Dynamics and Stability AE3-914

Sample problem—Week 1

Kinetic energy of helicopter rotor

Statement

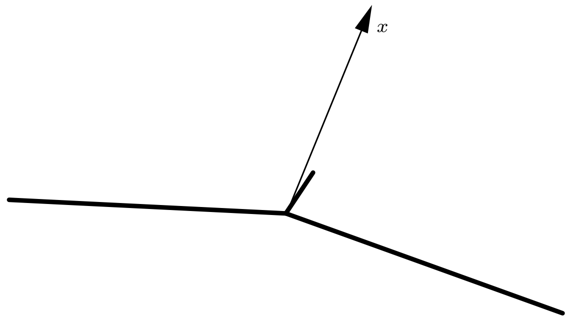
The rotor of the represented helicopter consists of three equally spaced blades and has an angular velocity with magnitude p counterclockwise about the rotor shaft, when observed from above. The helicopter is performing a turn to the left with radius R and speed v, without slip and under a constant bank angle θ. Each blade can be modelled as a bar with mass M and moment of inertia I_b with respect to an endpoint.

a. Choose an adequate coordinate system and set up the inertia tensor of the rotor with respect to its mass centre.

b. Find an expression for the kinetic energy of the rotor in terms of M, I_b, p, v, R and θ.

Inertia Tensor

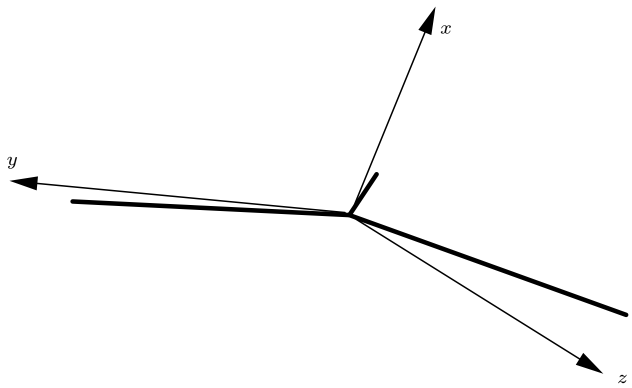
The first step when setting up the inertia tensor is to choose an adequate coordinate system. "Adequate" means here that the products of inertia should vanish if possible. It is observed that the rotor, when modelled by three bars, is a planar object. One can then apply the property that the products of inertia involving the perpendicular axis to the plane containing the object vanish and choose one axis to be perpendicular to this plane. To avoid "traditional" notations this axis will be labelled as the x-axis here:



One can now state that, independently of the position of the y and z-axes, that will be contained in the rotor plane, the following components of the tensor are determinate:

$$\begin{aligned} I_{xx} &= 3I_b \\ I_{xy} &= 0 \\ I_{xz} &= 0 \end{aligned} \tag{1}$$

The y and z-axes are now to be chosen. As the rotor has three equally spaced blades one can observe that it has three symmetry axes in the yz-plane. This means that all axes in the yz-plane are principal axes, that $I_{yy} = I_{zz}$ and that, as these are principal moments of inertia for any choice of the axes, $I_{yz} = 0$. A judicious choice is that for which it then becomes easy to express the angular velocity vectors, for example by setting the y-axis pointing forward and the z-axis according to the right-hand rule, thus pointing to the left side of the helicopter in this case:



Recalling the definition of the moments of inertia

$$\begin{aligned}
I_{xx} &= \int (y^2 + z^2)\, dm \\
I_{yy} &= \int (x^2 + z^2)\, dm \\
I_{zz} &= \int (x^2 + y^2)\, dm
\end{aligned} \tag{2}$$

together with the fact that the body is contained in the yz-plane and thus all integrals involving x vanish, i.e.,

$$\int x^2\, dm = 0 \tag{3}$$

one can state that

$$\begin{aligned}
I_{yy} + I_{zz} &= \int (x^2 + z^2)\, dm + \int (x^2 + y^2)\, dm \\
&= \int (y^2 + z^2)\, dm \\
&= I_{xx}
\end{aligned} \tag{4}$$

Combining this with the property that $I_{yy} = I_{zz}$ leads to the conclusion that

$$I_{yy} = I_{zz} = \frac{3}{2} I_b \tag{5}$$

and the inertia tensor thus reads

$$\mathbf{I}_G = \begin{bmatrix} 3I_b & 0 & 0 \\ 0 & \frac{3}{2}I_b & 0 \\ 0 & 0 & \frac{3}{2}I_b \end{bmatrix} \tag{6}$$

where the G subscript emphasises that this tensor is written with respect to the mass centre.

Kinetic energy

As there is no fixed point in the motion, the general expression must be used for the kinetic energy

$$T = \frac{1}{2} m v_G^2 + \frac{1}{2} \boldsymbol{\omega}^T \mathbf{I}_G \boldsymbol{\omega} \tag{7}$$

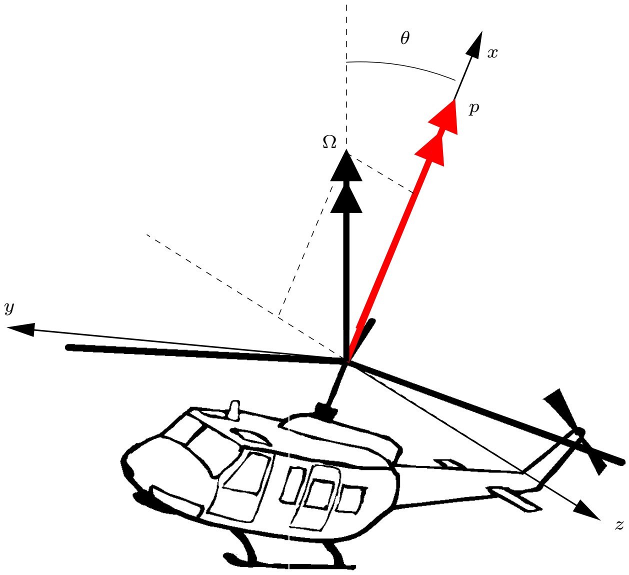
The speed of the mass centre can immediately be written as

$$v_G = v \tag{8}$$

and, as there is no slip, the rate of turn is given by

$$\Omega = \frac{v}{R} \tag{9}$$

where it is recalled that R is the radius of the flying path. Further analysis of the angular velocity vector on the chosen axes provides the following scheme, in which the bank angle θ plays a role:



The angular velocity vector is now set up as

$$\boldsymbol{\omega} = (p + \Omega\cos\theta)\mathbf{i} - \Omega\sin\theta\,\mathbf{k} \tag{10}$$

This renders the kinetic energy as

$$T = \frac{1}{2}3Mv^2 + \frac{1}{2}\begin{pmatrix} p+\Omega\cos\theta & 0 & -\Omega\sin\theta \end{pmatrix} \begin{bmatrix} 3I_b & 0 & 0 \\ 0 & \frac{3}{2}I_b & 0 \\ 0 & 0 & \frac{3}{2}I_b \end{bmatrix} \begin{pmatrix} p+\Omega\cos\theta \\ 0 \\ -\Omega\sin\theta \end{pmatrix} \tag{11}$$

which is elaborated as

$$T = \frac{3}{2}\left\{Mv^2 + I_b\left[p^2 + \frac{2pv\cos\theta}{R} + \frac{v^2(1+\cos^2\theta)}{2R^2}\right]\right\} \tag{12}$$