1

Integration in Banach spaces

When integrating a continuous function $f : [a, b] \to E$, where E is a Banach space, it usually suffices to use the Riemann integral. We shall be concerned frequently with E-valued functions defined on some abstract measure space (typically, a probability space), and in this context the notions of continuity and Riemann integral make no sense. For this reason we start this first lecture with generalising the Lebesgue integral to the E-valued setting.

1.1 Banach spaces

Throughout this lecture, E is a Banach space over the scalar field $\mathbb{K}$, which may be either $\mathbb{R}$ or $\mathbb{C}$ unless otherwise stated. The norm of an element $x \in E$ is denoted by $\|x\|_E$, or, if no confusion can arise, by $\|x\|$. We write

$$B_E = \{x \in E : \ \|x\| \leqslant 1\}$$

for the closed unit ball of E.

The *Banach space dual* of E is the vector space E^* of all continuous linear mappings from E to $\mathbb{K}$. This space is a Banach space with respect to the norm

$$\|x^*\|_{E^*} := \sup_{\|x\| \leqslant 1} |\langle x, x^* \rangle|.$$

Here, $\langle x, x^* \rangle := x^*(x)$ denotes the duality pairing of the elements $x \in E$ and $x^* \in E^*$. We shall simply write $\|x^*\|$ instead of $\|x^*\|_{E^*}$ if no confusion can arise. The elements of E^* are often called (linear) *functionals* on E. The *Hahn-Banach separation theorem* guarantees an ample supply of functionals on E: for every convex closed set $C \subseteq E$ and convex compact set $K \subseteq E$ such that $C \cap K = \varnothing$ there exist $x^* \in E^*$ and real numbers $a < b$ such that

$$\mathrm{Re}\langle x, x^* \rangle \leqslant a < b \leqslant \mathrm{Re}\langle y, x^* \rangle$$

for all $x \in C$ and $y \in K$. As is well-known, from this one derives the *Hahn-Banach extension theorem*: if F is a closed subspace of E, then for all $y^* \in F^*$ there exists an $x^* \in E^*$ such that $x^*|_F = y^*$ and $\|x^*\| = \|y^*\|$. This easily implies that for all $x \in E$ we have

$$\|x\| = \sup_{\|x^*\| \leqslant 1} |\langle x, x^* \rangle|.$$

A linear subspace F of E^* is called *norming* for a subset S of E if for all $x \in S$ we have

$$\|x\| = \sup_{\substack{x^* \in F \\ \|x^*\| \leqslant 1}} |\langle x, x^* \rangle|.$$

A subspace of E^* which is norming for E is simply called *norming*. The following lemma will be used frequently.

Lemma 1.1. *If E_0 is a separable subspace of E and F is a linear subspace of E^* which is norming for E_0, then F contains a sequence of unit vectors that is norming for E_0.*

Proof. Choose a dense sequence $(x_n)_{n=1}^\infty$ in E_0 and choose a sequence of unit vectors $(x_n^*)_{n=1}^\infty$ in F such that $|\langle x_n, x_n^* \rangle| \geqslant (1 - \varepsilon_n)\|x_n\|$ for all $n \geqslant 1$, where the numbers $0 < \varepsilon_n \leqslant 1$ satisfy $\lim_{n\to\infty} \varepsilon_n = 0$. The sequence $(x_n^*)_{n=1}^\infty$ is norming for E_0. To see this, fix an arbitrary $x \in E_0$ and let $\delta > 0$. Pick $n_0 \geqslant 1$ such that $0 < \varepsilon_{n_0} \leqslant \delta$ and $\|x - x_{n_0}\| \leqslant \delta$. Then,

$$\begin{aligned}(1-\delta)\|x\| \leqslant (1-\varepsilon_{n_0})\|x\| &\leqslant (1-\varepsilon_{n_0})\|x_{n_0}\| + (1-\varepsilon_{n_0})\delta \\ &\leqslant |\langle x_{n_0}, x_{n_0}^* \rangle| + \delta \leqslant |\langle x, x_{n_0}^* \rangle| + 2\delta.\end{aligned}$$

Since $\delta > 0$ was arbitrary it follows that $\|x\| \leqslant \sup_{n\geqslant 1} |\langle x, x_n^* \rangle|$. □

A linear subspace F of E^* is said to *separate the points* of a subset S of E if for every pair $x, y \in S$ with $x \neq y$ there exists an $x^* \in F$ with $\langle x, x^* \rangle \neq \langle y, x^* \rangle$. Clearly, norming subspaces separate points, but the converse need not be true.

Lemma 1.2. *If E_0 is a separable subspace of E and F is a linear subspace of E^* which separates the points of E_0, then F contains a sequence that separates the points of E_0.*

Proof. By the Hahn-Banach theorem, for each $x \in E_0 \setminus \{0\}$ there exists a vector $x^*(x) \in F$ such that $\langle x, x^*(x) \rangle \neq 0$. Defining

$$V_x := \{y \in E_0 \setminus \{0\} : \ \langle y, x^*(x) \rangle \neq 0\}$$

we obtain an open cover $\{V_x\}_{x \in E_0 \setminus \{0\}}$ of $E_0 \setminus \{0\}$. Since every open cover of a separable metric space admits a countable subcover it follows that there exists a sequence $(x_n)_{n=1}^\infty$ in $E_0 \setminus \{0\}$ such that $\{V_{x_n}\}_{n=1}^\infty$ covers $E_0 \setminus \{0\}$. Then the sequence $\{x^*(x_n)\}_{n=1}^\infty$ separates the points of E_0: indeed, every $x \in E_0 \setminus \{0\}$ belongs to some V_{x_n}, which means that $\langle x, x^*(x_n) \rangle \neq 0$. □

1.2 The Pettis measurability theorem

We begin with a discussion of weak and strong measurability of E-valued functions. The main result in this direction is the Pettis measurability theorem which states, roughly speaking, that an E-valued function is strongly measurable if and only if it is weakly measurable and takes its values in a separable subspace of E.

1.2.1 Strong measurability

Throughout this section $(A, \mathscr{A})$ denotes a *measurable space*, that is, A is a set and $\mathscr{A}$ is a *σ-algebra* in A, that is, a collection of subsets of A with the following properties:

1. $A \in \mathscr{A}$;
2. $B \in \mathscr{A}$ implies $\complement B \in \mathscr{A}$;
3. $B_1 \in \mathscr{A}, \ B_2 \in \mathscr{A}, \ \dots$ imply $\bigcup_{n=1}^{\infty} B_n \in \mathscr{A}$.

The first property guarantees that $\mathscr{A}$ is non-empty, the second expresses that $\mathscr{A}$ is closed under taking complements, and the third that $\mathscr{A}$ is closed under taking countable unions.

The *Borel σ-algebra* of a topological space T, notation $\mathscr{B}(T)$, is the smallest σ-algebra containing all open subsets of T. The sets in $\mathscr{B}(T)$ are the *Borel sets* of T.

Definition 1.3. *A function $f : A \to T$ is called* $\mathscr{A}$-measurable *if $f^{-1}(B) \in \mathscr{A}$ for all $B \in \mathscr{B}(T)$.*

The collection of all $B \in \mathscr{B}(T)$ satisfying $f^{-1}(B) \in \mathscr{A}$ is easily seen to be a σ-algebra. As a consequence, f is $\mathscr{A}$-measurable if and only if $f^{-1}(U) \in \mathscr{A}$ for all open sets U in T.

When T_1 and T_2 are topological spaces, a function $g : T_1 \to T_2$ is *Borel measurable* if $g^{-1}(B) \in \mathscr{B}(T_1)$ for all $B \in \mathscr{B}(T_2)$, that is, if g is $\mathscr{B}(T_1)$-measurable. Note that if $f : A \to T_1$ is $\mathscr{A}$-measurable and $g : T_1 \to T_2$ is Borel measurable, then the composition $g \circ f : A \to T_2$ is $\mathscr{A}$-measurable. By the above observation, every continuous function $g : T_1 \to T_2$ is Borel measurable.

It is a matter of experience that the notion of $\mathscr{A}$-measurability does not lead to a satisfactory theory from the point of view of vector-valued analysis. Indeed, the problem is that this definition does not provide the means for approximation arguments. It is for this reason that we shall introduce next another notion of measurability. We shall restrict ourselves to Banach space-valued functions, although some of the results proved below can be generalised to functions with values in metric spaces.

Let E be a Banach space and $(A, \mathscr{A})$ a measurable space. A function $f : A \to E$ is called *$\mathscr{A}$-simple* if it is of the form $f = \sum_{n=1}^{N} 1_{A_n} x_n$ with $A_n \in \mathscr{A}$ and $x_n \in E$ for all $1 \leqslant n \leqslant N$. Here 1_A denotes the *indicator function* of the set A, that is, $1_A(\xi) = 1$ if $\xi \in A$ and $1_A(\xi) = 0$ if $\xi \notin A$.

Definition 1.4. *A function $f : A \to E$ is* strongly $\mathscr{A}$-measurable *if there exists a sequence of $\mathscr{A}$-simple functions $f_n : A \to E$ such that* $\lim_{n\to\infty} f_n = f$ *pointwise on A.*

In order to be able to characterise strong $\mathscr{A}$-measurability of E-valued functions we introduce some terminology. A function $f : A \to E$ is called *separably valued* if there exists a separable closed subspace $E_0 \subseteq E$ such that $f(\xi) \in E_0$ for all $\xi \in A$, and *weakly $\mathscr{A}$-measurable* if the functions $\langle f, x^*\rangle : A \to \mathbb{K}$, $\langle f, x^*\rangle(\xi) := \langle f(\xi), x^*\rangle$, are $\mathscr{A}$-measurable for all $x^* \in E^*$.

Theorem 1.5 (Pettis measurability theorem, first version). *Let $(A, \mathscr{A})$ be a measurable space and let F be a norming subspace of E^*. For a function $f : A \to E$ the following assertions are equivalent:*

(1) *f is strongly $\mathscr{A}$-measurable;*
(2) *f is separably valued and $\langle f, x^*\rangle$ is $\mathscr{A}$-measurable for all $x^* \in E^*$;*
(3) *f is separably valued and $\langle f, x^*\rangle$ is $\mathscr{A}$-measurable for all $x^* \in F$.*

Proof. (1)⇒(2): Let $(f_n)_{n=1}^\infty$ be a sequence of $\mathscr{A}$-simple functions converging to f pointwise and let E_0 be the closed subspace spanned by the countably many values taken by these functions. Then E_0 is separable and f takes its values in E_0. Furthermore, each $\langle f, x^*\rangle$ is $\mathscr{A}$-measurable, being the pointwise limit of the $\mathscr{A}$-measurable functions $\langle f_n, x^*\rangle$.

(2)⇒(3): This implication is trivial.

(3)⇒(1): Using Lemma 1.1, choose a sequence $(x_n^*)_{n=1}^\infty$ of unit vectors in F that is norming for a separable closed subspace E_0 of E where f takes its values. By the $\mathscr{A}$-measurability of the functions $\langle f, x_n^*\rangle$, for each $x \in E_0$ the real-valued function

$$\xi \mapsto \|f(\xi) - x\| = \sup_{n\geqslant 1} |\langle f(\xi) - x, x_n^*\rangle|$$

is $\mathscr{A}$-measurable. Let $(x_n)_{n=1}^\infty$ be a dense sequence in E_0.

Define the functions $s_n : E_0 \to \{x_1, \dots, x_n\}$ as follows. For each $y \in E_0$ let $k(n, y)$ be the least integer $1 \leqslant k \leqslant n$ with the property that

$$\|y - x_k\| = \min_{1\leqslant j\leqslant n} \|y - x_j\|$$

and put $s_n(y) := x_{k(n,y)}$. Notice that

$$\lim_{n\to\infty} \|s_n(y) - y\| = 0 \quad \forall y \in E_0$$

since $(x_n)_{n=1}^\infty$ is dense in E_0. Now define $f_n : A \to E$ by

$$f_n(\xi) := s_n(f(\xi)), \qquad \xi \in A.$$

For all $1 \leqslant k \leqslant n$ we have

$$\begin{aligned}
&\{\xi \in A : \ f_n(\xi) = x_k\} \\
&\quad = \big\{\xi \in A : \ \|f(\xi) - x_k\| = \min_{1 \leqslant j \leqslant n} \|f(\xi) - x_j\|\big\} \\
&\qquad \cap \big\{\xi \in A : \ \|f(\xi) - x_l\| > \min_{1 \leqslant j \leqslant n} \|f(\xi) - x_j\| \text{ for } l = 1, \ldots, k-1\big\}.
\end{aligned}$$

Note that the sets on the right hand side are in $\mathscr{A}$. Hence each f_n is $\mathscr{A}$-simple, and for all $\xi \in A$ we have

$$\lim_{n\to\infty} \|f_n(\xi) - f(\xi)\| = \lim_{n\to\infty} \|s_n(f(\xi)) - f(\xi)\| = 0. \qquad \square$$

Corollary 1.6. *The pointwise limit of a sequence of strongly $\mathscr{A}$-measurable functions is strongly $\mathscr{A}$-measurable.*

Proof. Each function f_n takes its values in a separable subspace of E. Then f takes its values in the closed linear span of these spaces, which is separable. The measurability of the functions $\langle f, x^*\rangle$ follows by noting that each $\langle f, x^*\rangle$ is the pointwise limit of the measurable functions $\langle f_n, x^*\rangle$. $\square$

Corollary 1.7. *If an E-valued function f is strongly $\mathscr{A}$-measurable and $\phi : E \to F$ is continuous, where F is another Banach space, then $\phi \circ f$ is strongly $\mathscr{A}$-measurable.*

Proof. Choose simple functions f_n converging to f pointwise. Then $\phi \circ f_n \to \phi \circ f$ pointwise and the result follows from the previous corollary. $\square$

Proposition 1.8. *For a function $f : A \to E$, the following assertions are equivalent:*

(1) *f is strongly $\mathscr{A}$-measurable;*
(2) *f is separably valued and for all $B \in \mathscr{B}(E)$ we have $f^{-1}(B) \in \mathscr{A}$.*

Proof. (1)⇒(2): Let f be strongly $\mathscr{A}$-measurable. Then f is separably-valued. To prove that $f^{-1}(B) \in \mathscr{A}$ for all $B \in \mathscr{B}(E)$ it suffices to show that $f^{-1}(U) \in \mathscr{A}$ for all open sets U.

Let U be open and choose a sequence of $\mathscr{A}$-simple functions f_n converging pointwise to f. For $r > 0$ let $U_r = \{x \in U : \ d(x, \complement U) > r\}$, where $\complement U$ denotes the complement of U. Then $f_n^{-1}(U_r) \in \mathscr{A}$ for all $n \geqslant 1$, by the definition of an $\mathscr{A}$-simple function. Since

$$f^{-1}(U) = \bigcup_{m\geqslant 1} \bigcup_{n \geqslant 1} \bigcap_{k \geqslant n} f_k^{-1}(U_{\frac{1}{m}})$$

(the inclusion '$\subseteq$' being a consequence of the fact that U is open) it follows that also $f^{-1}(U) \in \mathscr{A}$.

(2)⇒(1): By assumption, f is $\mathscr{A}$-measurable, and therefore $\langle f, x^*\rangle$ is $\mathscr{A}$-measurable for all $x^* \in E^*$. The result now follows from the Pettis measurability theorem. $\square$

Thus if E is separable, then an E-valued function f is strongly $\mathscr{A}$-measurable if and only if it is $\mathscr{A}$-measurable.

1.2.2 Strong μ-measurability

So far, we have considered measurability properties of E-valued functions defined on a measurable space $(A, \mathscr{A})$. Next we consider functions defined on a *σ-finite measure space* $(A, \mathscr{A}, \mu)$, that is, μ is a non-negative measure on a measurable space $(A, \mathscr{A})$ and there exist sets $A^{(1)} \subseteq A^{(2)} \subseteq \dots$ in $\mathscr{A}$ with $\mu(A^{(n)}) < \infty$ for all $n \geqslant 1$ and $A = \bigcup_{n=1}^{\infty} A^{(n)}$.

A *μ-simple* function with values in E is a function of the form

$$f = \sum_{n=1}^{N} 1_{A_n} x_n,$$

where $x_n \in E$ and the sets $A_n \in \mathscr{A}$ satisfy $\mu(A_n) < \infty$.

We say that a property holds *μ-almost everywhere* if there exists a μ-null set $N \in \mathscr{A}$ such that the property holds on the complement $\complement N$ of N.

Definition 1.9. *A function $f : A \to E$ is* strongly μ-measurable *if there exists a sequence $(f_n)_{n \geqslant 1}$ of μ-simple functions converging to f μ-almost everywhere.*

Using the σ-finiteness of μ it is easy to see that every strongly $\mathscr{A}$-measurable function is strongly μ-measurable. Indeed, if f is strongly $\mathscr{A}$-measurable and $\lim_{n\to\infty} f_n = f$ pointwise with each f_n an $\mathscr{A}$-simple functions, then also $\lim_{n\to\infty} 1_{A^{(n)}} f_n = f$ pointwise, where $A = \bigcup_{n=1}^{\infty} A^{(n)}$ as before, and each $1_{A^{(n)}} f_n$ is μ-simple. The next proposition shows that in the converse direction, every strongly μ-measurable function is equal μ-almost everywhere to a strongly $\mathscr{A}$-measurable function.

Let us call two functions which agree μ-almost everywhere *μ-versions* of each other.

Proposition 1.10. *For a function $f : A \to E$ the following assertions are equivalent:*

(1) *f is strongly μ-measurable;*
(2) *f has a μ-version which is strongly $\mathscr{A}$-measurable.*

Proof. (1)⇒(2): Suppose that $f_n \to f$ outside the null set $N \in \mathscr{A}$, with each f_n μ-simple. Then we have $\lim_{n\to\infty} 1_{\complement N} f_n = 1_{\complement N} f$ pointwise on A, and since the functions $1_{\complement N} f_n$ are $\mathscr{A}$-simple, $1_{\complement N} f$ is strongly $\mathscr{A}$-measurable. It follows that $1_{\complement N} f$ is a strongly $\mathscr{A}$-measurable μ-version of f.

(2)⇒(1): Let $\widetilde{f}$ be a strongly $\mathscr{A}$-measurable μ-version of f and let $N \in \mathscr{A}$ be a null set such that $f = \widetilde{f}$ on $\complement N$. If $(\widetilde{f}_n)_{n=1}^{\infty}$ is a sequence of $\mathscr{A}$-simple functions converging pointwise to $\widetilde{f}$, then $\lim_{n\to\infty} \widetilde{f}_n = f$ on $\complement N$, which means that $\lim_{n\to\infty} \widetilde{f}_n = f$ μ-almost everywhere.

Write $A = \bigcup_{n=1}^{\infty} A^{(n)}$ with $A^{(1)} \subseteq A^{(2)} \subseteq \dots \in \mathscr{A}$ and $\mu(A^{(n)}) < \infty$ for all $n \geqslant 1$. The functions $f_n := 1_{A^{(n)}} \widetilde{f}_n$ are μ-simple and we have $\lim_{n\to\infty} f_n = f$ μ-almost everywhere. $\square$

We say that f is *μ-separably valued* if there exists a closed separable subspace E_0 of E such that $f(\xi) \in E_0$ for μ-almost all $\xi \in A$, and *weakly μ-measurable* if $\langle f, x^* \rangle$ is μ-measurable for all $x^* \in E^*$.

Theorem 1.11 (Pettis measurability theorem, second version). *Let $(A, \mathscr{A}, \mu)$ be a σ-finite measure space and let F be a norming subspace of E^*. For a function $f : A \to E$ the following assertions are equivalent:*

(1) *f is strongly μ-measurable;*
(2) *f is μ-separably valued and $\langle f, x^* \rangle$ is μ-measurable for all $x^* \in E^*$;*
(3) *f is μ-separably valued and $\langle f, x^* \rangle$ is μ-measurable for all $x^* \in F$.*

Proof. The implication (1)⇒(2) follows the corresponding implication in Theorem 1.5 combined with Proposition 1.10, and (2)⇒(3) is trivial. The implication (3)⇒(1) is proved in the same way as the corresponding implication in Theorem 1.5, observing that this time the functions f_n have μ-versions $\widetilde{f_n}$ that are $\mathscr{A}$-simple. If we write $A = \bigcup_{n=1}^{\infty} A^{(n)}$ as before with each $A^{(n)}$ of finite μ-measure, the functions $1_{A^{(n)}} \widetilde{f_n}$ are μ-simple and converge to f μ-almost everywhere. □

By combining Proposition 1.10 with Corollaries 1.6 and 1.7 we obtain:

Corollary 1.12. *The μ-almost everywhere limit of a sequence of strongly μ-measurable E-valued functions is strongly μ-measurable.*

Corollary 1.13. *If an E-valued function f is strongly μ-measurable and $\phi : E \to F$ is continuous, where F is another Banach space, then $\phi \circ f$ is strongly μ-measurable.*

The following result will be applied frequently.

Corollary 1.14. *If f and g are strongly μ-measurable E-valued functions which satisfy $\langle f, x^* \rangle = \langle g, x^* \rangle$ μ-almost everywhere for every $x^* \in F$, where F is subspace of E^* separating the points of E. Then $f = g$ μ-almost everywhere.*

Proof. Both f and g take values in a separable closed subspace E_0 μ-almost everywhere, say outside the μ-null set N. Since E_0 is separable, by Lemma 1.2 some countable family of elements $(x_n^*)_{n=1}^{\infty}$ in F separates the points of E_0. Since $\langle f, x_n^* \rangle = \langle g, x_n^* \rangle$ outside a μ-null set N_n, we conclude that f and g agree outside the μ-null set $N \cup \bigcup_{n=1}^{\infty} N_n$. □

1.3 The Bochner integral

The Bochner integral is the natural generalisation of the familiar Lebesgue integral to the vector-valued setting.

Throughout this section, $(A, \mathscr{A}, \mu)$ is a σ-finite measure space.

1.3.1 The Bochner integral

Definition 1.15. *A function $f : A \to E$ is μ-*Bochner integrable *if there exists a sequence of μ-simple functions $f_n : A \to E$ such that the following two conditions are met:*

(1) $\lim_{n\to\infty} f_n = f$ *μ-almost everywhere;*

(2) $\displaystyle\lim_{n\to\infty} \int_A \|f_n - f\| \, d\mu = 0.$

Note that f is strongly μ-measurable. The functions $\|f_n - f\|$ are μ-measurable by Corollary 1.13.

It follows trivially from the definitions that every μ-simple function is μ-Bochner integrable. For $f = \sum_{n=1}^N 1_{A_n} x_n$ we put

$$\int_A f \, d\mu := \sum_{n=1}^N \mu(A_n) x_n.$$

It is routine to check that this definition is independent of the representation of f. If f is μ-Bochner integrable, the limit

$$\int_A f \, d\mu := \lim_{n\to\infty} \int_A f_n \, d\mu$$

exists in E and is called the *Bochner integral* of f with respect to μ. It is routine to check that this definition is independent of the approximating sequence $(f_n)_{n=1}^\infty$.

If f is μ-Bochner integrable and g is a μ-version of f, then g is μ-Bochner integrable and the Bochner integrals of f and g agree. In particular, in the definition of the Bochner integral the function f need not be everywhere defined; it suffices that f be μ-almost everywhere defined.

If f is μ-Bochner integrable, then for all $x^* \in E^*$ we have the identity

$$\Big\langle \int_A f \, d\mu, x^* \Big\rangle = \int_A \langle f, x^* \rangle \, d\mu.$$

For μ-simple functions this is trivial, and the general case follows by approximating f with μ-simple functions.

Proposition 1.16. *A strongly μ-measurable function $f : A \to E$ is μ-Bochner integrable if and only if*

$$\int_A \|f\| \, d\mu < \infty,$$

and in this case we have

$$\Big\| \int_A f \, d\mu \Big\| \leqslant \int_A \|f\| \, d\mu.$$

Proof. First assume that f is μ-Bochner integrable. If the μ-simple functions f_n satisfy the two assumptions of Definition 1.15, then for large enough n we obtain

$$\int_A \|f\|\,d\mu \leqslant \int_A \|f - f_n\|\,d\mu + \int_A \|f_n\|\,d\mu < \infty.$$

Conversely let f be a strongly μ-measurable function satisfying $\int_A \|f\|\,d\mu < \infty$. Let g_n be μ-simple functions such that $\lim_{n\to\infty} g_n = f$ μ-almost everywhere and define

$$f_n := 1_{\{\|g_n\| \leqslant 2\|f\|\}} g_n.$$

Then f_n is μ-simple, and clearly we have $\lim_{n\to\infty} f_n = f$ μ-almost everywhere. Since we have $\|f_n\| \leqslant 2\|f\|$ pointwise, the dominated convergence theorem can be applied and we obtain

$$\lim_{n\to\infty} \int_A \|f_n - f\|\,d\mu = 0.$$

The final inequality is trivial for μ-simple functions, and the general case follows by approximation. □

As a simple application, note that if $f : A \to E$ is μ-Bochner integrable, then for all $B \in \mathscr{A}$ the truncated function $1_B f : A \to E$ is μ-Bochner integrable, the restricted function $f|_B : B \to E$ is $\mu|_B$-Bochner integrable, and we have

$$\int_A 1_B f\,d\mu = \int_B f|_B\,d\mu|_B.$$

Henceforth, both integrals will be denoted by $\int_B f\,d\mu$.

In the following result, $\mathrm{conv}(V)$ denotes the convex hull of a subset $V \subseteq E$, i.e., the set of all finite sums $\sum_{j=1}^k \lambda_j x_j$ with $\lambda_j \geqslant 0$ satisfying $\sum_{j=1}^k \lambda_j = 1$ and $x_j \in V$ for $j = 1, \ldots, k$. The closure of this set is denoted by $\overline{\mathrm{conv}}(V)$.

Proposition 1.17. *Let $f : A \to E$ be a μ-Bochner integrable function. If $\mu(A) = 1$, then*

$$\int_A f\,d\mu \in \overline{\mathrm{conv}}\{f(\xi) :\ \xi \in A\}.$$

Proof. Let us say that an element $x \in E$ is *strictly separated* from a set $V \subseteq E$ by a functional $x^* \in E^*$ if there exists a number $\delta > 0$ such that

$$|\mathrm{Re}\langle x, x^*\rangle - \mathrm{Re}\langle v, x^*\rangle| \geqslant \delta \quad \forall v \in V.$$

The Hahn-Banach separation theorem asserts that if V is convex and $x \notin \overline{V}$, then there exists a functional $x^* \in E^*$ which strictly separates x from V.

For $x^* \in E^*$, let

$$m(x^*) := \inf\{\mathrm{Re}\langle f(\xi), x^*\rangle :\ \xi \in A\},$$
$$M(x^*) := \sup\{\mathrm{Re}\langle f(\xi), x^*\rangle :\ \xi \in A\},$$

allowing these values to be $-\infty$ and ∞, respectively. Then, since $\mu(A) = 1$,

$$\mathrm{Re}\Big\langle \int_A f \, d\mu, x^* \Big\rangle = \int_A \mathrm{Re}\langle f, x^* \rangle \, d\mu \in [m(x^*), M(x^*)].$$

This shows that $\int_A f \, d\mu$ cannot be strictly separated from the convex set $\mathrm{conv}\{f(\xi) : \ \xi \in A\}$ by functionals in E^*. Therefore the conclusion follows by an application of the Hahn-Banach separation theorem. $\square$

As a rule of thumb, results from the theory of Lebesgue integration carry over to the Bochner integral as long as there are no non-negativity assumptions involved. For example, there are no analogues of the Fatou lemma and the monotone convergence theorem, but we do have the following analogue of the dominated convergence theorem:

Proposition 1.18 (Dominated convergence theorem). *Let $f_n : A \to E$ be a sequence of functions, each of which is μ-Bochner integrable. Assume that there exist a function $f : A \to E$ and a μ-Bochner integrable function $g : A \to \mathbb{K}$ such that:*

(1) $\lim_{n\to\infty} f_n = f$ *μ-almost everywhere;*
(2) $\|f_n\| \leqslant |g|$ *μ-almost everywhere.*

Then f is μ-Bochner integrable and we have

$$\lim_{n\to\infty} \int_A \|f_n - f\| \, d\mu = 0.$$

In particular we have

$$\lim_{n\to\infty} \int_A f_n \, d\mu = \int_A f \, d\mu.$$

Proof. We have $\|f_n - f\| \leqslant 2|g|$ μ-almost everywhere, and therefore the result follows from the scalar dominated convergence theorem. $\square$

It is immediate from the definition of the Bochner integral that if $f : A \to E$ is μ-Bochner integrable and T is a bounded linear operator from E into another Banach space F, then $Tf : A \to F$ is μ-Bochner integrable and

$$T \int_A f \, d\mu = \int_A Tf \, d\mu.$$

This identity has a useful extension to a suitable class of unbounded operators. A linear operator T, defined on a linear subspace $\mathscr{D}(T)$ of E and taking values in another Banach space F, is said to be *closed* if its *graph*

$$\mathscr{G}(T) := \{(x, Tx) : \ x \in \mathscr{D}(T)\}$$

is a closed subspace of $E \times F$. If T is closed, then $\mathscr{D}(T)$ is a Banach space with respect to the *graph norm*

$$\|x\|_{\mathscr{D}(T)} := \|x\| + \|Tx\|$$

and T is a bounded operator from $\mathscr{D}(T)$ to E.

The *closed graph theorem* asserts that if $T : E \to F$ is a closed operator with domain $\mathscr{D}(T) = E$, then T is bounded.

Theorem 1.19 (Hille). *Let $f : A \to E$ be μ-Bochner integrable and let T be a closed linear operator with domain $\mathscr{D}(T)$ in E taking values in a Banach space F. Assume that f takes its values in $\mathscr{D}(T)$ μ-almost everywhere and the μ-almost everywhere defined function $Tf : A \to F$ is μ-Bochner integrable. Then $\int_A f\,d\mu \in \mathscr{D}(T)$ and*

$$T\int_A f\,d\mu = \int_A Tf\,d\mu.$$

Proof. We begin with a simple observation which is a consequence of Proposition 1.16 and the fact that the coordinate mappings commute with Bochner integrals: if E_1 and E_2 are Banach spaces and $f_1 : A \to E_1$ and $f_2 : A \to E_2$ are μ-Bochner integrable, then $f = (f_1, f_2) : A \to E_1 \times E_2$ is μ-Bochner integrable and

$$\int_A f\,d\mu = \Big(\int_A f_1\,d\mu,\ \int_A f_2\,d\mu\Big).$$

Turning to the proof of the proposition, by the preceding observation the function $g : A \to E \times F$, $g(\xi) := (f(\xi), Tf(\xi))$, is μ-Bochner integrable. Moreover, since g takes its values in the graph $\mathscr{G}(T)$, we have $\int_A g(\xi)\,d\mu(\xi) \in \mathscr{G}(T)$. On the other hand,

$$\int_A g(\xi)\,d\mu(\xi) = \Big(\int_A f(\xi)\,d\mu(\xi),\ \int_A Tf(\xi)\,d\mu(\xi)\Big).$$

The result follows by combining these facts. $\square$

We finish this section with a result on integration of E-valued functions which may fail to be Bochner integrable.

Theorem 1.20 (Pettis). *Let $(A, \mathscr{A}, \mu)$ be a finite measure space and let $1 < p < \infty$ be fixed. If $f : A \to E$ is strongly μ-measurable and satisfies $\langle f, x^*\rangle \in L^p(A)$ for all $x^* \in E^*$, then there exists a unique $x_f \in E$ satisfying*

$$\langle x_f, x^*\rangle = \int_A \langle f, x^*\rangle\,d\mu.$$

Proof. We may assume that f is strongly $\mathscr{A}$-measurable.

It is easy to see that the linear mapping $S : E^* \to L^p(A)$, $Sx^* := \langle f, x^*\rangle$ is closed. Hence S is bounded by the closed graph theorem.

Put $A_n := \{\|f\| \leqslant n\}$. Then $A_n \in \mathscr{A}$ and by Proposition 1.16 the integral $\int_{A_n} f\,d\mu$ exists as a Bochner integral in E. For all $x^* \in E^*$ and $n \geqslant m$, by Hölder's inequality we have

$$\Big|\Big\langle \int_{A_n\setminus A_m} f\,d\mu(x), x^*\Big\rangle\Big| \leqslant \big(\mu(A_n\setminus A_m)\big)^{\frac{1}{q}}\Big(\int_A |\langle f, x^*\rangle|^p\,d\mu(x)\Big)^{\frac{1}{p}}$$
$$\leqslant \big(\mu(A_n\setminus A_m)\big)^{\frac{1}{q}}\|S\|\,\|x^*\|.$$

Taking the supremum over all $x^* \in E^*$ with $\|x^*\| \leqslant 1$ we see that

$$\limsup_{m,n\to\infty}\Big\|\int_{A_n\setminus A_m} f\,d\mu\Big\| \leqslant \lim_{m,n\to\infty}\big(\mu(A_n\setminus A_m)\big)^{\frac{1}{q}}\|S\| = 0.$$

Hence the limit $x_f := \lim_{n\to\infty}\int_{A_n} f\,d\mu$ exists in E. Clearly,

$$\langle x_f, x^*\rangle = \lim_{n\to\infty}\int_{A_n}\langle f, x^*\rangle\,d\mu = \int_A \langle f, x^*\rangle\,d\mu$$

for all $x^* \in E^*$. Uniqueness is obvious by the Hahn-Banach theorem. $\square$

The element x_f is called the *Pettis integral* of f with respect to μ.

1.3.2 The Lebesgue-Bochner spaces $L^p(A;E)$

Let $(A,\mathscr{A},\mu)$ be a σ-finite measure space. For $1 \leqslant p < \infty$ we define $L^p(A;E)$ as the linear space of all (equivalence classes of) strongly μ-measurable functions $f : A \to E$ for which

$$\int_A \|f\|^p\,d\mu < \infty,$$

identifying functions which are equal μ-almost everywhere. Endowed with the norm

$$\|f\|_{L^p(A;E)} := \Big(\int_A \|f\|^p\,d\mu\Big)^{\frac{1}{p}},$$

the space $L^p(A;E)$ is a Banach space; the proof for the scalar case carries over verbatim. Repeating the second part of the proof of Proposition 1.16 we see that the μ-simple functions are dense in $L^p(A;E)$.

Note that the elements of $L^1(A;E)$ are precisely the equivalence classes of μ-Bochner integrable functions.

We define $L^\infty(A;E)$ as the linear space of all (equivalence classes of) strongly μ-measurable functions $f : A \to E$ for which there exists a number $r \geqslant 0$ such that $\mu\{\|f\| > r\} = 0$. Endowed with the norm

$$\|f\|_{L^\infty(A;E)} := \inf\Big\{r \geqslant 0 :\ \mu\{\|f\| > r\} = 0\Big\},$$

the space $L^\infty(A;E)$ is a Banach space.

Example 1.21. For each $1 \leqslant p \leqslant \infty$, the Fubini theorem establishes a canonical isometric isomorphism

$$L^p(A_1; L^p(A_2;E)) \simeq L^p(A_1 \times A_2; E),$$

which is uniquely defined by the mapping $1_{A_1} \otimes (1_{A_2} \otimes x) \mapsto 1_{A_1\times A_2} \otimes x$ and linearity. Here $1_A \otimes y \in L^p(A;F)$ is defined by $(1_A \otimes y)(\xi) := 1_A(\xi)y$.

1.4 Exercises

1. (!)[1] Let E be a separable Banach space and let C be a closed convex subset of E. Prove that there exists a sequence $(x_n^*)_{n=1}^\infty$ of norm one elements in E^* and a sequence $(F_n)_{n=1}^\infty$ of closed sets in $\mathbb{K}$ such that
$$C = \bigcap_{n=1}^{\infty} \big\{x \in E : \langle x, x_n^* \rangle \in F_n\big\}.$$
Hint: Separate C from the elements of a dense sequence in its complement $\complement C$ using the Hahn-Banach separation theorem.

2. Prove that the function $f : (0,1) \to L^\infty(0,1)$ defined by $f(t) = 1_{(0,t)}$ is weakly measurable, but not strongly measurable.
Hint: In the real case, elements in the dual of $L^\infty(0,1)$ can be decomposed into a positive and negative part. The complex case, consider real and imaginary parts separately.

3. Let E be a Banach space and $f : [0,1] \to E$ a continuous function. Show that f is Bochner integrable, and that its Bochner integral coincides with its Riemann integral.

4. A familiar theorem of calculus asserts that
$$\frac{d}{dx}\int_0^1 f(x,y)\,dy = \int_0^1 \frac{\partial f}{\partial x}(x,y)\,dy$$
for suitable functions $f : [0,1]\times[0,1] \to \mathbb{K}$. Show that this is a special case of Hille's theorem and deduce a set of rigorous conditions for this result.

5. Let $(A, \mathscr{A}, \mu)$ be a σ-finite measure space and let $1 \leqslant p,q \leqslant \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Let E be a Banach space and let F be a norming subspace of E^*. Prove that $L^q(A;F)$ is a norming subspace of $(L^p(A;E))^*$ with respect to the duality pairing
$$\langle f, g\rangle = \int_A \langle f(\xi), g(\xi)\rangle \, d\mu(\xi), \quad f \in L^p(A;E),\ g \in L^q(A;E^*).$$
Hint: First find simple functions in $L^q(A;F)$ which norm simple functions in $L^p(A;E)$.

Notes. The material in this lecture is standard and can be found in many textbooks. More complete discussions of measurability in Banach spaces can be found in the monographs by Bogachev [8] and Vakhania, Tarieladze, Chobanyan [105]. Systematic expositions of the Bochner integral are presented in Arendt, Batty, Hieber, Neubrander [3], Diestel and Uhl [36], Dunford and Schwartz [37] and Lang [66].

[1] Results proved in the exercises marked with (!) are needed in the main text.

The Pettis measurability theorems 1.5 and 1.11 as well as Theorem 1.20 are due to Pettis [90]. Both versions of the Pettis measurability theorem remain correct if we only assume f to be weakly measurable with respect to the functionals from a subspace F of E^* which separates the points of E, but the proof is more involved. For more details we refer to [36] and [105].