter $F(f) = 1/S(f)$ but first multiplying both the numerator and the denominator by the complex conjugate of the input-signal spectrum, $S^*(f)$, and since the denominator is now real we can add a small (real) constant ϵ to it. Thus instead of $1/S(f)$, we apply the filter:

$$F(f) = \frac{S^*(f)}{S(f)S^*(f) + \epsilon^2}. \tag{1.21}$$

Often we take ϵ as a fraction of the maximum value in $|S(f)|$, e.g. $\epsilon = \alpha MAX(|S(f)|)$ with α in the order of 0.01 - 0.1. In this way we have controlled the noise, but it can still be large outside the bandwidth of $S(f)$ (see Figure (1.7)). As an example, Figure (1.8) shows the result for deconvolution.

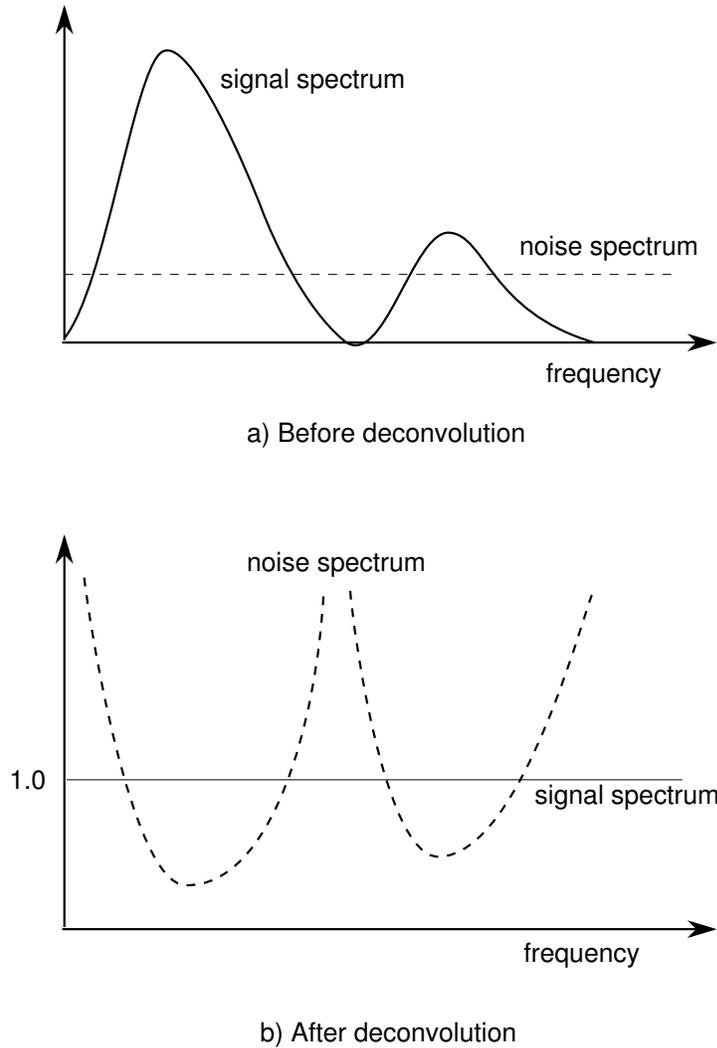


Figure 1.7: The effect of deconvolution in the frequency domain in the presence of noise.

The other way of dealing with the blowing up of the noise is only doing the division in a certain bandwidth which is equivalent to shaping the input signal $s(t)$ into a shorter one, which we call $d(t)$. In this case we do not apply the filter $1/S(f)$ but instead we use $D(f)/S(f)$. Then the deconvolution amounts to:

$$\frac{X(f)D(f)}{S(f)} = G(f)D(f) + \frac{N(f)D(f)}{S(f)}, \quad (1.22)$$

where $|D(f)|$ is approximately equal to $|S(f)|$, i.e.:

$$a < \frac{|D(f)|}{|S(f)|} < b, \quad (1.23)$$

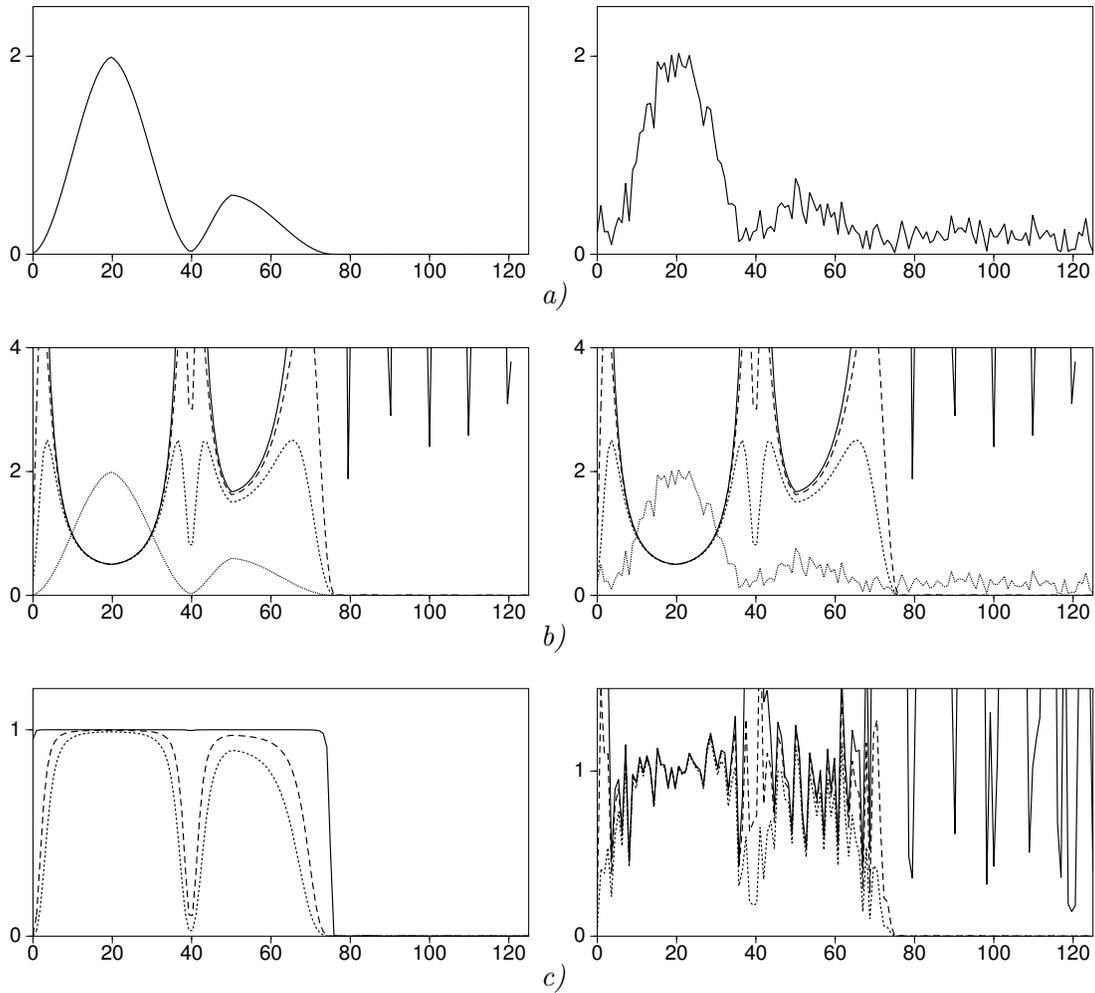


Figure 1.8: Applying stabilized inversion in the frequency domain, left for a noise free input signal, right for a noisy input signal. a) Spectrum of signal to be inverted. b) Spectra of inverse operators with 3 stabilization constants ($\epsilon = 0, 0.05, 0.1$). c) Multiplication of inverse filters with original spectrum of a), i.e. the deconvolution results.

in which b/a is less than 10, say. Often in seismics, we would like to end up with a signal that is short in the time domain. This means that the spectrum of $D(f)$ must be smooth compared to the true input-signal spectrum $S(f)$. Note that a short signal in time corresponds with a smooth (i.e. oversampled) signal in frequency, as the major part of the time signal will be zero. Practically this means when we know the spectrum we can design some smooth envelope around the spectrum $S(f)$, or we can just pick a few significant points in the spectrum and let a smooth interpolator go through these picked points. An example of designing such a window is given in Figure (1.9).

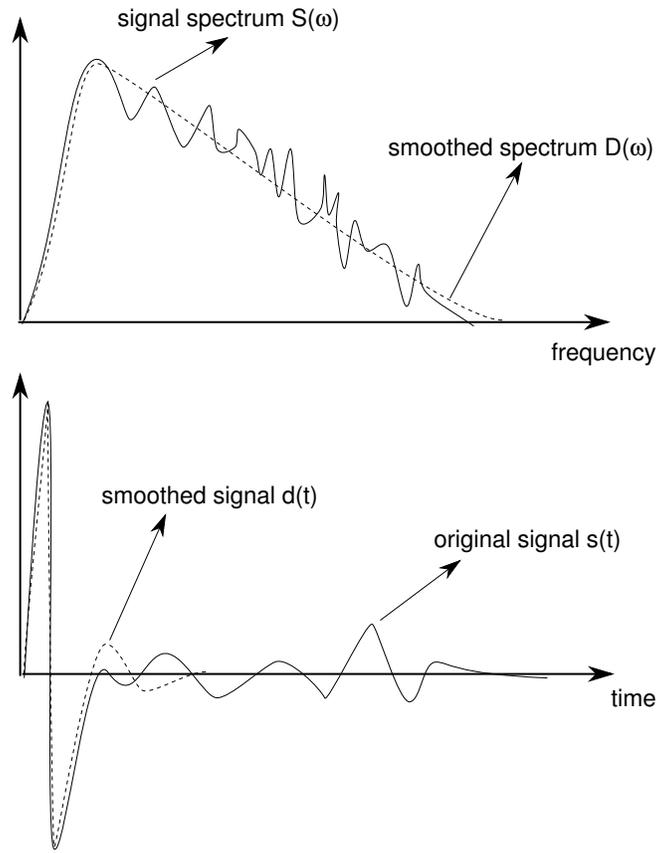


Figure 1.9: Designing a desired input signal via smoothing in the frequency domain.

As a last remark of deconvolution in the frequency domain it can be said that in practice both ways of control over the division by $S(f)$ are used. We then apply a filter :

$$F(f) = \frac{D(f)S^*(f)}{S(f)S^*(f) + \epsilon^2} \quad (1.24)$$

to the output signal $x(t)$, resulting in:

$$\frac{X(f)D(f)S^*(f)}{S(f)S^*(f) + \epsilon^2} = \frac{G(f)D(f)S(f)S^*(f)}{S(f)S^*(f) + \epsilon^2} + \frac{N(f)D(f)S^*(f)}{S(f)S^*(f) + \epsilon^2}. \quad (1.25)$$

This is about the best we can do given the constraints of bandwidth and signal-to-noise ratio.

1.8 Time- and frequency characteristics

In the table we below, we list the characteristics that we will use throughout these lecture notes. Some of them have been discussed in this chapter, others will be discussed in the coming chapters.

time domain	frequency domain
discretisation with Δt $a(t = k\Delta t)$	making periodic with $\frac{1}{\Delta t}$ $\sum_{m=-\infty}^{\infty} A_{\text{Continuous}}(f + \frac{m}{\Delta t})$
convolution of signals $\int_{-\infty}^{+\infty} s(\tau)h(t - \tau)d\tau$	multiplication of spectra $S(f)H(f)$
correlation with $\int_{-\infty}^{+\infty} a(\tau)b^*(\tau - t)d\tau$	multiplication by complex conjugate $A(f)B^*(f)$
purely symmetric $a(t) = a(-t)$	zero phase (imaginary part zero) $ A(f) $
time shift $\delta(t - T)$	linear phase $\exp(-2\pi ifT)$
signal and inverse both causal	minimum phase
deconvolution $f(t) * s(t) = \delta(t)$	division $F(f) = \frac{1}{S(f)}$

Table 1.1: Some characteristics in time- and frequency domain.