5

γ-Radonifying operators

Experience has taught that many results in analysis involving L^2-techniques, such as the Plancherel theorem in harmonic analysis and the Itô isometry in stochastic analysis, carry over without difficulty to the Hilbert space-valued setting. Often this fact characterises Hilbert spaces among all Banach spaces.

It has only been recently realised that many results do generalise beyond the Hilbert space case if one does three things:

- Replace 'functions' by 'γ-radonifying (integral) operators';
- Replace 'uniform boundedness' by 'γ-boundedness';
- Replace 'orthogonality' by 'unconditionality'.

This paradigm has had enormous impact in the areas of (parabolic) evolution equations and harmonic analysis, more recently, in the theory of stochastic (parabolic) evolution equations.

In this lecture we address the first item in the list and investigate properties of γ-radonifying operators. These operators will be used in the next lecture to give necessary and sufficient conditions for stochastic integrability, the main idea being that the L^2-norms occurring in the Itô isometry are replaced by the γ-radonifying norms of associated integral operators.

5.1 γ-Summing operators

We begin with a discussion of the class of γ-summing operators. In the next section, γ-radonifying operators are defined as the γ-summing operators which can be approximated in the γ-summing norm by finite rank operators.

Continuing the notational conventions of the previous lectures, $(\gamma_n)_{n=1}^{\infty}$ always denotes a Gaussian sequence, H is a Hilbert space (with inner product $[\cdot,\cdot]$), and E is a Banach space. Although we have made the standing assumption that all spaces are real, most results of this lecture extend with only minor changes to complex scalars.

Definition 5.1. *A linear operator $S : H \to E$ is called* γ-summing *if for some (equivalently, for all) $1 \leqslant p < \infty$,*

$$\|S\|_{\gamma_p^\infty(H,E)} := \sup \Big(\mathbb{E}\Big\| \sum_{j=1}^k \gamma_j \, Sh_j \Big\|^p\Big)^{\frac{1}{p}} < \infty,$$

the supremum being taken over all finite orthonormal systems $\{h_1, \dots, h_k\}$.

By considering singletons $\{h\}$ we see that every γ-summing operator is bounded and satisfies $\|S\| \leqslant \|S\|_{\gamma_p^\infty(H,E)}$.

With respect to any one of the norms $S \mapsto \|S\|_{\gamma_p^\infty(H,E)}$, which are mutually equivalent by the Kahane-Khintchine inequalities, the linear space $\gamma^\infty(H,E)$ of all γ-summing operators from H to E is a normed space. Unless otherwise stated we shall write

$$\|S\|_{\gamma^\infty(H,E)} := \|S\|_{\gamma_2^\infty(H,E)}.$$

Proposition 5.2. *The space $\gamma^\infty(H,E)$ is a Banach space.*

Proof. If $(S_n)_{n=1}^\infty$ is Cauchy in $\gamma^\infty(H,E)$, then $\sup_{n\geqslant 1} \|S_n\|_{\gamma^\infty(H,E)} < \infty$. Let us denote this supremum by C. Since $(S_n)_{n=1}^\infty$ is a Cauchy sequence in $\mathscr{L}(H,E)$ it tends to an operator S in $\mathscr{L}(H,E)$. We will prove that $S \in \gamma^\infty(H,E)$ and that $\lim_{n\to\infty} S_n = S$ in the norm of $\gamma^\infty(H,E)$.

If $\{h_1, \dots, h_k\}$ is an orthonormal system in H, then by Fatou's lemma,

$$\mathbb{E}\Big\| \sum_{j=1}^k \gamma_j \, Sh_j \Big\|^2 \leqslant \liminf_{n\to\infty} \mathbb{E}\Big\| \sum_{j=1}^k \gamma_j \, S_n h_j \Big\|^2 \leqslant C.$$

It follows that $S \in \gamma^\infty(H,E)$ and $\|S\|_{\gamma^\infty(H,E)} \leqslant C$.

Next we check that $\lim_{n\to\infty} S_n = S$ in the norm of $\gamma^\infty(H,E)$. Given $\varepsilon > 0$, we choose $N \geqslant 1$ such that $\|S_n - S_m\|_{\gamma^\infty(H,E)} < \varepsilon$ for all $m,n \geqslant N$. Let $\{h_1, \dots, h_k\}$ be an orthonormal system in H. By another application of the Fatou lemma,

$$\mathbb{E}\Big\| \sum_{j=1}^k \gamma_j \, (S_n - S)h_j \Big\|^2 \leqslant \liminf_{m\to\infty} \mathbb{E}\Big\| \sum_{j=1}^k \gamma_j \, (S_n - S_m)h_j \Big\|^2 < \varepsilon^2.$$

Therefore, $\|S_n - S\|_{\gamma^\infty(H,E)} \leqslant \varepsilon$ for all $n \geqslant N$. □

Proposition 5.3 (γ-Fatou lemma). *Let $(S_n)_{n=1}^\infty$ be a bounded sequence in $\gamma_p^\infty(H,E)$. If $S \in \mathscr{L}(H,E)$ is an operator such that*

$$\lim_{n\to\infty} \langle S_n h, x^* \rangle = \langle Sh, x^* \rangle \quad \forall h \in H, \ x^* \in E^*,$$

then $S \in \gamma^\infty(H,E)$ and for all $1 \leqslant p < \infty$ we have

$$\|S\|_{\gamma_p^\infty(H,E)} \leqslant \liminf_{n\to\infty} \|S_n\|_{\gamma_p^\infty(H,E)}.$$

Proof. Let $\{h_1, \ldots, h_k\}$ be an orthonormal system in H. Let $(x_n^*)_{n=1}^\infty$ be a sequence of unit vectors in E^* which is norming for the linear span of $\{Sh_1, \ldots, Sh_k\}$. For all $M \geqslant 1$ we have, by the Fatou lemma,

$$\begin{aligned}\mathbb{E} \sup_{m=1,\ldots,M} \Big|\Big\langle \sum_{j=1}^k \gamma_j\, Sh_j, x_m^*\Big\rangle\Big|^p &\leqslant \liminf_{n\to\infty} \mathbb{E} \sup_{m=1,\ldots,M} \Big|\Big\langle \sum_{j=1}^k \gamma_j\, S_n h_j, x_m^*\Big\rangle\Big|^p \\ &\leqslant \liminf_{n\to\infty} \|S_n\|^p_{\gamma_p^\infty(H,E)}.\end{aligned}$$

Taking the limit $M \to \infty$ we obtain, by the monotone convergence theorem,

$$\mathbb{E}\Big\| \sum_{j=1}^k \gamma_j\, Sh_j\Big\|^p \leqslant \liminf_{n\to\infty} \|S_n\|^p_{\gamma_p^\infty(H,E)}$$

and the proposition follows. □

The next result shows that the class of γ-summing operators enjoys a certain ideal property:

Proposition 5.4 (Ideal property I). *Let $S \in \gamma^\infty(H,E)$. If H' is another Hilbert space and E' another Banach space, then for all $T \in \mathscr{L}(H',H)$ and $U \in \mathscr{L}(E,E')$ we have $UST \in \gamma^\infty(H',E')$ and for all $1 \leqslant p < \infty$ we have*

$$\|UST\|_{\gamma_p^\infty(H',E')} \leqslant \|U\|\, \|S\|_{\gamma_p^\infty(H,E)} \|T\|.$$

Proof. It suffices to prove that $ST \in \gamma^\infty(H',E)$ and $\|ST\|_{\gamma^\infty(H',E)} \leqslant \|S\|_{\gamma^\infty(H,E)}\|T\|$, the assertions concerning U being trivial.

Let $\{h_1', \ldots, h_k'\}$ be an orthonormal system in H'. We denote by $\widetilde{H}'$ and $\widetilde{H}$ the spans in H' and H of $\{h_1', \ldots, h_k'\}$ and $\{Th_1', \ldots, Th_k'\}$, respectively. Let $\widetilde{E}$ be the span in E of $\{STh_1', \ldots, STh_k'\}$. Then S and T restrict to operators $\widetilde{S} : \widetilde{H} \to \widetilde{E}$ and $\widetilde{T} : \widetilde{H'} \to \widetilde{H}$.

Let $\{h_1, \ldots, h_N\}$ be an orthonormal basis for $\widetilde{H}$. For all $x^* \in \widetilde{E}^*$ we have

$$\sum_{j=1}^k \langle \widetilde{S}\widetilde{T}h_j', x^*\rangle^2 = \|\widetilde{T}^*\widetilde{S}^*x^*\|^2_{\widetilde{H}'} \leqslant \|\widetilde{T}^*\|^2\, \|\widetilde{S}^*x^*\|^2_{\widetilde{H}} = \|\widetilde{T}\|^2 \sum_{n=1}^N \langle \widetilde{S}h_n, x^*\rangle^2.$$

Hence, by Theorem 3.9,

$$\mathbb{E}\Big\|\sum_{j=1}^k \gamma_j\, STh_j'\Big\|^p \leqslant \|T\|^p\, \mathbb{E}\Big\|\sum_{n=1}^N \gamma_n\, Sh_n\Big\|^p \leqslant \|T\|^p\, \|S\|^p_{\gamma_p^\infty(H,E)}$$

and the result follows. □

As a corollary we observe that we may ignore the kernel of S:

Corollary 5.5. *If $S \in \gamma^\infty(H,E)$ and H_0 is a closed subspace of H containing $(\ker S)^\perp$, then the restriction S_0 of S to H_0 belongs to $\gamma^\infty(H_0,E)$ and for all $1 \leqslant p < \infty$,*

$$\|S_0\|_{\gamma_p^\infty(H_0,E)} = \|S\|_{\gamma_p^\infty(H,E)}.$$

Proof. The only nontrivial thing to prove is the inequality $\|S\|_{\gamma_p^\infty(H,E)} \leqslant \|S_0\|_{\gamma_p^\infty(H_0,E)}$. Let P_0 be the orthonormal projection of H onto H_0. Then $S = S_0 P_0$ and the desired inequality follows from Proposition 5.4. □

We are now in a position to prove the following characterisation of γ-summing operators in terms of orthonormal bases. We formulate the result for separable infinite-dimensional spaces; for finite-dimensional spaces the same result holds with a slightly simpler proof.

Proposition 5.6. *If H is separable and $(h_n)_{n=1}^\infty$ is an orthonormal basis for H, then an operator $S \in \mathscr{L}(H,E)$ belongs to $\gamma^\infty(H,E)$ if and only if for some (equivalently, for all) $1 \leqslant p < \infty$,*

$$\sup_{N\geqslant 1} \mathbb{E}\Big\| \sum_{n=1}^N \gamma_n \, Sh_n \Big\|^p < \infty.$$

In this case,

$$\|S\|^p_{\gamma_p^\infty(H,E)} = \sup_{N\geqslant 1} \mathbb{E}\Big\| \sum_{n=1}^N \gamma_n \, Sh_n \Big\|^p.$$

Proof. Let $\{h_1', \dots, h_k'\}$ be an orthonormal system in H. For $K \geqslant 1$ let P_K denote the orthogonal projection onto the span of $\{h_1, \dots, h_K\}$. For all $x^* \in E^*$ and $K \geqslant k$ we have

$$\sum_{j=1}^k \langle SP_K h_j', x^*\rangle^2 \leqslant \|P_K S^* x^*\|^2 = \sum_{n=1}^K \langle Sh_n, x^*\rangle^2.$$

Let $1 \leqslant p < \infty$. From Theorem 3.9 it follows that

$$\mathbb{E}\Big\| \sum_{j=1}^k \gamma_j \, SP_K h_j' \Big\|^p \leqslant \mathbb{E}\Big\| \sum_{n=1}^K \gamma_n \, Sh_n \Big\|^p \leqslant \sup_{N\geqslant 1} \mathbb{E}\Big\| \sum_{n=1}^N \gamma_n \, Sh_n \Big\|^p.$$

Hence by Fatou's lemma,

$$\mathbb{E}\Big\| \sum_{j=1}^k \gamma_j \, Sh_j' \Big\|^p \leqslant \liminf_{K\to\infty} \mathbb{E}\Big\| \sum_{j=1}^k \gamma_j \, SP_K h_j' \Big\|^p \leqslant \sup_{N\geqslant 1} \mathbb{E}\Big\| \sum_{n=1}^N \gamma_n \, Sh_n \Big\|^p.$$

It follows that

$$\|S\|^p_{\gamma_p^\infty(H,E)} \leqslant \sup_{N\geqslant 1} \mathbb{E}\Big\| \sum_{n=1}^N \gamma_n \, Sh_n \Big\|^p.$$

The converse inequality trivially holds and the proof is complete. □

5.2 γ-Radonifying operators

When h is an element of a Hilbert space H and x an element of a Banach space E, we denote by $h \otimes x$ the operator in $\mathscr{L}(H,E)$ defined by

$$(h \otimes x)h' := [h, h']x, \qquad h' \in H.$$

An operator in $\mathscr{L}(H,E)$ is said to be of *finite rank* if it is a linear combination of operators of the above form. It is a trivial observation that every finite rank operator from H to E belongs to $\gamma^\infty(H,E)$. In fact we have:

Lemma 5.7. *If $S = \sum_{n=1}^N h_n \otimes x_n$ is a finite rank operator with $h_1, \dots, h_N$ orthonormal in H and $x_1, \dots, x_N \in E$ arbitrary, then $S \in \gamma^\infty(H,E)$ and for all $1 \leqslant p < \infty$ we have*

$$\|S\|^p_{\gamma_p^\infty(H,E)} = \mathbb{E}\Big\| \sum_{n=1}^N \gamma_n\, x_n \Big\|^p.$$

Proof. Testing on $h_1, \dots, h_N$ gives the inequality '$\geqslant$'. To prove the inequality '$\leqslant$' let P be the orthogonal projection from H onto the span $\widetilde{H}$ of $\{h_1, \dots, h_N\}$ and define $\widetilde{S} \in \mathscr{L}(\widetilde{H}, E)$ by $\widetilde{S} = SP^*$. The inequality then follows from Proposition 5.4 applied to $S = \widetilde{S}P$, and Proposition 5.6 applied to $\widetilde{S}$. ☐

In view of this observation the following definition makes sense.

Definition 5.8. *The space $\gamma(H,E)$ is defined as the closure in $\gamma^\infty(H,E)$ of all finite rank operators. The operators in $\gamma(H,E)$ are called* γ-radonifying.

By definition, $\gamma(H,E)$ is a Banach space with respect to the norm inherited from $\gamma^\infty(H,E)$. For notational simplicity, for $R \in \gamma(H,E)$ we shall write

$$\|R\|_{\gamma(H,E)} := \|R\|_{\gamma^\infty(H,E)}$$

and more generally $\|R\|_{\gamma_p(H,E)} := \|R\|_{\gamma_p^\infty(H,E)}$ for $1 \leqslant p < \infty$.

A bounded operator is *compact* if the image of the unit ball is relatively compact. Every γ-radonifying operator R is compact: if $\lim_{n\to\infty} \|R_n - R\|_{\gamma(H,E)} = 0$ with each R_n of finite rank, then $\lim_{n\to\infty} \|R_n - R\| = 0$ and the claim follows since each R_n is compact. Here we use that the uniform limit of a sequence of compact operators is compact.

Without proof we mention the following theorem, which rephrases a famous result due to Hoffmann-Jorgensen and Kwapień on the almost sure convergence of random sums whose partial sums are almost surely bounded.

Theorem 5.9 (Hoffmann-Jorgensen and Kwapień). *Let H be an infinite-dimensional Hilbert space. For a Banach space E the following assertions are equivalent:*

(1) $\gamma^\infty(H,E) = \gamma(H,E)$;
(2) E *does not contain a closed subspace isomorphic to* c_0.

An explicit example of an operator which is γ-summing but not γ-radonifying is the multiplication operator $R : \ell^2 \to c_0$ defined by

$$R\big((\alpha_n)_{n=1}^\infty\big) := (\alpha_n/\sqrt{\log(n+1)})_{n=1}^\infty$$

The proof of this statement depends on some subtle estimates for Gaussian sums and is omitted.

As an immediate consequence of Definition 5.8, every $R \in \gamma(H,E)$ is 'supported' on a separable closed subspace of H:

Proposition 5.10. *If* $R \in \gamma(H,E)$, *then* $(\ker(R))^\perp$ *is separable.*

Proof. Suppose that $R = \lim_{n\to\infty} R_n$ in $\gamma(H,E)$ with each R_n of finite rank, say $R_n h = \sum_{j=1}^{k_n} [h, h_{jn}] x_{jn}$. Let H_0 denote the closed linear span of all vectors h_{jn}, $n \geqslant 1$, $1 \leqslant j \leqslant k_n$. Then H_0 is separable and if $h \perp H_0$, then $R_n h = 0$ for all $n \geqslant 1$ and consequently $Rh = 0$. □

The ideal property of $\gamma^\infty(H,E)$ carries over to $\gamma(H,E)$:

Proposition 5.11 (Ideal property II). *Let* $R \in \gamma(H,E)$. *If* H' *is another Hilbert space and* E' *another Banach space, then for all* $T \in \mathscr{L}(H',H)$ *and* $U \in \mathscr{L}(E,E')$ *we have* $URT \in \gamma(H',E')$ *and for all* $1 \leqslant p < \infty$ *we have*

$$\|URT\|_{\gamma_p(H',E')} \leqslant \|U\|\,\|R\|_{\gamma_p(H,E)}\|T\|.$$

Proof. If R is of finite rank, then also URT is of finite rank. Moreover if $\lim_{n\to\infty} R_n = R$ in $\gamma_p^\infty(H,E)$, then $\|U(R-R_n)T\|_{\gamma_p^\infty(H',E')} \leqslant \|U\|\,\|R - R_n\|_{\gamma_p^\infty(H,E)}\|T\|$ and therefore $URT \in \gamma(H',E')$. The estimate follows from the corresponding estimate for the γ-summing norms. □

We mention a simple but useful application.

Proposition 5.12 (Convergence by right multiplication). *If* H_1 *and* H_2 *are Hilbert spaces and* $S_1, S_2, \dots$ *and* S *are operators in* $\mathscr{L}(H_1,H_2)$ *satisfying* $S^*h = \lim_{n\to\infty} S_n^* h$ *for all* $h \in H_2$, *then for all* $R \in \gamma(H_2,E)$ *we have* $\lim_{n\to\infty} RS_n = RS$ *in* $\gamma(H_1,E)$.

Proof. The uniform boundedness principle implies that $\sup_{n\geqslant 1}\|S_n\| < \infty$. Hence, by the estimate $\|RT\|_{\gamma(H_1,E)} \leqslant \|R\|_{\gamma(H_2,E)}\|T\|$ for $T \in \mathscr{L}(H_1,H_2)$, it suffices to consider finite rank operators $R \in \gamma(H_2,E)$. Fix such an operator, say $R = \sum_{m=1}^M h_m' \otimes x_m$, and let $(h_j)_{j=1}^k$ be orthonormal in H_1. Then, by the triangle inequality in $L^2(\Omega;E)$,

$$\begin{aligned}\Big(\mathbb{E}\Big\|\sum_{j=1}^{k}\gamma_j R(S-S_n)h_j\Big\|^2\Big)^{\frac12} &= \Big(\mathbb{E}\Big\|\sum_{m=1}^{M}\sum_{j=1}^{k}\gamma_j[S^*h'_m - S_n^*h'_m, h_j]x_m\Big\|^2\Big)^{\frac12}\\ &\leqslant \sum_{m=1}^{M}\Big(\mathbb{E}\Big|\sum_{j=1}^{k}\gamma_j[S^*h'_m - S_n^*h'_m, h_j]\Big|^2\Big)^{\frac12}\|x_m\|\\ &= \sum_{m=1}^{M}\Big(\sum_{j=1}^{k}|[S^*h'_m - S_n^*h'_m, h_j]|^2\Big)^{\frac12}\|x_m\|\\ &\leqslant \sum_{m=1}^{M}\|S^*h'_m - S_n^*h'_m\|\|x_m\|.\end{aligned}$$

Hence,

$$\|R(S-S_n)\|_{\gamma(H_1,E)} \leqslant \sum_{m=1}^{M}\|S^*h'_m - S_n^*h'_m\|\|x_m\|,$$

and by assumption the right hand side tends to zero as $n\to\infty$. □

Here is a simple illustration:

Example 5.13. Consider an operator $R\in\gamma(H,E)$ and let $(h_n)_{n=1}^\infty$ be an orthonormal basis for $(\ker(R))^\perp$. Let P_n denote the orthogonal projection in H onto the span of $\{h_1,\dots,h_n\}$. Then $\lim_{n\to\infty}RP_n = R$ in $\gamma(H,E)$.

Proposition 5.14 (Measurability). *Let $(A,\mathscr{A},\mu)$ be a σ-finite measure space and H a separable Hilbert space. For a function $\Phi : A\to\gamma(H,E)$ define $\Phi h : A\to E$ by $(\Phi h)(\xi) := \Phi(\xi)h$ for $h\in H$. The following assertions are equivalent:*

(1) *Φ is strongly μ-measurable;*
(2) *Φh is strongly μ-measurable for all $h\in H$.*

Proof. It suffices to prove that (2) implies (1). If $(h_n)_{n=1}^\infty$ is an orthonormal basis for H, then with the notations of the Example 5.13 for all $\xi\in A$ we have

$$\Phi(\xi) = \lim_{n\to\infty}\Phi(\xi)P_n = \lim_{n\to\infty}\sum_{j=1}^{n}[\,\cdot\,,h_j]\Phi(\xi)h_j,$$

with convergence in the norm of $\gamma(H,E)$. □

We proceed with the main result of this section which states, loosely speaking, that an operator is γ-radonifying if and only if it maps orthonormal sequences into γ-summable sequences.

Theorem 5.15. *If H is separable, then for an operator $R\in\mathscr{L}(H,E)$ the following assertions are equivalent:*

(1) *$R\in\gamma(H,E)$;*

(2) *for all orthonormal bases* $(h_n)_{n=1}^\infty$ *in* H *and all* $1 \leqslant p < \infty$ *the sum* $\sum_{n=1}^\infty \gamma_n\, Rh_n$ *converges in* $L^p(\Omega; E)$*;*
(3) *for some orthonormal basis* $(h_n)_{n=1}^\infty$ *in* H *and some* $1 \leqslant p < \infty$ *the sum* $\sum_{n=1}^\infty \gamma_n\, Rh_n$ *converges in* $L^p(\Omega; E)$*.*

In this situation, the sums in (2) *and* (3) *converge almost surely and define an* E*-valued Gaussian random variable with covariance operator* RR^**. For all orthonormal bases* $(h_n)_{n=1}^\infty$ *of* H *and* $1 \leqslant p < \infty$ *we have*

$$\|R\|_{\gamma_p(H,E)}^p = \mathbb{E}\Big\| \sum_{n=1}^\infty \gamma_n\, Rh_n \Big\|^p.$$

Proof. (1)⇒(2): Fix $R \in \gamma(H,E)$ and $1 \leqslant p < \infty$, and let $(h_n)_{n=1}^\infty$ be an orthonormal basis of H. Let P_n denote the orthogonal projection in H onto the linear span of $\{h_1, \dots, h_n\}$. By Proposition 5.12 we have $\lim_{n\to\infty} RP_n = R$ in $\gamma(H,E)$, and by Proposition 5.6 for all $m < n$ we have

$$\mathbb{E}\Big\| \sum_{j=m+1}^n \gamma_j Rh_j \Big\|^p = \|RP_n - RP_m\|_{\gamma_p(H,E)}^p.$$

Since the right-hand side tends to 0 as $m, n \to \infty$, this proves the convergence of the sum $\sum_{n=1}^\infty \gamma_n Rh_n$ in $L^p(\Omega; E)$.

(2)⇒(3): This implication is trivial.

(3)⇒(1): With the notations as before, by Proposition 5.6 we have

$$\lim_{m,n\to\infty} \|RP_n - RP_m\|_{\gamma_p(H,E)}^p = \lim_{m,n\to\infty} \mathbb{E}\Big\| \sum_{j=m+1}^n \gamma_j Rh_j \Big\|^p = 0.$$

It follows that $(RP_n)_{n=1}^\infty$ is a Cauchy sequence in $\gamma(H,E)$. Its limit equals R, since $\lim_{n\to\infty} RP_n h = Rh$ for all $h \in H$.

This proves the equivalence of (1), (2), (3) as well as the final identity. The almost sure convergence in (2) and (3) follows from the Itô-Nisio theorem. □

We are now ready to characterise Gaussian covariance operators in terms of γ-radonifying operators.

Theorem 5.16. *Suppose* $Q \in \mathscr{L}(E^*, E)$ *and* $R \in \mathscr{L}(H, E)$ *satisfy* $Q = RR^*$*. The following assertions are equivalent:*

(1) Q *is a Gaussian covariance operator;*
(2) $R \in \gamma(H,E)$*.*

If X *is an* E*-valued random variable with covariance operator* Q*, then*

$$\mathbb{E}\|X\|^p = \|R\|_{\gamma_p(H,E)}^p, \quad 1 \leqslant p < \infty.$$

Proof. (1)⇒(2): Let X be E-valued Gaussian with covariance Q. By Theorem 4.12 the Hilbert space H_X is separable, and from the identities $\mathbb{E}\langle X, x^*\rangle^2 = \langle Qx^*, x^*\rangle = \|R^*x^*\|^2$ it follows that the mapping $j_X : \langle X, x^*\rangle \mapsto R^*x^*$ extends uniquely to an isometry from H_X onto $\widetilde{H} := \overline{\operatorname{ran}(R^*)}$.

Let $(\gamma_n)_{n=1}^\infty$ be an orthonormal basis of H_X and put $h_n := j_X\gamma_n$. Then $(h_n)_{n=1}^\infty$ is an orthonormal basis of $\widetilde{H}$. By the Karhunen-Loève theorem (Theorem 4.12) we have $X = \sum_{n=1}^\infty \gamma_n i_X \gamma_n$, where $i_X : H_X \to E$ is given by

$$i_X\langle X, x^*\rangle = Qx^* = RR^*x^* = Rj_X\langle X, x^*\rangle.$$

It follows that $i_X = Rj_X$, and therefore

$$\sum_{n=1}^\infty \gamma_n Rh_n = \sum_{n=1}^\infty \gamma_n Rj_X\gamma_n = \sum_{n=1}^\infty \gamma_n i_X \gamma_n = X. \tag{5.1}$$

Let $\widetilde{R}$ denote the restriction of R to $\widetilde{H}$. By the implication (3)⇒(1) of Theorem 5.15, we have proved that $\widetilde{R} \in \gamma(\widetilde{H}, E)$. Since $R = 0$ on $\widetilde{H}^\perp = \ker(R)$, we have $R = \widetilde{R}P$ where P is the orthogonal projection from H onto $\widetilde{H}$. From Proposition 5.11 we infer that $R \in \gamma(H, E)$.

(2)⇒(1): Using Proposition 5.10, let $(h_n)_{n=1}^\infty$ be an orthonormal basis of the separable Hilbert space $\widetilde{H} = (\ker(R))^\perp$. The E-valued random variable $X := \sum_{n=1}^\infty \gamma_n Rh_n$ is Gaussian and has covariance operator $RR^* = Q$.

The final identity follows from (5.1) and Theorem 5.15. □

We continue with a domination result for γ-radonifying operators.

Theorem 5.17 (Domination). *Let H_1 and H_2 be Hilbert spaces and let $R_1 \in \mathscr{L}(H_1, E)$ and $R_2 \in \mathscr{L}(H_2, E)$. If*

$$\|R_1^*x^*\| \leqslant \|R_2^*x^*\| \qquad \forall x^* \in E^*,$$

then $R_2 \in \gamma(H_2, E)$ implies $R_1 \in \gamma(H_1, E)$ and for all $1 \leqslant p < \infty$ we have

$$\|R_1\|_{\gamma_p(H_1,E)} \leqslant \|R_2\|_{\gamma_p(H_2,E)}.$$

Proof. Put $\widetilde{H}_1 = \overline{\operatorname{ran}(R_1^*)}$ and $\widetilde{H}_2 = \overline{\operatorname{ran}(R_2^*)}$. By assumption, the mapping $j : R_2^*x^* \mapsto R_1^*x^*$ extends to a contraction from $\widetilde{H}_2$ to $\widetilde{H}_1$. For all $h_1 \in \widetilde{H}_1$ and $x^* \in E^*$ we have $\langle R_2 j^* h_1, x^*\rangle = [h_1, jR_2^*x^*] = [h_1, R_1^*x^*] = \langle R_1 h_1, x^*\rangle$. Hence $R_2 j^* P = R_1$, where P is the orthogonal projection of H_1 onto $\widetilde{H}_1$, and the result follows from Proposition 5.11. □

Corollary 5.18 (Covariance domination). *Let X_1 and X_2 be E-valued Gaussian random variables satisfying*

$$\mathbb{E}\langle X_1, x^*\rangle^2 \leqslant \mathbb{E}\langle X_2, x^*\rangle^2 \qquad \forall x^* \in E^*.$$

Then, for all $1 \leqslant p < \infty$,

$$\mathbb{E}\|X_1\|^p \leqslant \mathbb{E}\|X_2\|^p.$$

Proof. Combine Theorems 5.16 and 5.17. □

5.3 Examples of γ-radonifying operators

For certain range spaces, a complete characterisation of γ-radonifying operators can be given in non-probabilistic terms. The simplest example occurs when the range space is a Hilbert space.

Theorem 5.19 (Operators into Hilbert spaces). *If E is a Hilbert space, then $R \in \gamma(H, E)$ if and only if $R \in \mathscr{L}_2(H, E)$, and in this case we have*

$$\|R\|_{\gamma(H,E)} = \|R\|_{\mathscr{L}_2(H,E)}.$$

Here, $\mathscr{L}_2(H, E)$ denotes the space of all *Hilbert-Schmidt operators* from H to E, that is, completion of the space of all finite rank operators $R \in \mathscr{L}(H, E)$ with respect to the norm

$$\|R\|^2_{\mathscr{L}_2(H,E)} := \sum_{n=1}^{N} \|x_n\|^2,$$

where $R = \sum_{n=1}^{N} h_n \otimes x$ with the $h_1, \dots, h_N$ orthonomal in H.

Proof. This is trivial, since for $R = \sum_{n=1}^{N} h_n \otimes x$ with $h_1, \dots, h_N$ orthonomal in H we have

$$\|R\|^2_{\gamma(H,E)} = \mathbb{E}\Big\| \sum_{n=1}^{N} \gamma_n x_n \Big\|^2 = \sum_{n=1}^{N} \|x_n\|^2 = \|R\|^2_{\mathscr{L}_2(H,E)}.$$

□

In what follows we shall use the notation $A \eqsim_p B$ to express the fact that there exist constants $0 < c \leqslant C < \infty$, depending only on p, such that $cA \leqslant B \leqslant CA$. The notation $A \lesssim_p B$ has a similar meaning.

The next result shows that an operator from a separable Hilbert space into an L^p-space is γ-radonifying if and only if it satisfies a *square function estimate.*

Theorem 5.20 (Operators into L^p-spaces). *Let $(A, \mathscr{A}, \mu)$ be a σ-finite measure space, let H be a separable Hilbert space, and let $1 \leqslant p < \infty$. For an operator $R \in \mathscr{L}(H, L^p(A))$ the following assertions are equivalent:*

(1) *$R \in \gamma(H, L^p(A))$;*
(2) *For all orthonormal bases $(h_n)_{n=1}^{\infty}$ of H the function $\big(\sum_{n=1}^{\infty} |Rh_n|^2\big)^{\frac{1}{2}}$ belongs to $L^p(A)$;*
(3) *For some orthonormal basis $(h_n)_{n=1}^{\infty}$ of H the function $\big(\sum_{n=1}^{\infty} |Rh_n|^2\big)^{\frac{1}{2}}$ belongs to $L^p(A)$.*

In this case we have $\|R\|_{\gamma(H,L^p(A))} \eqsim_p \big\|\big(\sum_{n=1}^{\infty} |Rh_n|^2\big)^{\frac{1}{2}}\big\|$.

Proof. Applying the identity $\sum_{n=M}^N |c_n|^2 = \mathbb{E}|\sum_{n=M}^N c_n\gamma_n|^2$ with $c_n = f_n(\xi)$, $\xi \in A$, then applying the scalar Kahane-Khintchine inequality, then Fubini's theorem, and finally the Kahane-Khintchine inequality in $L^p(A)$, for all $M \leqslant N$ and $f_M, \dots, f_N \in L^p(A)$ we obtain

$$\Big\|\Big(\sum_{n=M}^N |f_n|^2\Big)^{\frac12}\Big\|_p = \Big\|\Big(\mathbb{E}\Big|\sum_{n=M}^N \gamma_n f_n\Big|^2\Big)^{\frac12}\Big\|_p \eqsim_p \Big\|\Big(\mathbb{E}\Big|\sum_{n=M}^N \gamma_n f_n\Big|^p\Big)^{\frac1p}\Big\|_p$$
$$= \Big(\mathbb{E}\Big\|\sum_{n=M}^N \gamma_n f_n\Big\|_p^p\Big)^{\frac1p} \eqsim_p \Big(\mathbb{E}\Big\|\sum_{n=M}^N \gamma_n f_n\Big\|_p^2\Big)^{\frac12}.$$

The equivalences (1)$\Leftrightarrow$(2), (1)$\Leftrightarrow$(3), and the final two-sided estimate now follow by taking $f_n := Rh_n$, where $(h_n)_{n=1}^\infty$ is an orthonormal basis of H. $\square$

Here is a neat application:

Corollary 5.21. *Let $(A, \mathscr{A}, \mu)$ be a finite measure space and H a separable Hilbert space. For all $T \in \mathscr{L}(H, L^\infty(A))$ and $1 \leqslant p < \infty$ we have $T \in \gamma(H, L^p(A))$ and*

$$\|T\|_{\gamma(H,L^p(A))} \lesssim_p \|T\|_{\mathscr{L}(H,L^\infty(A))}.$$

Proof. Let $(h_n)_{n=1}^\infty$ be an orthonormal basis of H. Fixing $N \geqslant 1$ and $c \in \mathbb{R}^N$, for μ-almost all $\xi \in A$ we have

$$\Big|\sum_{n=1}^N c_n(Th_n)(\xi)\Big| \leqslant \Big\|\sum_{n=1}^N c_n Th_n\Big\|_\infty$$
$$\leqslant \|T\|_{\mathscr{L}(H,L^\infty(A))}\Big\|\sum_{n=1}^N c_n h_n\Big\|_H = \|T\|_{\mathscr{L}(H,L^\infty(A))}\|c\|.$$

Taking the supremum over a countable dense set in the unit ball of $\mathbb{R}^N$ we obtain the following estimate, valid for μ-almost all $\xi \in A$:

$$\Big(\sum_{n=1}^N |(Th_n)(\xi)|^2\Big)^{\frac12} \leqslant \|T\|_{\mathscr{L}(H,L^\infty(A))}.$$

Now apply Theorem 5.20. $\square$

Every $f \in L^p(A; H)$ defines a bounded operator $R_f \in \mathscr{L}(H, L^p(A))$ by putting

$$(R_f h)(\xi) := [f(\xi), h], \qquad \xi \in A,\ h \in H.$$

The next result shows that $R_f \in \gamma(H, L^p(A))$, and that every $R \in \gamma(H, L^p(A))$ is of this form; this gives an alternative description of $\gamma(H, L^p(A))$. For later use it will be useful to formulate this result in a more slightly more general form. The isomorphism $\gamma(H, L^p(A)) \simeq L^p(A; H)$ is obtained in the special case $E = \mathbb{R}$ in the next theorem.

Theorem 5.22 (γ-Fubini isomorphism). *Let $(A, \mathscr{A}, \mu)$ be a σ-finite measure space, let H be a Hilbert space, and let $1 \leqslant p < \infty$. The mapping $U : L^p(A; \gamma(H, E)) \to \mathscr{L}(H, L^p(A; E))$ defined by*

$$((Uf)h)(\xi) := f(\xi)h, \qquad \xi \in A,\ h \in H,$$

defines an isometry U from $L^p(A; \gamma_p(H, E))$ onto $\gamma_p(H, L^p(A; E))$.

Proof. Let $f \in L^p(A; \gamma_p(H, E))$ be a simple function of the form $f = \sum_{m=1}^M 1_{A_m} \otimes U_m$, where the operators U_m are of the form $\sum_{n=1}^N h_n \otimes x_{mn}$ for some orthonormal system $\{h_1, \dots, h_N\}$ in H. Let $\widetilde{H}$ be the span of $\{h_1, \dots, h_N\}$. Using Corollary 5.5, Lemma 5.7, and Fubini's theorem we obtain

$$\begin{aligned}
\|Uf\|_{\gamma_p(H, L^p(A;E))} &= \|Uf\|_{\gamma_p(\widetilde{H}, L^p(A;E))} \\
&= \Big(\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n\,(Uf)h_n\Big\|^p_{L^p(A;E)}\Big)^{\frac1p} = \Big(\int_A \mathbb{E}\Big\|\sum_{n=1}^N \gamma_n\, f h_n\Big\|^p\,d\mu\Big)^{\frac1p} \\
&= \Big(\int_A \|f\|^p_{\gamma_p(\widetilde{H}, E)}\,d\mu\Big)^{\frac1p} = \Big(\int_A \|f\|^p_{\gamma_p(H,E)}\,d\mu\Big)^{\frac1p} \\
&= \|f\|_{L^p(A;\gamma_p(H,E))}.
\end{aligned}$$

Since the simple functions f of the above form are dense, these estimates imply that U extends to an isomorphism of $L^p(A; \gamma_p(H, E))$ onto a closed subspace of $\gamma_p(H, L^p(A; E))$. To show that this operator is surjective it is enough to show that its range is dense. But

$$U\Big(\sum_{n=1}^N 1_{A_n} \otimes \Big(\sum_{k=1}^K h_k \otimes x_{kn}\Big)\Big) = \sum_{k=1}^K h_k \otimes \Big(\sum_{n=1}^N 1_{A_n} \otimes x_{kn}\Big),$$

for all $A_n \in \mathscr{A}$ with $\mu(A_n) < \infty$, orthonormal $h_1, \dots, h_K \in H$, and arbitrary $x_{kn} \in E$. The elements on the right hand side are dense in $\gamma_p(H, L^p(A; E))$. □

The final example is important in the theory of Brownian motion.

Theorem 5.23 (Indefinite integration). *The operator $I_T : L^2(0, T) \to C[0, T]$ defined by*

$$(I_T f)(t) := \int_0^t f(s)\,ds, \qquad f \in L^2(0, T),\ t \in [0, T],$$

is γ-radonifying.

A proof is outlined in Exercise 5.

5.4 Exercises

1. Let $1 \leqslant p < \infty$. Determine for which scalar sequences $a = (a_n)_{n=1}^\infty$ the diagonal operator $u_n \mapsto a_n u_n$ defines a γ-radonifying operator from ℓ^2 to ℓ^p. Here $u_n = (0, \dots, 0, 1, 0, \dots)$, with the '1' in the n-th entry, is the n-th unit vector of ℓ^p.

   Hint: Apply Theorem 5.20.

2. Let $(h_n)_{n=1}^\infty$ be a *Hilbert sequence* in a Hilbert space H, that is, there exists a constant $C \geqslant 0$ such that for all scalars $\alpha_1, \dots, \alpha_N$,

$$\Big\| \sum_{n=1}^N \alpha_n h_n \Big\| \leqslant C \Big(\sum_{n=1}^N |\alpha_n|^2 \Big)^{\frac{1}{2}}.$$

   Show that if $R \in \gamma(H, E)$, then $\sum_{n=1}^\infty \gamma_n R h_n$ converges in $L^2(\Omega; E)$ and

$$\mathbb{E} \Big\| \sum_{n=1}^\infty \gamma_n R h_n \Big\|^2 \leqslant C^2 \|R\|_{\gamma(H,E)}^2.$$

3. (!) Let $(A, \mathscr{A}, \mu)$ be a σ-finite measure space and define $\Phi : A \to \gamma(H, E)$ by $\Phi := \phi \otimes U$, where $\phi \in L^2(A)$ and $U \in \gamma(H, E)$. Prove that the operator $R_\Phi : L^2(A; H) \to E$,

$$R_\Phi f := \int_A \Phi(\xi) f(\xi) \, d\mu(\xi) = \int_A \phi(\xi) U f(\xi) \, d\mu(\xi)$$

   belongs to $\gamma(L^2(A; H), E)$ with norm

$$\|R_\Phi\|_{\gamma(L^2(A;H),E)} = \|\phi\|_{L^2(A)} \|U\|_{\gamma(H,E)}.$$

4. (!) Let again $(A, \mathscr{A}, \mu)$ be a σ-finite measure space. For μ-simple functions $\phi : A \to \gamma(H, E)$ we define $R_\phi : L^2(A; H) \to E$ by

$$R_\phi f := \int_A \phi(\xi) f(\xi) \, d\mu(\xi).$$

   By the previous exercise, $R_\phi \in \gamma(L^2(A; H), E)$.

   a) Prove that if E has type 2, then the mapping $\phi \mapsto R_\phi$ has a unique extension to a continuous embedding $L^2(A; \gamma(H, E)) \hookrightarrow \gamma(L^2(A; H), E)$.
   Hint: Consider simple functions whose values are finite rank operators.

   b) Prove the following converse for $H = \mathbb{R}$ and $A = (0, 1)$: if the mapping $\phi \mapsto R_\phi$, defined for simple functions $\phi : (0, 1) \to E$, extends to a bounded operator R from $L^2(0, 1; E)$ to $\gamma(L^2(0, 1), E)$, then E has type 2.
   Hint: Consider step functions.

Examples of Banach space with type 2 are Hilbert spaces and L^p-spaces for $2 \leqslant p < \infty$ (see Exercise 3.4).

Remark: The following 'dual' result also holds, with a similar proof: if E has cotype 2, then the mapping $R_\Phi \mapsto \Phi$ is well defined and has a unique extension to a continuous embedding $\gamma(L^2(A;H),E) \hookrightarrow L^2(A;\gamma(H,E))$. Conversely, if the mapping $R_\phi \mapsto \phi$ extends to a continuous embedding $\gamma(L^2(0,1),E) \hookrightarrow L^2(0,1;E)$, then E has cotype 2.

5. We present a proof of Theorem 5.23 due to Ciesielski. Another proof will be outlined in the next lecture.
   Without loss of generality we take $T = 1$ and set $I_T = I_1 =: I$.
   a) Let γ be a standard Gaussian variable. Prove that
   $$\mathbb{P}\{|\gamma| > t\} \leqslant \frac{2}{t\sqrt{2\pi}} e^{-\frac{1}{2}t^2}.$$
   b) Let $(\gamma_n)_{n=1}^\infty$ be a Gaussian sequence. Use a) and the Borel-Cantelli lemma to prove that for any $\alpha > 1$, almost surely we have
   $$|\gamma_n| \leqslant \sqrt{2\alpha \log(n+1)}$$
   for all but at most finitely many $n \geqslant 1$.

   The *Haar basis* of $L^2(0,1)$ is defined by $h_1 \equiv 1$ and $h_n := \phi_{jk}$ for $n \geqslant 2$, where $n = 2^j + k$ with $j = 0, 1, 2, \ldots$ and $k = 1, \ldots, 2^j$, and
   $$\phi_{jk} = 2^{j/2} 1_{\left(\frac{k-1}{2^j}, \frac{k-1/2}{2^j}\right)} - 2^{j/2} 1_{\left(\frac{k-1/2}{2^j}, \frac{k}{2^j}\right)}.$$
   c) Prove that $(h_n)_{n=1}^\infty$ is an orthonormal basis for $L^2(0,1)$.
   d) Prove that, almost surely, the sum $\sum_{n=1}^\infty \gamma_n (Ih_n)(t)$ converges absolutely and uniformly with respect to $t \in [0,1]$.
   Hint: Use b) together with the observation that for all $j \geqslant 0$ and $t \in [0,1]$, we have $I\phi_{jk}(t) = 0$ for all but at most one $k \in \{1, \ldots, 2^j\}$ and that for this k we have $0 \leqslant I\phi_{jk}(t) \leqslant 2^{-j/2-1}$.
   e) Combine d) with Theorem 5.15 and the final assertion of the Itô-Nisio theorem to deduce that I is γ-radonifying from $L^2(0,1)$ to $C[0,1]$.

Notes. The class of γ-summing operators was introduced by Linde and Pietsch [70]. A detailed study of γ-summing operators is presented in Diestel, Jarchow, Tonge [35, Chapter 12]. The notion of a γ-radonifying operator is older and has its origins in the work of Gross [43]. Frequently H is assumed to be separable and the equivalent conditions (2) and (3) of Theorem 5.15 are taken as the definition of a γ-radonifying operator.

To explain the name 'γ-radonifying', let us first introduce some terminology. A probability measure μ on a topological space E is called a *Radon measure* if for all Borel sets $B \subseteq E$ and $\varepsilon > 0$ there exists a compact subset $K \subseteq B$ such that $\mu(B \setminus K) < \varepsilon$. If μ is the distribution of a random variable with values in a Banach space E, then μ is a Radon measure on E; this can be deduced from Proposition 2.3 and some additional thought. Now Theorem 5.16 can be interpreted as saying that *a bounded operator* $T : H \to E$ *is* γ*-radonifying if and only if it maps the finitely additive standard Gaussian measure* γ_H (see the discussion in the Notes of Lecture 4) *to a Radon measure* μ *on* E (viz., the Gaussian measure μ with covariance operator TT^*).

In some sense, the class of γ-radonifying operators is the Gaussian analogue of the class of p-absolutely summing operators, a fact with indicates its importance from the point of view of Banach space theory. The intermediate notion of p-radonifying operators has been studied thoroughly by the French school. We refer to Vakhania, Tarieladze, Chobanyan [105] for more information and references to the literature.

The γ-Fatou lemma is essentially due to Kalton and Weis [58]. The authors used γ-radonifying norms to extend certain results in spectral theory involving square functions to the Banach space-valued setting. Propositions 5.12 and 5.14 are taken from [82].

Corollary 5.18 can be improved as follows: if X and Y are E-valued Gaussian random variables satisfying $\mathbb{E}\langle X, x^*\rangle^2 \leqslant \mathbb{E}\langle Y, x^*\rangle^2$ for all $x^* \in E^*$ and $C \subseteq E$ is closed, convex, and symmetric, then

$$\mathbb{P}\{X \notin C\} \leqslant \mathbb{P}\{Y \notin C\}. \tag{5.2}$$

This result is due to Anderson [2].

Without proof we mention the following result, essentially due to Neidhardt [85], which can be proved using Prokhorov's theorem (Theorem 2.19) and a Anderson's inequality (see the Notes of Lecture 4):

Theorem 5.24 (γ-Dominated convergence). *Suppose* $(T_n)_{n=1}^\infty$ *is a sequence in* $\mathscr{L}(H, E)$ *and assume that there exist* $R \in \gamma(H, E)$ *and* $T \in \mathscr{L}(H, E)$ *such that for all* $x^* \in E^*$ *we have:*

(1) $\|T_n^* x^*\| \leqslant \|R^* x^*\|$,
(2) $\lim_{n\to\infty} T_n^* x^* = T^* x^*$ *in* H.

Then $T \in \gamma(H, E)$ *and* $\lim_{n\to\infty} T_n = T$ *in the norm of* $\gamma(H, E)$.

The main idea is as follows. If $\mathscr{X}$ is a family E-valued Gaussian random variables whose covariances are dominated by R in the sense of (1), then by using Anderson's inequality (5.2) it can be shown that $\mathscr{X}$ is uniformly tight, and Prokhorov's theorem can be applied.

The square function characterisation of γ-radonifying operators into L^p-spaces of Theorem 5.20 is taken from [83]. For $p = 2$, Corollary 5.21 asserts that if $(A, \mathscr{A}, \mu)$ is a finite measure space, then every bounded operator from H to $L^2(A)$ which factors through $L^\infty(A)$ is Hilbert-Schmidt. In its present form, the corollary was suggested to us by Haase. A related result is contained in [83]; see also [106, Lemma 8.7.2]. The γ-Fubini isomorphism is taken from [82].

Exercise 2 is from [44]. Exercise 4 goes back to HOFFMANN-JORGENSEN and PISIER [49] and ROSIŃSKI and SUCHANECKI [96]. In its present form it was noted in [84]. From KWAPIEŃ's theorem (see the notes of Lecture 3 and Exercise 4) we deduce that the mapping $\phi \mapsto R_\phi$ induces an isomorphism of Banach spaces

$$L^2(0,1;E) \simeq \gamma(L^2(0,1),E)$$

if and only if E is isomorphic to a Hilbert space.

The proof of Theorem 5.23 sketched in Exercise 5 is due to CIESIELSKI. He used the uniform convergence of the sum $\sum_{n=1}^{\infty} \gamma_n I_T h_n$ to give an elementary proof that a Brownian motion admits a version with continuous trajectories; we return to this point in the lext lecture. According to Theorem 5.16, the operator $I_T I_T^*$ is the covariance of a Gaussian measure w on $C[0,T]$, the so-called *Wiener measure*. A straightforward computation shows that

$$\langle I_T I_T^* \delta_s, \delta_t \rangle = \int_{C[0,T]} f(s)f(t)\,dw(f) = \min\{s,t\}, \quad s,t \in [0,T].$$

Here δ_s and δ_t are the Dirac measures concentrated at s and t. We refer to the textbook of STEELE [99] for a discussion of CIESIELKI's result as well as some of its ramifications.