Fourier Analysis

(Ta3520)

Fourier Analysis

- Continuous Fourier transform
- Discrete Fourier Transform and Sampling Theorem
- Linear Time-Invariant (LTI) systems and Convolution
- Convolution Theorem
- Filters
- Correlation
- Deconvolution

Continuous Fourier transform

Definition:

$$A(f) = \int_{-\infty}^{+\infty} a(t) \exp(-2\pi i f t)\, dt$$

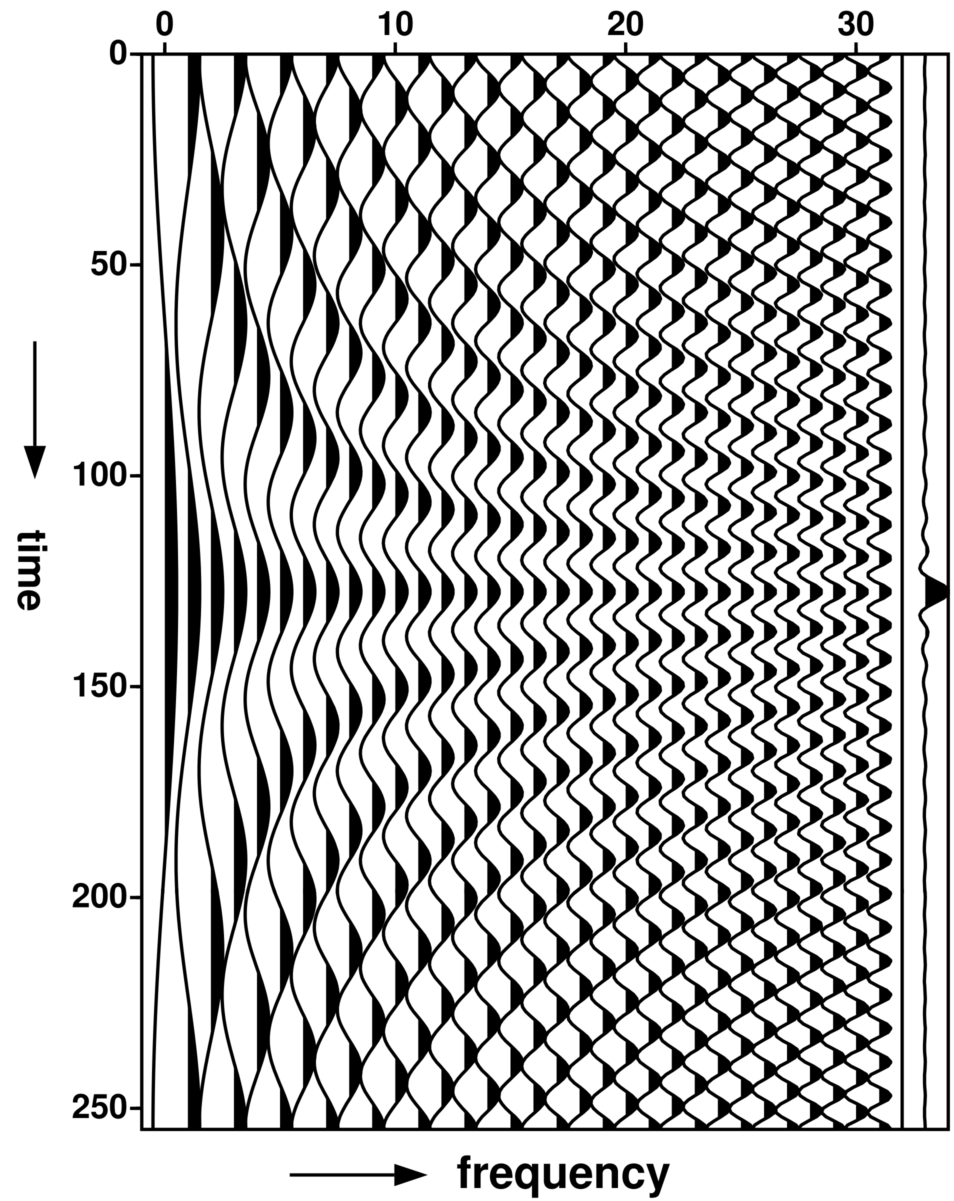
Inverse transform:

$$a(t) = \int_{-\infty}^{+\infty} A(f) \exp(2\pi i f t)\, df$$

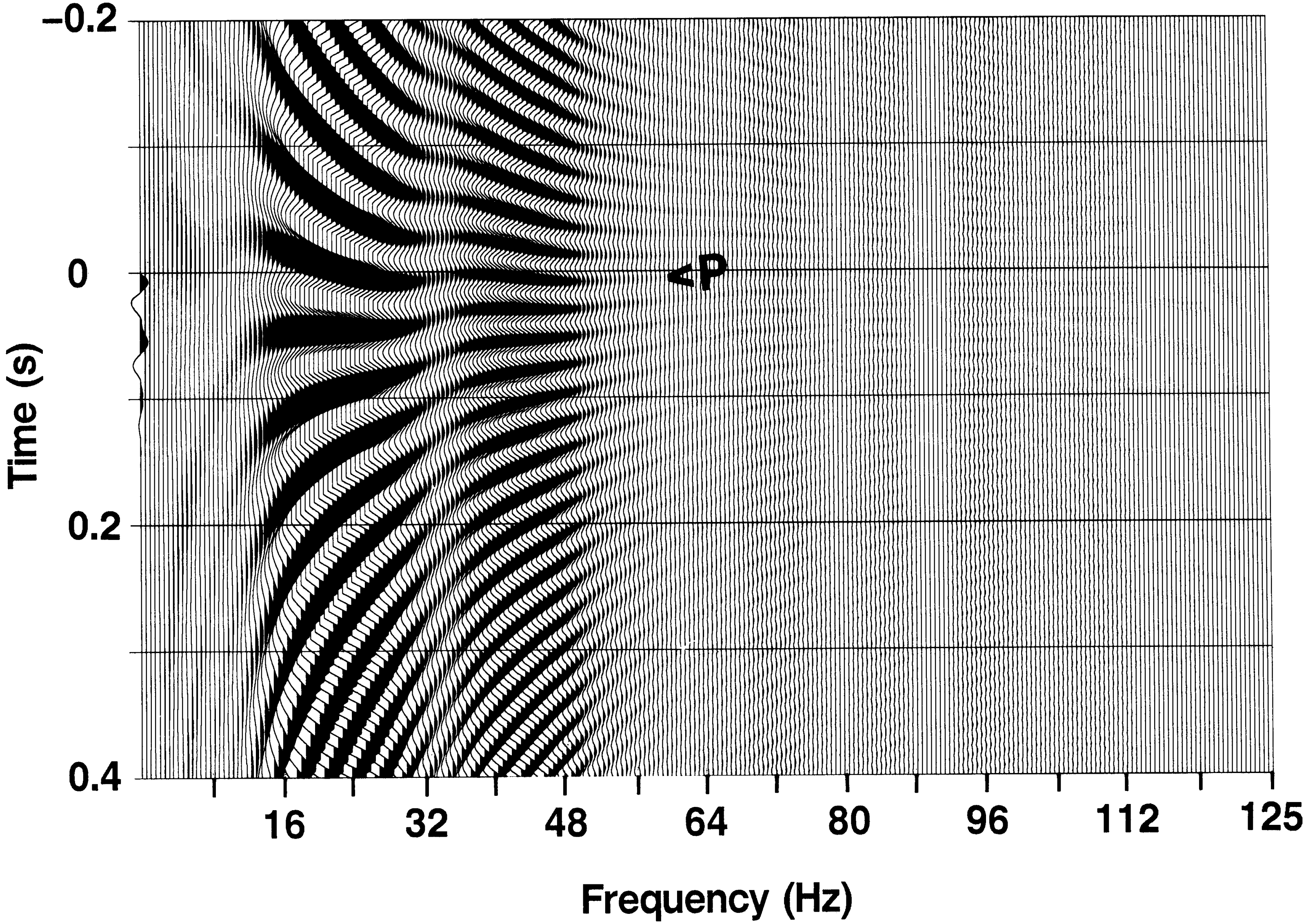
$f =$ frequency (Hz); $t =$ time (s)

Decomposition of signal into sines and cosines

Decomposition into cosines



Decomposition into sines and cosines



Discretisation of signal

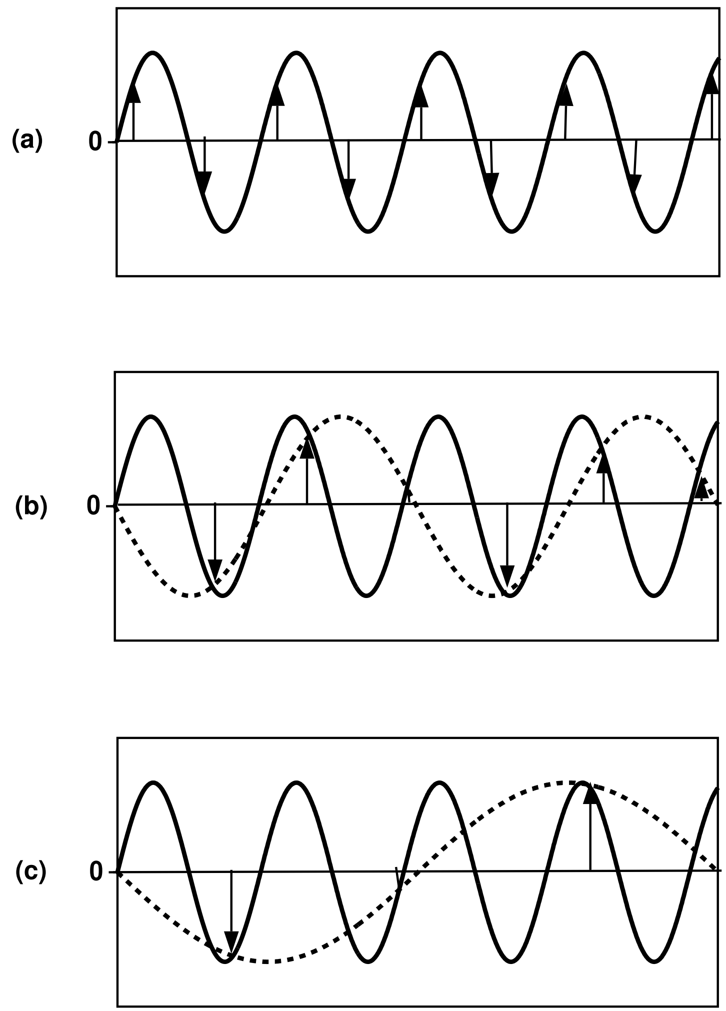
Discretisation in time makes spectrum periodic:

$$A_{\text{Discrete}}(f) = \sum_{m=-\infty}^{+\infty} A_{\text{Continuous}}(f + \frac{m}{\Delta t})$$

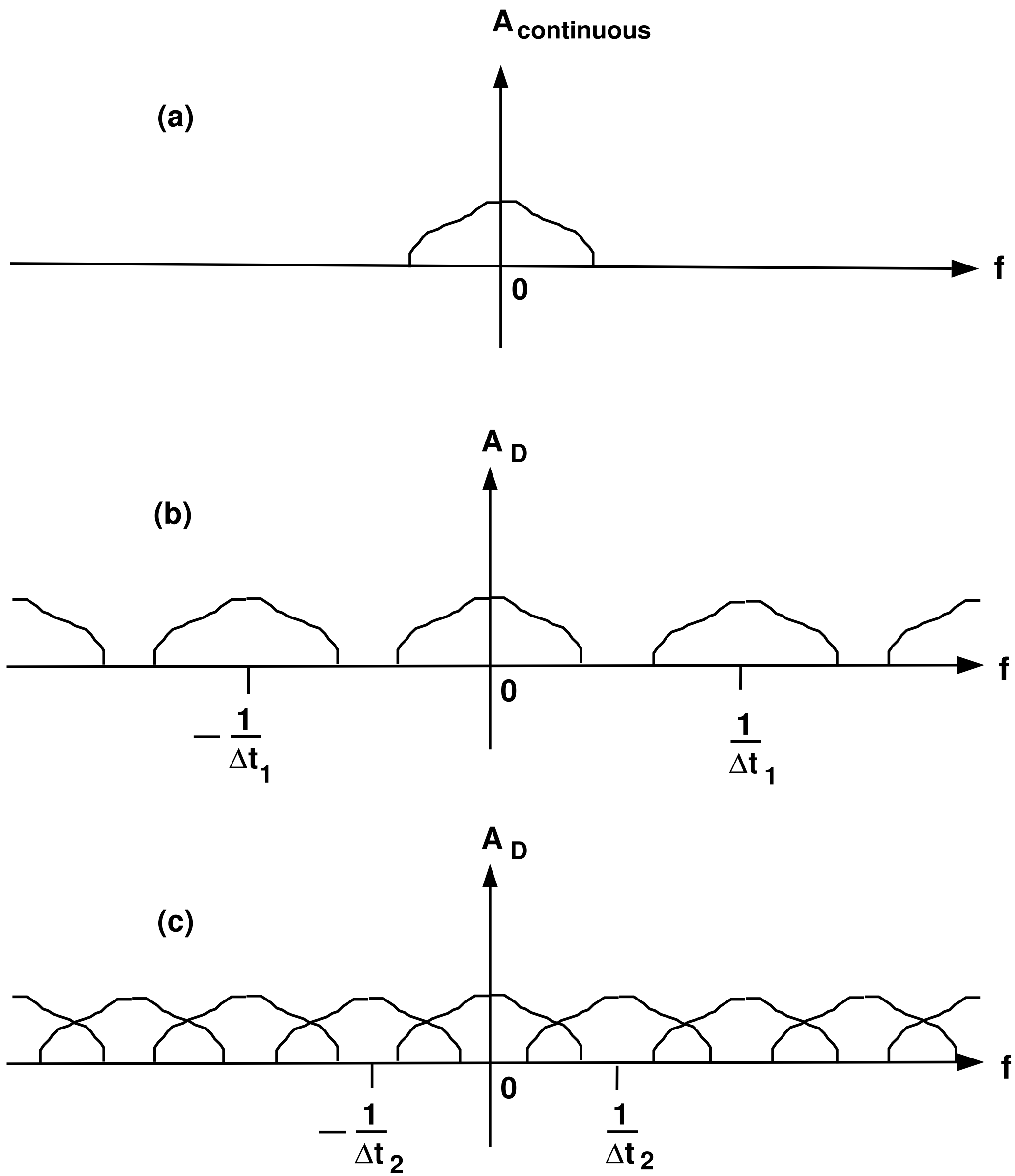
At least two samples per frequency/wavelength.

Otherwise: aliasing.

Aliasing in time domain



Aliasing = overlapping spectra



Discrete Fourier transform

$$A_n = \Delta t \sum_{k=0}^{N-1} a_k \exp(-2\pi i n k/N) \quad \text{for } n = 0, 1, ..., N-1$$

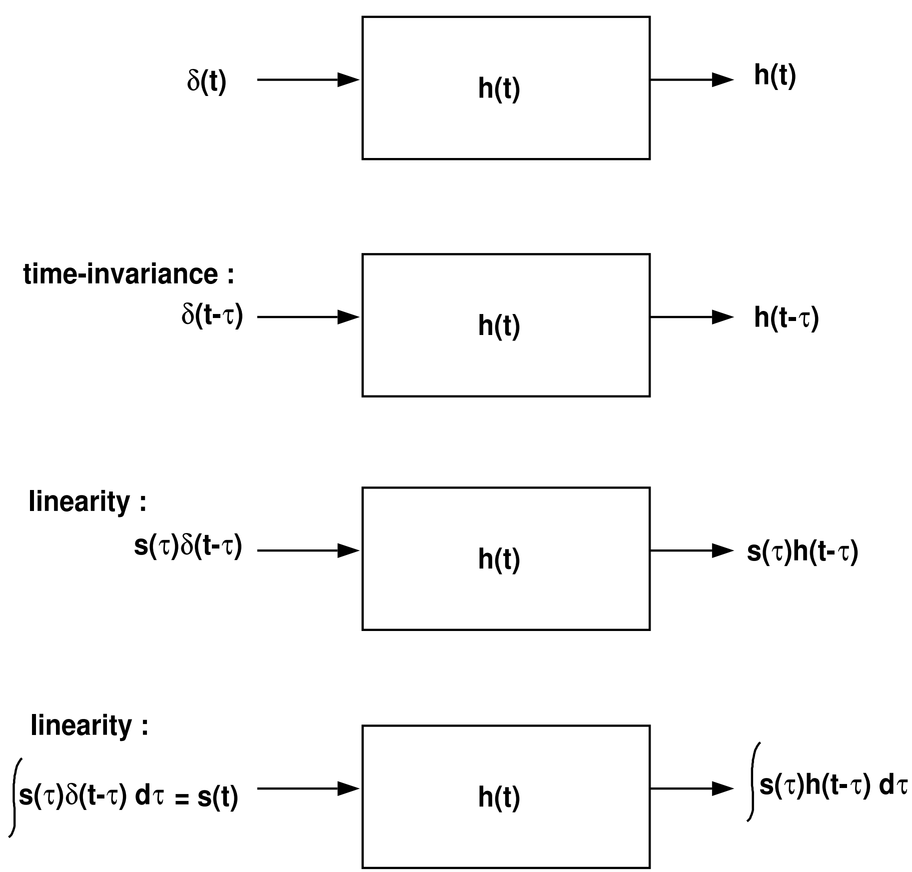
k = index for time-domain coefficient
n = index for Fourier-domain coefficient

$$a_k = \Delta f \sum_{n=0}^{N-1} A_n \exp(2\pi i n k/N) \quad \text{for } k = 0, 1, ..., N-1$$

Basic relation: $N\Delta t \Delta f = 1$
Nyquist criterion: $f_N = \frac{1}{2\Delta t}$

Linear Time-Invariant (LTI) systems and convolution

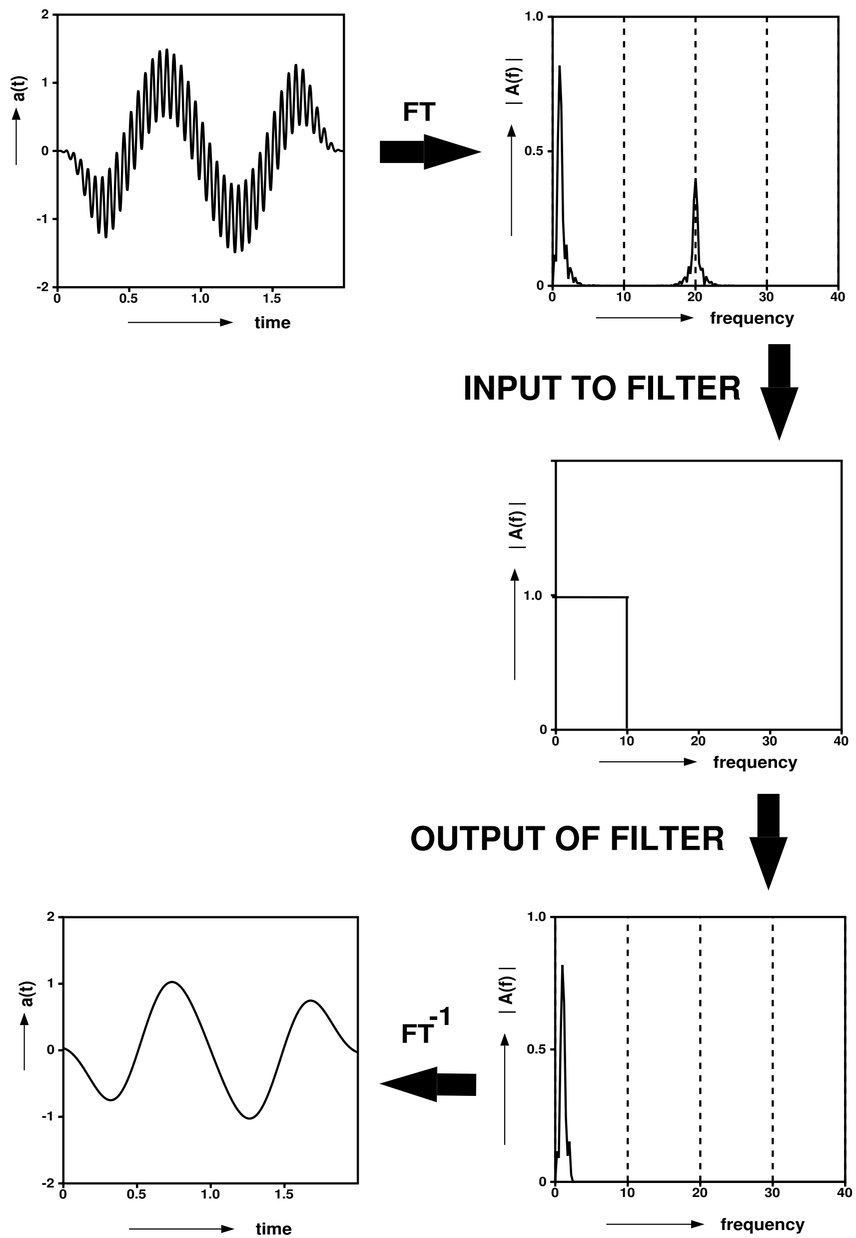
Output of linear time-invariant system = convolution of input with impulse response of system:



Convolution theorem

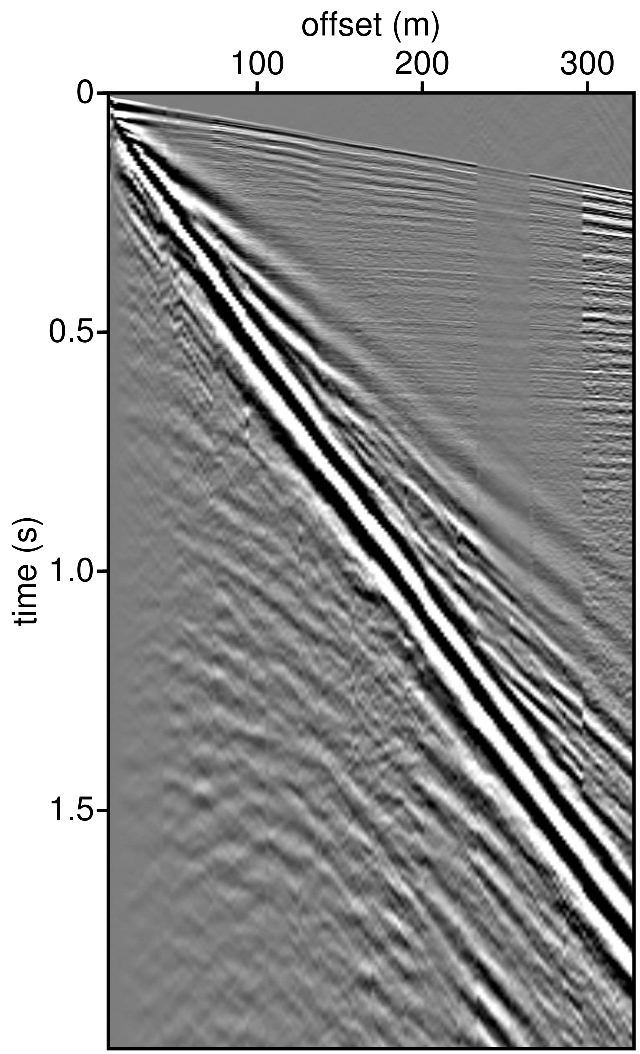
$$\begin{aligned}\mathcal{F}_t\left(\int_{-\infty}^{+\infty} h(t')g(t-t')dt'\right) &= \mathcal{F}_t\left(h(t) * g(t)\right) \\ &= H(f)G(f)\end{aligned}$$

Filters

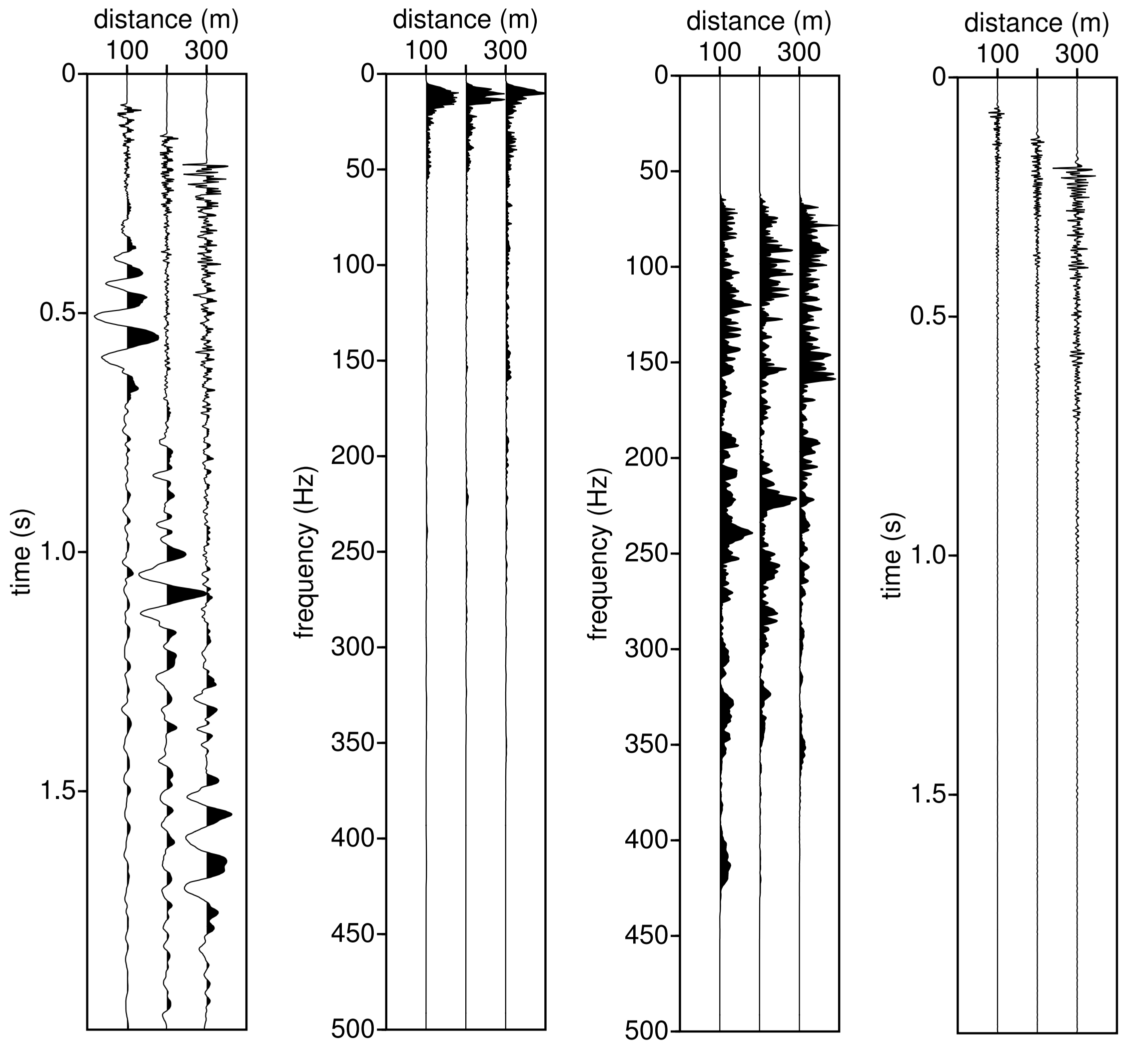


a(t)
time
FT
| A(f) |
frequency
INPUT TO FILTER
| A(f) |
frequency
OUTPUT OF FILTER
| A(f) |
frequency
FT-1
a(t)
time

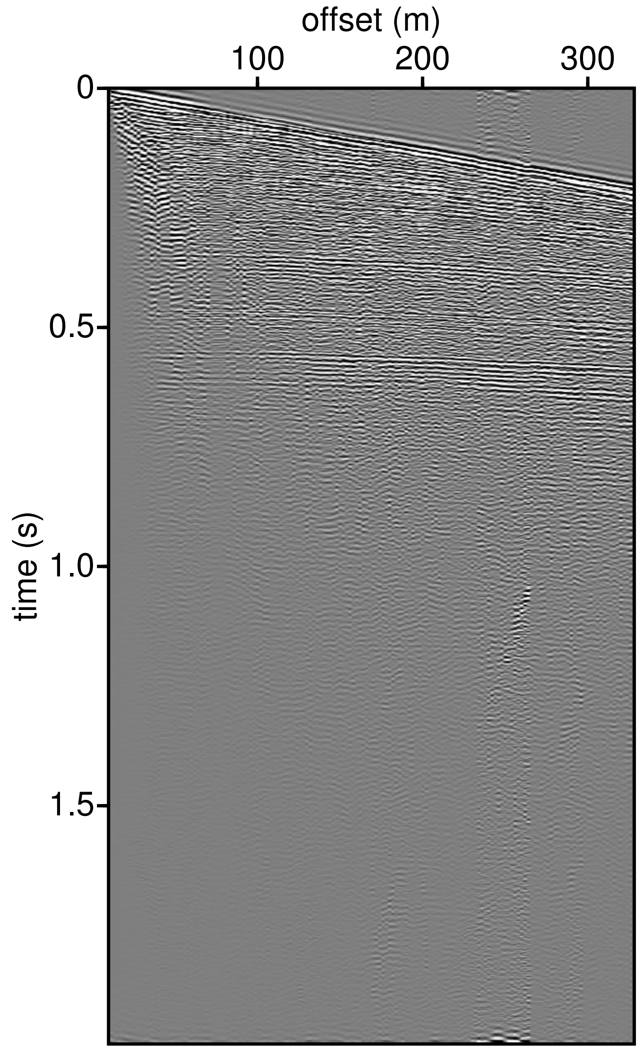
Filters: original Wassenaar data



Filters: Wassenaar data



Filters: filtered Wassenaar data



Correlation

Auto-correlation:

$$\mathcal{F}_t \left(\int_{-\infty}^{+\infty} a(\tau) a^*(\tau - t) d\tau \right) = A(f) A^*(f) = |A(f)|^2$$

Cross-correlation:

$$\mathcal{F}_t \left(\int_{-\infty}^{+\infty} a(\tau) b^*(\tau - t) d\tau \right) = A(f) B^*(f)$$