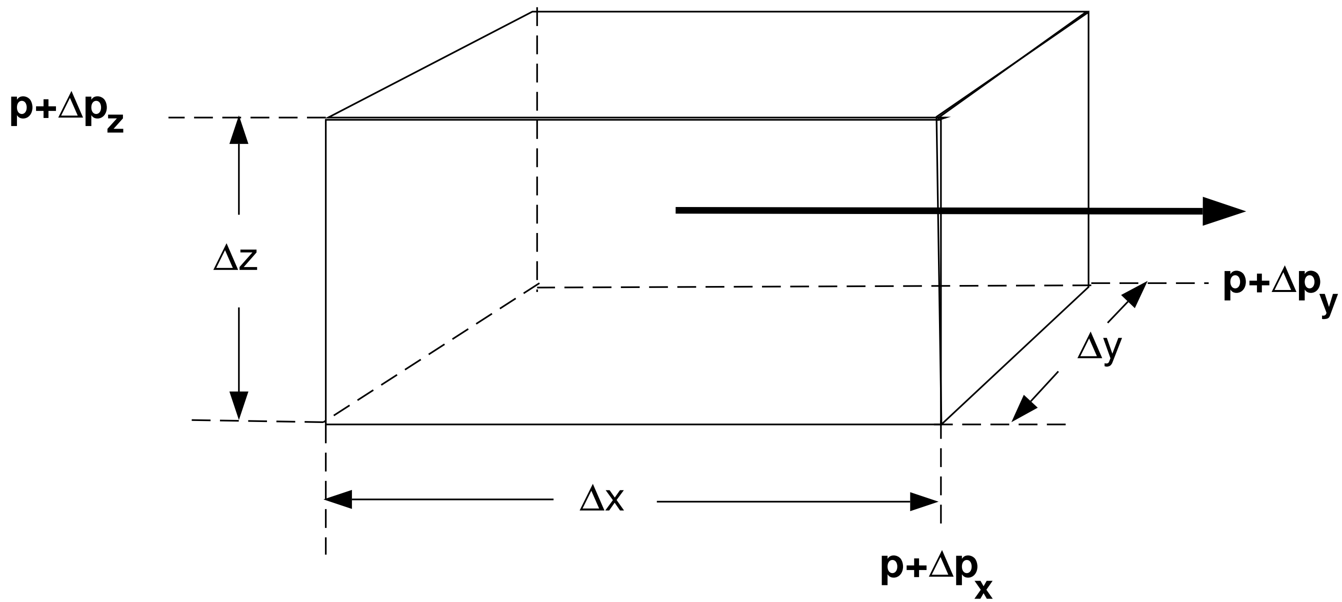
Derivation of Wave Equation

(Ta3520)

Derivation wave equation

Consider small cube of mass with volume ΔV:



Desired: equations in terms of pressure p and particle velocity $\mathbf{v}$

Deformation Equation

Conservation of mass:

$$\begin{aligned}\rho(t_0)\Delta V(t_0) &= \rho(t_0+dt)\Delta V(t_0+dt)\\ \rho_0\Delta V &\simeq (\rho_0+d\rho)(\Delta V+dV)\end{aligned}$$

or:

$$\frac{d\rho}{\rho_0} \simeq -\frac{dV}{\Delta V},$$

neglecting term $d\rho dV$.

Deformation Equation

Assume linear relation between density ρ and pressure p:

$$\frac{d\rho}{\rho_0} = \frac{dp}{K}$$

where K is bulk modulus.

This is one. Now $dV/\Delta V$.

1-Dimensional motion (in x-direction):

$$\frac{dV}{\Delta V} \simeq \frac{dx}{\Delta x}$$

dx is difference in change of particle displacement du_x:

$$\begin{aligned} dx &= (du_x)_{x+\Delta x} - (du_x)_x \\ &= \frac{\partial (du_x)}{\partial x}\Delta x = \frac{\partial v_x}{\partial x} dt\, \Delta x \end{aligned}$$

where v_x is x–component of particle velocity.

Substitute $d\rho/\rho_0$ and $dV/\Delta V$:

$$\frac{dp}{K} = -\frac{\partial v_x}{\partial x} dt$$

Assuming low-velocity approximation:

$$\frac{dp}{dt} \approx \frac{\partial p}{\partial t}$$

Gives **deformation equation:**

$$\frac{1}{K}\frac{\partial p}{\partial t} = -\frac{\partial v_x}{\partial x}$$

Equation of Motion

Newton's law applied to volume ΔV :

$$\Delta F_x = \rho \Delta V \frac{dv_x}{dt}$$

where F_x is force in x-direction (1-dimensional motion)

$$\begin{aligned} \Delta F_x &= -\Delta p_x \Delta S_x \\ &= -\left(\frac{\partial p}{\partial x} \Delta x + \frac{\partial p}{\partial t} dt \right) \Delta S_x \\ &\simeq -\frac{\partial p}{\partial x} \Delta V \qquad \text{(for } dt \to 0\text{)} \end{aligned}$$

Assuming low-velocity approximation:

$$\frac{dv_x}{dt} \approx \frac{\partial v_x}{\partial t}$$

Then Newton's Law becomes:

$$-\frac{\partial p}{\partial x} = \rho_0 \frac{\partial v_x}{\partial t}$$

This is the **equation of motion.**

Wave Equation

Combine deformation equation and equation of motion.

Let operator $(\partial/\partial x)$ work on equation of motion and assume ρ constant:

$$-\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}\right) = \rho\frac{\partial}{\partial t}\left(\frac{\partial v_x}{\partial x}\right)$$

Substitute deformation equation:

Wave Equation

$$\frac{\partial^2 p}{\partial x^2} - \frac{\rho_0}{K}\frac{\partial^2 p}{\partial t^2} = 0$$

or:

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$$

This is the **wave equation**.

c is velocity of sound: $c = \sqrt{K/\rho}$

Need to know for exam

Conservation of mass, with some approximations, leads to deformation equation:

$$\frac{1}{K}\frac{\partial p}{\partial t} = -\frac{\partial v_x}{\partial x}$$

Conservation of forces, with some approximations, leads to equation of motion:

$$-\frac{\partial p}{\partial x} = \rho_0 \frac{\partial v_x}{\partial t}$$

From then on, everything.