6

Stochastic integration I: the Wiener integral

The hard work in the previous lectures will pay off in this lecture, which is devoted to stochastic integration. In view of future applications to stochastic Cauchy problems we shall consider a setting where the integrands take values in the space of operators $\mathscr{L}(H,E)$, where H is a Hilbert space and E a Banach space, and the integrator is a H-cylindrical Brownian motion on a probability space $(\Omega,\mathscr{F},\mathbb{P})$. It is advisable, however, to keep in mind the special case $H=\mathbb{R}$ which concerns the stochastic integration of E-valued functions with respect to a real-valued Brownian motion (cf. Corollary 6.18).

In this lecture we only consider stochastic integrals of functions $\Phi : (0,T) \to \mathscr{L}(H,E)$; such integrals are sometimes called Wiener integrals. The more delicate problem of stochastic integration of stochastic processes $\Phi : (0,T)\times\Omega \to \mathscr{L}(H,E)$ will be considered later on in this course. The theory developed in the present lecture suffices for applications to linear stochastic evolution equations with additive noise, which is the topic of the next couple of lectures.

6.1 Brownian motion

An E-valued *stochastic process* (briefly, an E-valued *process*) indexed by a set I is a family of E-valued random variables $(X(i))_{i\in I}$ defined on some underlying probability space $(\Omega,\mathscr{F},\mathbb{P})$.

Definition 6.1. *An E-valued process $(X(i))_{i\in I}$ is called* Gaussian *if for all $N \geqslant 1$ and $i_1,\dots,i_N \in I$ the E^N-valued random variable $(X(i_1),\dots,X(i_N))$ is Gaussian.*

6.1.1 Brownian motion

We start with the definition.

Definition 6.2. *A real-valued process* $(W(t))_{t\in[0,T]}$ *is called a* Brownian motion *if it enjoys the following properties:*

(i) $W(0) = 0$ *almost surely;*
(ii) $W(t) - W(s)$ *is Gaussian with variance* $t - s$ *for all* $0 \leqslant s \leqslant t \leqslant T$*;*
(iii) $W(t) - W(s)$ *is independent of* $\{W(r) : \ 0 \leqslant r \leqslant s\}$ *for all* $0 \leqslant s \leqslant t \leqslant T$.

In some texts, Brownian motions are called *Wiener processes*.

Proposition 6.3. *Every Brownian motion is a Gaussian process.*

Proof. Fix $t_1, \dots, t_N \in [0, T]$. By independence, the $\mathbb{R}^N$-valued random variable $(W(t_1),\ W(t_2) - W(t_1),\ \dots, W(t_N) - W(t_{N-1}))$ is Gaussian, and the random variable $(W(t_1), \dots, W(t_N))$ is obtained from it under the linear transformation $(\rho_1, \dots, \rho_N) \mapsto (\rho_1, \rho_1 + \rho_2, \dots, \rho_1 + \dots + \rho_N)$. □

Here is a simple way to recognise Brownian motions:

Proposition 6.4. *A real-valued Gaussian process* $(W(t))_{t\in[0,T]}$ *is a Brownian motion if and only if*

$$\mathbb{E}(W(s)W(t)) = \min\{s, t\} \qquad \forall 0 \leqslant s, t \leqslant T.$$

Proof. Let us first prove the 'if' part. Property (i) follows from $\mathbb{E}(W(0))^2 = 0$. To prove (ii) let $0 \leqslant s \leqslant t \leqslant T$. Then

$$\mathbb{E}(W(t) - W(s))^2 = t - 2\min\{s, t\} + s = t - s.$$

For (iii) we must prove that $W(t) - W(s)$ is independent of $(W(r_1), \dots, W(r_N))$ whenever $0 \leqslant r_1, \dots, r_N \leqslant s \leqslant t \leqslant T$ (cf. Definition 3.4). Noting that $(W(r_1), \dots, W(r_N))$ is the image of $(W(r_1), W(r_2) - W(r_1), \dots, W(r_N) - W(r_{N-1}))$ under a linear transformation, it suffices to prove that $W(t) - W(s)$ is independent of $(W(r_1), W(r_2) - W(r_1), \dots, W(r_N) - W(r_{N-1}))$. For this, in turn, it is enough to check that the random variables $W(r_1), W(r_2) - W(r_1), \dots, W(r_N) - W(r_{N-1}), W(t) - W(s)$ are independent. By Proposition 4.10, all we have to check is their orthogonality in $L^2(\Omega)$. But this follows from a simple computation using $\mathbb{E}(W(s)W(t)) = \min\{s, t\}$.

To prove the 'only if' part let $(W(t))_{t\in[0,T]}$ be a Brownian motion. Then for all $0 \leqslant s \leqslant t \leqslant T$,

$$\begin{aligned} 2\mathbb{E}(W(s)W(t)) &= \mathbb{E}W(s)^2 + \mathbb{E}W(t)^2 - \mathbb{E}(W(t) - W(s))^2 \\ &= s + t - (t - s) = 2s = 2\min\{s, t\}. \end{aligned}$$

□

In order to prove the existence of Brownian motions it will be helpful to introduce the notion of an isonormal process.

Let $\mathscr{H}$ be a Hilbert space with inner product $[\cdot, \cdot]$.

Definition 6.5. *An* $\mathscr{H}$-isonormal process *on* Ω *is a mapping* $\mathscr{W} : \mathscr{H} \to L^2(\Omega)$ *with the following two properties:*

(i) *For all* $h \in \mathscr{H}$ *the random variable* $\mathscr{W}h$ *is Gaussian;*
(ii) *For all* $h_1, h_2 \in \mathscr{H}$ *we have* $\mathbb{E}(\mathscr{W}h_1 \cdot \mathscr{W}h_2) = [h_1, h_2]$.

From (ii) it follows that for all scalars c_1, c_2 and all $h_1, h_2 \in \mathscr{H}$ one has

$$\mathbb{E}(\mathscr{W}(c_1h_1 + c_2h_2) - (c_1\mathscr{W}(h_1) + c_2\mathscr{W}(h_2)))^2 = 0.$$

As a consequence, $\mathscr{H}$-isonormal processes are linear. By linearity we have $\sum_{n=1}^N c_n \mathscr{W}h_n = \mathscr{W}(\sum_{n=1}^N c_n h_n)$, which shows that for all $h_1, \dots, h_N \in \mathscr{H}$ the $\mathbb{R}^N$-valued random variable $(\mathscr{W}h_1, \dots, \mathscr{W}h_N)$ is Gaussian. Stated differently, $(\mathscr{W}h)_{h \in H}$ is a Gaussian process.

Example 6.6. If $\mathscr{H}$ is a separable Hilbert space with orthonormal basis $(h_n)_{n=1}^\infty$ and $(\gamma_n)_{n=1}^\infty$ is a Gaussian sequence, then $\mathscr{W}h := \sum_{n=1}^\infty \gamma_n [h, h_n]$ defines an $\mathscr{H}$-isonormal process $\mathscr{W}$. The verification is an easy exercise.

The next theorem provides the existence of Brownian motions:

Theorem 6.7. *If* $\mathscr{W}$ *is an* $L^2(0,T)$*-isonormal process, then* $W(t) := \mathscr{W}1_{[0,t]}$ *defines a Brownian motion on* $[0,T]$.

Proof. By the observation preceding Example 6.6, $(W(t))_{t \in [0,T]}$ is a Gaussian process. Since it satisfies $\mathbb{E}(W(s)W(t)) = [1_{[0,s]}, 1_{[0,t]}]_{L^2(0,T)} = \min\{s,t\}$, it is a Brownian motion by Proposition 6.4. □

The Brownian motion constructed in Theorem 6.7 is given explicitly by

$$W(t) = \sum_{n=1}^\infty \gamma_n [h_n, 1_{[0,t]}] = \sum_{n=1}^\infty \gamma_n \int_0^t h_n(s)\, ds, \tag{6.1}$$

where $(\gamma_n)_{n=1}^\infty$ is a Gaussian sequence and $(h_n)_{n=1}^\infty$ is an orthonormal basis for $L^2(0,T)$. This formula gives a profound connection between Brownian motions and the integration operator $I_T : L^2(0,T) \to C[0,T]$ of Theorem 5.23. We return to this point in Exercise 2.

So far, we have never worried about the distinction between a pointwise defined random variable $X : \Omega \to E$ and its equivalence class modulo null sets. When considering stochastic processes $(X(i))_{i \in I}$, however, one is often interested in properties of the *trajectories* $i \mapsto X(i,\omega) := (X(i))(\omega)$, where $\omega \in \Omega$. Of course these are well-defined only if the $X(i)$ are defined pointwise. Since random variables are often given only as equivalence classes (for instance, when they are constructed as elements of $L^p(\Omega; E)$), one is confronted with the problem of selecting, for each $i \in I$, a pointwise defined representative of $X(i)$. The question then arises whether these representatives can be chosen in a way that the trajectories have 'good' properties.

This discussion leads naturally to the following definition.

Definition 6.8. *Two (pointwise defined) processes* $X = (X(i))_{i\in I}$ *and* $\widetilde{X} = (\widetilde{X}(i))_{i\in I}$ *are* versions *of each other if for all* $i \in I$ *we have* $X(i) = \widetilde{X}(i)$ *almost surely.*

Stated differently, X and $\widetilde{X}$ are versions of each other if and only if $X(i)$ and $\widetilde{X}(i)$ define the same equivalence class for each $i \in I$. From now we shall tacitly assume that processes are always pointwise defined.

The next result, due to Kolmogorov, gives a sufficient condition for the existence of a (Hölder) continuous version of an E-valued process $(X(t))_{t\in[0,T]}$.

Theorem 6.9 (Kolmogorov). *Let* $(X(t))_{t\in[0,T]}$ *be an* E*-valued process on* Ω *with the property that there exist real constants* $C \geqslant 0$, $\alpha > 0$, $\beta > 0$, *such that*

$$\mathbb{E}\|X(t) - X(s)\|^{\alpha} \leqslant C(t-s)^{1+\beta} \qquad \forall 0 \leqslant s \leqslant t \leqslant T.$$

Then for all $0 \leqslant \gamma < \frac{\beta}{\alpha}$, X *has a version* $\widetilde{X}$ *with Hölder continuous trajectories of exponent* γ, *that is, for all* $\omega \in \Omega$ *there is a constant* $\widetilde{C}(\omega) \geqslant 0$ *such that*

$$\|\widetilde{X}(t,\omega) - \widetilde{X}(s,\omega)\| \leqslant \widetilde{C}(\omega)|t-s|^{\gamma} \qquad \forall 0 \leqslant s,t \leqslant T.$$

Proof. We may assume that $T = 1$ for notational simplicity. For $j = 0, 1, \dots$ put

$$Y_j := \sup_{0\leqslant k\leqslant 2^j-1} \|X_{(k+1)2^{-j}} - X_{k2^{-j}}\|.$$

Clearly,

$$\mathbb{E}Y_j^{\alpha} \leqslant \sum_{k=0}^{2^j-1} \mathbb{E}\|X_{(k+1)2^{-j}} - X_{k2^{-j}}\|^{\alpha} \leqslant 2^j \cdot C2^{-(1+\beta)j} = C2^{-\beta j}.$$

Set $D_j := \{k2^{-j} : k = 0, \dots, 2^j - 1\}$ and $D := \bigcup_{j=0}^{\infty} D_j$. Fix $j \geqslant 0$ and $s, t \in D$ satisfying $|t-s| \leqslant 2^{-j}$. For each $n \geqslant 0$ let s_n and t_n be the largest elements in D_n such that $s_n \leqslant s$ and $t_n \leqslant t$. Then either $s_n = t_n$ or $|t_n - s_n| = 2^{-n}$. Similarly, $s_{n+1} - s_n$ and $t_{n+1} - t_n$ can only take the values 0 or $2^{-(n+1)}$. Moreover, eventually $s_n = s$ and $t_n = t$. Hence,

$$\begin{aligned}\|X_t - X_s\| &\leqslant \|X_{s_j} - X_{t_j}\| + \sum_{n=j}^{\infty} \|X_{t_{n+1}} - X_{t_n}\| + \sum_{n=j}^{\infty} \|X_{s_{n+1}} - X_{s_n}\| \\ &\leqslant Y_j + 2\sum_{n=j+1}^{\infty} Y_n \leqslant 2\sum_{n=j}^{\infty} Y_n,\end{aligned}$$

where all sums are actually finite. Fixing $0 \leqslant \gamma < \frac{\beta}{\alpha}$ we obtain

$$\begin{aligned}Z &:= \sup\{\|X_t - X_s\|/|t-s|^{\gamma} :\ s,t \in D,\ s \neq t\} \\ &\leqslant \sup_{j\geqslant 0} \Big\{2^{(j+1)\gamma} \sup_{2^{-(j+1)}<|t-s|\leqslant 2^{-j}} \|X_t - X_s\| :\ s,t \in D,\ s \neq t\Big\} \\ &\leqslant \sup_{j\geqslant 0} \Big(2^{(j+1)\gamma} \cdot 2\sum_{n=j}^{\infty} Y_n\Big) \leqslant 2^{\gamma+1} \sum_{n=0}^{\infty} 2^{\gamma n} Y_n.\end{aligned}$$

In case $\alpha \geqslant 1$, the triangle inequality in $L^\alpha(\Omega)$ gives

$$(\mathbb{E}Z^\alpha)^{\frac{1}{\alpha}} \leqslant 2^{\gamma+1} \sum_{n=0}^{\infty} 2^{\gamma n} (\mathbb{E}Y_n^\alpha)^{\frac{1}{\alpha}} \leqslant 2^{\gamma+1} \sum_{n=0}^{\infty} 2^{\gamma n} (C 2^{-\beta n})^{\frac{1}{\alpha}},$$

which is finite since we assumed that $\gamma < \beta/\alpha$. For $0 < \alpha < 1$ we reason similarly, replacing the triangle inequality by the inequality $(\sum_{n=0}^\infty |c_n|)^\alpha \leqslant \sum_{n=0}^\infty |c_n|^\alpha$. In either case, it follows that $Z < \infty$ almost surely.

In particular, almost surely X is uniformly continuous on D. On the set $\{Z < \infty\}$ we define $\widetilde{X}_t = \lim_{\substack{s \to t \\ s \in D}} X_s$ and on the remaining null set we set $\widetilde{X}_t := 0$. The process $\widetilde{X}$ thus obtained has Hölder continuous trajectories of exponent γ. By Fatou's lemma and the assumption of the theorem, for all $t \in [0,1]$ we have $\widetilde{X}_t = X_t$ almost surely. Therefore $\widetilde{X}$ is a version of X. □

Corollary 6.10. *Every Brownian motion has a version with Hölder continuous trajectories for any exponent* $\gamma < \frac{1}{2}$.

Proof. From $\mathbb{E}|W(t) - W(s)|^2 = |t - s|$ and Exercise 1 (or the Kahane-Khintchine inequality), for $k = 1, 2, \ldots$ we obtain

$$\mathbb{E}|W(t) - W(s)|^{2k} = C_k |t - s|^k,$$

and the result follows from Kolmogorov's theorem upon letting $k \to \infty$. □

6.1.2 Cylindrical Brownian motion

Definition 6.11. *An* $L^2(0,T;H)$*-isonormal process is called an* H-cylindrical Brownian motion *on* $[0,T]$.

H-Cylindrical Brownian motions will be denoted by W_H. For $t \in [0,T]$ and $h \in H$ we put

$$W_H(t)h := W_H(1_{(0,t)} \otimes h).$$

For each fixed $h \in H$ the process $(W_H(t)h)_{h \in H}$ is a Brownian motion, which is standard if and only if $\|h\|_H = 1$.

Example 6.12. If $(W^{(n)})_{n=1}^\infty$ is a sequence of independent Brownian motions and H is a separable Hilbert space with orthonormal basis $(h_n)_{n=1}^\infty$, then

$$W_H(t)h := \sum_{n=1}^{\infty} W^{(n)}(t)[h, h_n]$$

defines an H-cylindrical Brownian motion $(W_H(t))_{t \in [0,T]}$. The easy proof is left as an exercise.

Remark 6.13. Let $H = L^2(D)$, where D is an open subset of $\mathbb{R}^d$. An $L^2(D)$-cylindrical Brownian motion provides the mathematical model for 'space-time white noise' on $[0,T] \times D$. This explains why H-cylindrical Brownian motions appear naturally in the context of stochastic partial differential equations. We will return to this in later lectures.

6.2 The stochastic Wiener integral

After these preliminaries we turn to the problem of defining a stochastic integral of suitable functions $\Phi : (0,T) \to \mathscr{L}(H,E)$ with respect to an H-cylindrical Brownian motion W_H.

For an $\mathscr{L}(H,E)$-valued step function of the form $\Phi = 1_{(a,b)} \otimes (h \otimes x)$ with $0 \leqslant a < b \leqslant T$ and $h \in H$, $x \in E$, we define the random variable $\int_0^T \Phi \, dW_H \in L^2(\Omega; E)$ by

$$\int_0^T \Phi \, dW_H := W_H(1_{(a,b)} \otimes h) \otimes x = (W_H(b)h - W_H(a)h) \otimes x$$

and extend this definition by linearity to step functions with values in the finite rank operators in $\mathscr{L}(H,E)$; such functions will be called *finite rank step functions*. In order to extend the stochastic integral to a broader class of $\mathscr{L}(H,E)$-valued functions, just as in the classical scalar-valued theory we shall compute its square expectation.

We make the preliminary observation that any step function $\Phi : (0,T) \to \mathscr{L}(H,E)$ uniquely defines a bounded operator $R_\Phi \in \mathscr{L}(L^2(0,T;H),E)$ by the formula

$$R_\Phi f := \int_0^T \Phi(t) f(t) \, dt, \qquad f \in L^2(0,T;H).$$

Theorem 6.14 (Itô isometry). *For all finite rank step functions Φ : $(0,T) \to \mathscr{L}(H,E)$ we have $R_\Phi \in \gamma(L^2(0,T;H),E)$, the stochastic integral $\int_0^T \Phi \, dW_H$ is a Gaussian random variable, and*

$$\mathbb{E}\Big\| \int_0^T \Phi \, dW_H \Big\|^2 = \|R_\Phi\|^2_{\gamma(L^2(0,T;H),E)}.$$

Proof. Let $\Phi := \sum_{n=1}^N 1_{(t_{n-1},t_n)} \otimes U_n$ with $0 \leqslant t_0 < \cdots < t_N \leqslant T$ and the operators $U_n \in \mathscr{L}(H,E)$ of finite rank. It is an easy exercise in linear algebra to check that there is no loss of generality in assuming that $U_n = \sum_{j=1}^k h_j \otimes x_{jn}$, where the vectors $h_1, \ldots, h_k \in H$ are orthonormal (and do not depend on n). Since R_Φ is of finite rank, it belongs to $\gamma(L^2(0,T;H),E)$.

Put $\phi_n := c_n 1_{(t_{n-1},t_n)}$, where the normalising constant $c_n := 1/\sqrt{t_n - t_{n-1}}$ assures that the functions $\phi_1, \ldots, \phi_N$ are orthonormal in $L^2(0,T)$. The sequence $(\phi_n \otimes h_j)_{\substack{1\leqslant j\leqslant k \\ 1\leqslant n\leqslant N}}$ is orthonormal in $L^2(0,T;H)$, and from Lemma 5.7 we obtain that

$$\begin{aligned}\|R_\Phi\|^2_{\gamma(L^2(0,T;H);E)} &= \mathbb{E}\Big\|\sum_{j=1}^k\sum_{n=1}^N \gamma_{jn} R_\Phi(\phi_n\otimes h_j)\Big\|^2\\ &= \mathbb{E}\Big\|\sum_{j=1}^k\sum_{n=1}^N \gamma_{jn}\int_0^T c_n 1_{(t_{n-1},t_n)}(t)U_n h_j\,dt\Big\|^2\\ &= \mathbb{E}\Big\|\sum_{j=1}^k\sum_{n=1}^N \gamma_{jn}\sqrt{t_n-t_{n-1}}\,x_{jn}\Big\|^2,\end{aligned}$$

where $(\gamma_{jn})_{\substack{1\leqslant j\leqslant k\\ 1\leqslant n\leqslant N}}$ is a Gaussian sequence. On the other hand,

$$\begin{aligned}\mathbb{E}\Big\|\int_0^T \Phi\,dW_H\Big\|^2 &= \mathbb{E}\Big\|\sum_{j=1}^k\sum_{n=1}^N (W_H(t_n)h_j - W_H(t_{n-1})h_j)\otimes x_{jn}\Big\|^2\\ &= \mathbb{E}\Big\|\sum_{j=1}^k\sum_{n=1}^N \frac{W_H(t_n)h_j - W_H(t_{n-1})h_j}{\sqrt{t_n-t_{n-1}}}\otimes\sqrt{t_n-t_{n-1}}\,x_{jn}\Big\|^2.\end{aligned}$$

Putting $\gamma'_{jn} := (W_H(t_n)h_j - W_H(t_{n-1})h_j)/\sqrt{t_n-t_{n-1}}$, the desired identity now follows since $(\gamma'_{jn})_{\substack{1\leqslant j\leqslant k\\ 1\leqslant n\leqslant N}}$ is a Gaussian sequence. ☐

As a consequence, the linear mapping $J_T^{W_H} : R_\Phi \mapsto \int_0^T \Phi\,dW_H$ uniquely extends to an isometric embedding

$$J_T^{W_H} : \gamma(L^2(0,T;H),E) \to L^2(\Omega;E).$$

Accordingly, the stochastic integral of an operator $R\in\gamma(L^2(0,T;H),E)$ can be defined as $J_T^{W_H}(R)$. In order for this to be useful we need a way to recognise those $\mathscr{L}(H,E)$-valued functions which 'represent' an operator in $\gamma(L^2(0,T;H),E)$. To this problem we turn next.

For a function $\Phi:(0,T)\to\mathscr{L}(H,E)$ and elements $h\in H$ and $x^*\in E^*$ we define $\Phi h:(0,T)\to E$ and $\Phi^* x^*:(0,T)\to H$ by $(\Phi h)(t) := \Phi(t)h$ and $(\Phi^*x^*)(t) := \Phi^*(t)x^*$ (where of course $\Phi^*(t) := (\Phi(t))^*$).

Definition 6.15. *A function $\Phi:(0,T)\to\mathscr{L}(H,E)$ is said to be* stochastically integrable *with respect to the H-cylindrical Brownian motion W_H if there exists a sequence of finite rank step functions $\Phi_n:(0,T)\to\mathscr{L}(H,E)$ such that:*

(i) *for all $h\in H$ we have $\lim_{n\to\infty}\Phi_n h = \Phi h$ in measure;*
(ii) *there exists an E-valued random variable X such that $\displaystyle\lim_{n\to\infty}\int_0^T \Phi_n\,dW_H = X$ in probability.*

The stochastic integral *of a stochastically integrable function $\Phi:(0,T)\to\mathscr{L}(H,E)$ is then defined as the limit in probability*

$$\int_0^T \Phi \, dW_H := \lim_{n\to\infty} \int_0^T \Phi_n \, dW_H.$$

Three remarks are in order.

(a) Condition (i) means $\lim_{n\to\infty} |\{t \in (0,T) : \ \|\Phi_n(t)h - \Phi(t)h\| > r\}| = 0$ for all $h \in H$ and $r > 0$, where $|B|$ denotes the Lebesgue measure of B.
(b) The stochastic integral is well defined in the sense that it is independent of the approximating sequence.
(c) From Theorem 4.15 it follows that the convergence in probability in condition (ii) is equivalent to convergence in $L^p(\Omega; E)$ for some (all) $1 \leqslant p < \infty$.

In the special case $E = \mathbb{R}$ we may identify $\mathscr{L}(H, \mathbb{R}) = H^*$ with H by the Riesz representation theorem. Under this identification, Theorem 6.14 reduces to the statement that the stochastic integral of a step function $\phi : (0,T) \to H$ satisfies

$$\mathbb{E}\Big\| \int_0^T \phi \, dW_H \Big\|^2 = \|\phi\|^2_{L^2(0,T;H)}. \tag{6.2}$$

From this it is immediate that a strongly measurable function $\phi : (0,T) \to H$ is stochastically integrable with respect to W_H if and only if $\phi \in L^2(0,T;H)$, and the isometry (6.2) extends to functions $\phi \in L^2(0,T;H)$.

Definition 6.16. *A function $\Phi : (0,T) \to \mathscr{L}(H,E)$ is called* H-strongly measurable *if for each $h \in H$ the function $\Phi h : (0,T) \to E$ is strongly measurable.*

By Theorem 6.14 and a limiting argument, we see that if a function Φ is stochastically integrable with respect to W_H, then the integral operator R_Φ associated with Φ is well-defined and γ-radonifying. Interestingly, the converse is true as well. These two statements are contained in the next theorem, which is the main result of this lecture.

Theorem 6.17. *Let W_H be an H-cylindrical Brownian motion. For an H-strongly measurable function $\Phi : (0,T) \to \mathscr{L}(H,E)$ the following assertions are equivalent:*

(1) *Φ is stochastically integrable with respect to W_H;*
(2) *$\Phi^* x^* \in L^2(0,T;H)$ for all $x^* \in E^*$, and there exists an E-valued random variable X such that for all $x^* \in E^*$, almost surely we have*

$$\langle X, x^* \rangle = \int_0^T \Phi^* x^* \, dW_H;$$

(3) *$\Phi^* x^* \in L^2(0,T;H)$ for all $x^* \in E^*$, and there exists an operator $R \in \gamma(L^2(0,T;H), E)$ such that for all $f \in L^2(0,T;H)$ and $x^* \in E^*$ we have*

$$\langle Rf, x^* \rangle = \int_0^T \langle \Phi(t) f(t), x^* \rangle \, dt.$$

If these equivalent conditions are satisfied, the random variable X and the operator R are uniquely determined, we have $X = \int_0^T \Phi \, dW_H$ almost surely, and

$$\mathbb{E}\Big\| \int_0^T \Phi \, dW_H \Big\|^2 = \|R\|^2_{\gamma(L^2(0,T;H),E)}.$$

In the situation of (3) we say that Φ *represents* the operator R. Note that condition (3) does not depend on the particular choice of W_H.

Proof. We shall prove the implications (1)⇒(2)⇒(4)⇒(3)⇒(1), where

(4) $\Phi^* x^* \in L^2(0,T;H)$ for all $x^* \in E^*$, and there exists a γ-radonifying operator $\widetilde{R}$ from a Hilbert space $\widetilde{H}$ to E such that for all $x^* \in E^*$ we have

$$\|\Phi^* x^*\|_{L^2(0,T;H)} \leqslant \|\widetilde{R}^* x^*\|_{\widetilde{H}}.$$

(1)⇒(2): Let $(\Phi_n)_{n=1}^\infty$ be an approximating sequence of finite rank step functions for Φ and take $X := \int_0^T \Phi \, dW_H$. As we have already observed, $\lim_{n\to\infty} \int_0^T \Phi_n \, dW_H = X$ in $L^2(\Omega; E)$. Hence,

$$\lim_{n\to\infty} \int_0^T \Phi_n^* x^* \, dW_H = \lim_{n\to\infty} \Big\langle \int_0^T \Phi_n \, dW_H, x^* \Big\rangle = \langle X, x^* \rangle$$

in $L^2(\Omega)$, where the first identity is verified by writing out the definitions. By the special case of the Itô isometry contained in (6.2), the sequence $(\Phi_n^* x^*)_{n=1}^\infty$ is Cauchy in $L^2(0,T;H)$. Let f be its limit. Since $\lim_{n\to\infty} \langle \Phi_n h, x^* \rangle = \langle \Phi h, x^* \rangle$ in measure, it follows that $f = \Phi^* x^*$ in $L^2(0,T;H)$. Once more by (6.2),

$$\lim_{n\to\infty} \int_0^T \Phi_n^* x^* \, dW_H = \int_0^T \Phi^* x^* \, dW_H.$$

(2)⇒(4): Let $i_X \in \gamma(H_X, E)$ be defined by (4.3). Then, by (6.2),

$$\int_0^T \|\Phi^*(t) x^*\|^2 \, dt = \mathbb{E}\Big\| \int_0^T \Phi^* x^* \, dW_H \Big\|^2 = \mathbb{E}\langle X, x^* \rangle^2 = \|i_X^* x^*\|^2.$$

(4)⇒(3): The formula

$$(R_\Phi f)(x^*) := \int_0^T [f(t), \Phi^*(t) x^*] \, dt, \qquad f \in L^2(0,T;H), \ x^* \in E^*,$$

defines a bounded operator R_Φ from $L^2(0,T;H)$ to E^{**}. Once we know that R_Φ maps $L^2(0,T;H)$ into E, Theorem 5.17 shows that $R_\Phi \in \gamma(L^2(0,T;H),E)$.

For the proof that R_Φ takes values in E we invoke Theorem 1.20. By assumption, for all $h \in H$ the function Φh is strongly measurable and the functions $\langle \Phi h, x^* \rangle = [h, \Phi^* x^*]$ are square integrable. It follows that Φh is Pettis integrable. Therefore, for step functions f, the element $R_\Phi f \in E^{**}$ is given by the Pettis integral $\int_0^T \Phi(t)f\,dt$ in E. Thus, $R_\Phi f \in E$ for all step functions $f:(0,T)\to H$. Since these functions are dense in $L^2(0,T;H)$, a limiting argument implies that $R_\Phi f \in E$ for all $f\in L^2(0,T;H)$.

(3)⇒(1): We split the proof into three steps.

Step 1 - We begin by constructing an E-valued random variable X, which will turn out later to be the stochastic integral $\int_0^T \Phi\,dW_H$.

By Proposition 5.10 there is a separable closed subspace $\mathscr{H}_0$ of $L^2(0,T;H)$ such that $Rf=0$ for all $f\in \mathscr{H}_0^\perp$. Choose a separable closed subspace H_0 of H such that $\mathscr{H}_0 \subseteq L^2(0,T;H_0)$. Note that the range of R^* is contained in $\mathscr{H}_0$, hence in $L^2(0,T;H_0)$.

Let $(f_m)_{m=1}^\infty$ and $(h_n)_{n=1}^\infty$ be orthonormal bases for $L^2(0,T)$ and H_0, respectively. The functions $\phi_{mn} := f_m\otimes h_n$ define an orthonormal basis $(\phi_{mn})_{m,n=1}^\infty$ for $L^2(0,T;H_0)$. By (6.2) the random variables $\gamma_{mn} := \int_0^T \phi_{mn}\,dW_H$ are standard Gaussian, and the linearity of the stochastic integral implies that they are jointly Gaussian. The orthonormality of the ϕ_{mn} implies that the γ_{mn} are orthonormal in $L^2(\Omega)$, and therefore independent by Proposition 4.10. Thus we have shown that $(\gamma_{mn})_{m,n=1}^\infty$ is a Gaussian sequence.

Put

$$X := \sum_{m,n=1}^\infty \gamma_{mn} R\phi_{mn}.$$

This sum converges in $L^2(\Omega;E)$ by Theorem 5.15. Moreover, the identity $\langle R\phi_{mn}, x^*\rangle = [\Phi^*x^*, \phi_{mn}]_{L^2(0,T;H_0)}$ implies $\Phi^*x^* = R^*x^* \in L^2(0,T;H_0)$ and

$$\begin{aligned}\langle X, x^*\rangle &= \sum_{m,n=1}^\infty \int_0^T \langle R\phi_{mn}, x^*\rangle \phi_{mn}\,dW_H \\ &= \int_0^T \sum_{m,n=1}^\infty \langle R\phi_{mn}, x^*\rangle \phi_{mn}\,dW_H = \int_0^T \Phi^*x^*\,dW_H,\end{aligned} \tag{6.3}$$

where the second identity follows from $L^2(0,T;H)$-convergence and (6.2).

Step 2 - Define the operators $\Phi_k(t)\in\mathscr{L}(H,E)$ by

$$\Phi_k(t)h := \sum_{j=1}^{2^k} 1_{(\frac{(j-1)T}{2^k},\frac{jT}{2^k})}(t) RU_{jk}h,$$

where $U_{jk}\in \mathscr{L}(H, L^2(0,T;H))$ is given by $U_{jk}h := \frac{2^k}{T} 1_{(\frac{(j-1)T}{2^k},\frac{jT}{2^k})}\otimes h$. Note that $RU_{jk}\in\gamma(H,E)$ by the ideal property. Hence, each Φ_k is an $\gamma(H,E)$-valued step function. The identity

$$\langle \Phi_k(t)h, x^*\rangle = \sum_{j=1}^{2^k} 1_{(\frac{(j-1)T}{2^k},\frac{jT}{2^k})} \frac{2^k}{T}\int_{\frac{(j-1)T}{2^k}}^{\frac{jT}{2^k}} \langle \Phi(t)h, x^*\rangle\,dt$$

shows that Φ_k is obtained from Φ by averaging. We will show that

(i) $\lim_{k\to\infty} \Phi_k h = \Phi h$ in measure for all $h \in H$,
(ii) $\lim_{k\to\infty} \int_0^T \Phi_k \, dW_H = X$ in probability, where X is as in Step 1.

To prove (i) fix $h \in H$ and assume that $\|h\| = 1$. To get around the difficulty that we cannot be sure that $\Phi h \in L^2(0,T;E)$ we do a trunction argument.

Fix an arbitrary $\varepsilon > 0$. For $r > 0$ define $S^{(r)} \in \mathscr{L}(L^2(0,T;H))$ by $S^{(r)} f := 1_{\{\|\Phi(t)h\| \leqslant r\}} f$. Using Proposition 5.12 choose $r_0 > 0$ so large that

$$\|R - RS^{(r_0)}\|_{\gamma(L^2(0,T;H),E)} < \varepsilon, \quad |\{t \in (0,T) : \ \|\Phi(t)h - f^{(r_0)}(t)\| > \varepsilon\}| < \varepsilon,$$

where $f^{(r_0)}(t) := 1_{\{\|\Phi(t)h\| \leqslant r_0\}} \Phi(t)h$. Since $f^{(r_0)} \in L^2(0,T;E)$, by the properties of averaging operators (see Exercise 3) we have

$$f^{(r_0)} = \lim_{k\to\infty} \sum_{j=1}^{2^k} 1_{(\frac{(j-1)T}{2^k}, \frac{jT}{2^k})} \frac{2^k}{T} \int_{\frac{(j-1)T}{2^k}}^{\frac{jT}{2^k}} f^{(r_0)}(t) \, dt = \lim_{k\to\infty} f_k^{(r_0)} \tag{6.4}$$

in $L^2(0,T;E)$, where $f_k^{(r_0)} := \sum_{j=1}^{2^k} 1_{(\frac{(j-1)T}{2^k}, \frac{jT}{2^k})} RS^{(r_0)} U_{jk} h$.

If $s \in (\frac{(j-1)T}{2^k}, \frac{jT}{2^k})$, then

$$\|f_k^{(r_0)}(s) - \Phi_k(s)h\| = \|RS^{(r_0)} U_{jk} h - RU_{jk} h\| \leqslant \|R - RS^{(r_0)}\|_{\gamma(L^2(0,T;H),E)} < \varepsilon.$$

Hence,

$$\begin{aligned} &|\{t \in (0,T) : \ \|\Phi(t)h - \Phi_k(t)h\| > 3\varepsilon\}| \\ &\quad \leqslant \varepsilon + |\{t \in (0,T) : \ \|f^{(r_0)}(t) - f_k^{(r_0)}(t)\| > \varepsilon\}| \\ &\qquad + |\{t \in (0,T) : \ \|f_k^{(r_0)}(t) - \Phi_k(t)h\| > \varepsilon\}| \\ &\quad = \varepsilon + |\{t \in (0,T) : \ \|f^{(r_0)}(t) - f_k^{(r_0)}\| > \varepsilon\}|. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, (i) follows from (6.4) by letting $k \to \infty$.

We continue with the proof of (ii). Put

$$X_1 = \int_0^T \Phi_1 \, dW_H, \qquad X_n = \int_0^T (\Phi_n - \Phi_{n-1}) \, dW_H \quad \text{for} \quad n \geqslant 2.$$

We claim that the random variables X_n are independent. By the linearity of the stochastic integral, the random variables X_n are jointly Gaussian and therefore by Proposition 4.10 it suffices to check that $\mathbb{E}\langle X_m, x^*\rangle \langle X_n, y^*\rangle = 0$ for $m \neq n$ and $x^*, y^* \in E^*$. By (6.2) and linearity, the expectation equals

$$\int_0^T [\Phi_m^*(t)x^* - \Phi_{m-1}^*(t)x^*, \ \Phi_n^*(t)y^* - \Phi_{n-1}^*(t)y^*] \, dt$$

using the convention that $\Phi_0 = 0$. By a direct computation using the properties of the averaging operators, this expression equals 0.

Put $S_N := \sum_{n=1}^N X_n = \int_0^T \Phi_N \, dW_H$. By (6.3), (6.2), and the properties of averaging operators, for all $x^* \in E^*$ we have

$$\lim_{N\to\infty} \mathbb{E}\langle X - S_N, x^*\rangle^2 = \lim_{N\to\infty} \|\Phi^* x^* - \Phi_N^* x^*\|^2_{L^2(0,T;H)} = 0$$

and therefore $\lim_{N\to\infty}\langle S_N, x^*\rangle = \langle X, x^*\rangle$ in probability. The Itô-Nisio theorem implies that $\lim_{N\to\infty} S_N = X$ in probability.

Step 3 – So far, we have found a sequence of $\gamma(H,E)$-valued step functions $(\Phi_n)_{n=1}^\infty$ with the convergence properties as required in Definition 6.15. To conclude the proof we approximate the values of the functions Φ_n by finite rank operators. □

Corollary 6.18. *A strongly measurable function $\phi : (0,T) \to E$ is stochastically integrable with respect to a real-valued Brownian motion if and only if ϕ represents an operator $R \in \gamma(L^2(0,T), E)$.*

As an application of Theorem 6.17 we have the following domination criterion for stochastic integrability.

Theorem 6.19. *Suppose that $\Phi_1, \Phi_2 : (0,T) \to \mathscr{L}(H,E)$ are H-strongly measurable functions, and assume that Φ_2 stochastically integrable with respect to the H-cylindrical Brownian motion W_H. If*

$$\int_0^T \|\Phi_1^*(t)x^*\|^2\,dt \leqslant \int_0^T \|\Phi_2^*(t)x^*\|^2\,dt \qquad \forall x^* \in E^*,$$

then Φ_1 is stochastically integrable with respect to W_H, and for all $1 \leqslant p < \infty$ we have

$$\mathbb{E}\Big\| \int_0^T \Phi_1\,dW_H\Big\|^p \leqslant \mathbb{E}\Big\| \int_0^T \Phi_2\,dW_H\Big\|^p.$$

Proof. First note that by Theorem 6.17, for all $x^* \in E^*$ the function $\Phi_2^* x^*$ belongs to $L^2(0,T;H)$. By (4) in the proof of Theorem 6.17, Φ_2 represents an operator $R_2 \in \gamma(L^2(0,T;H),E)$. In view of $R_2^* x^* = \Phi_2^* x^*$ we have

$$\int_0^T \|\Phi_2^*(t)x^*\|^2\,dt = \|R_2^* x^*\|^2_{L^2(0,T;H)}.$$

Let $R_{\Phi_1} \in \gamma(L^2(0,T;H),E)$ denote the operator representing Φ_1 whose existence is assured by Theorem 6.17 (3). The first assertion follows by applying Theorem 6.17 to Φ_1 and the L^p-inequality follows from Corollary 5.18. □

6.3 Exercises

1. (!) Let γ be a Gaussian random variable with variance $\mathbb{E}\gamma^2 = q$. Compute $\mathbb{E}\gamma^{2k}$, $k = 1, 2, \ldots$
   Hint: Express $\mathbb{E}\gamma^{2k+2}$ in terms of $\mathbb{E}\gamma^{2k}$.

2. In view of the identity (6.1), Theorem 5.23 provides another proof of the existence of a continuous version for Brownian motions. In this exercise we show that in the converse direction Theorem 5.23 can be deduced from the path continuity of Brownian motions.
Let W be a Brownian motion and let $\widetilde{W}$ be a version of it with continuous trajectories.
   a) Use the Pettis measurability theorem to prove that the function $X_T : \Omega \to C[0,T]$ defined by $(X_T(\omega))(t) := \widetilde{W}(t,\omega)$ is strongly measurable.
   Hint: The Dirac measures span a norming subspace in $(C[0,T])^*$.
   b) Show that the random variable X_T is Gaussian.
   c) Show that the covariance operator Q_T of X_T is given by $Q_T = I_T I_T^*$, where $I_T : L^2(0,T) \to C[0,T]$ is the integration operator of Theorem 5.23, and deduce from this that I_T is γ-radonifying.

3. Fix $1 \leqslant p < \infty$. For $n = 0, 1, 2, \dots$ define the linear operators $A_n : L^p(0,T;E) \to L^p(0,T;E)$ by

$$A_n f := \sum_{j=1}^{2^n} 1_{(\frac{(j-1)T}{2^n}, \frac{jT}{2^n})} \otimes x_{jn},$$

where

$$x_{jn} := \frac{2^n}{T} \int_{\frac{(j-1)T}{2^n}}^{\frac{jT}{2^n}} f(t)\, dt.$$

   a) Show that each A_n is bounded and satisfies $\|A_n\| = 1$.
   b) Show that $\lim_{n\to\infty} A_n f = f$ in $L^p(0,T;E)$ for all $f \in L^p(0,T;E)$.
   Hint: What happens if f is a dyadic step function?
   c) Prove the assertion involving averaging operators in Step 3 (ii) of the proof of (3)⇒(1) of Theorem 6.17.

4. Let the function $\Phi : (0,T) \to \mathscr{L}(H,E)$ be stochastically integrable with respect to W_H.
   a) Show that for all $t \in [0,T]$ the restriction of $\Phi|_{(0,t)}$ is stochastically integrable on $(0,t)$ with respect to W_H, that $1_{(0,t)}\Phi$ is stochastically integrable on $(0,T)$ with respect to W_H, and that almost surely

$$\int_0^t \Phi\, dW_H = \int_0^T 1_{(0,t)}\Phi\, dW_H.$$

We consider the E-valued process X, where $X_t = \int_0^t \Phi\, dW_H$ for $t \in [0,T]$.
   b) Show that X is a Gaussian process.
   c) Show that X has trajectories in $L^p(0,T;E)$ almost surely for every $1 \leqslant p < \infty$.
   Hint: Prove the stronger statement that $\mathbb{E} \int_0^T \|X(t)\|^p\, dt < \infty$.

Remark: Using martingale techniques it can be shown that X has a continuous version. We return to this later on.

5. Suppose that $\Phi : (0,T) \to \mathscr{L}(H,E)$ is stochastically integrable with respect to the H-cylindrical Brownian motion W_H.
   a) Show that for each $h \in H$ function Φh is stochastically integrable with respect to the Brownian motions $W_H h$.
   b) Prove the following series expansion: if H is separable, then for any orthonormal basis $(h_n)_{n=1}^\infty$ of H we have
$$\int_0^T \Phi\, dW_H = \sum_{n=1}^\infty \int_0^T \Phi h_n \, dW_H h_n,$$
   with convergence almost surely and in $L^p(\Omega;E)$ for all $1 \leqslant p < \infty$.
   Hint: First consider the functions ΦP_n, where P_n is the orthogonal projection in H onto the span of $\{h_1, \dots, h_n\}$.

Notes. The notion of Brownian motion has its origin in the observations by the botanist Brown (1773-1858) who observed that small particles suspended in a fluid display random movements. The first rigorous mathematical treatment was given by Wiener in the 1920s.

The proof of Theorem 6.9 is taken from Revuz and Yor [94]. Its Corollary 6.10 is nearly optimal in the following sense: almost surely, one has
$$\limsup_{\delta \downarrow 0} \max_{|t-s| \leqslant \delta} \frac{|W(t)-W(s)|}{\sqrt{2|t-s| \ln(1/|t-s|)}} = 1.$$
This is a classical result of Lévy. In particular it shows that almost surely the paths of a Brownian motion are nowhere Hölder continuous of exponent $\frac{1}{2}$. For proofs and further results on Brownian motion we refer to Karatzas and Shreve [59] and Revuz and Yor [94]. A more recent result of Ciesielski [23] asserts that almost surely, the trajectories of Brownian motions belong to the Besov space $B^{\frac{1}{2}}_{p,\infty}$ for all $1 \leqslant p < \infty$. This result was extended to Banach space-valued Brownian motions, with a simpler proof, by Hytönen and Veraar [51].

For accessible introductions to the classical (scalar-valued) theory of stochastic integration we refer to the books by Chung and Williams [22], Kuo [63], Oksendal [86], and Steele [99]. For scalar-valued functions, the isometry of Theorem 6.14 goes back to Wiener and was generalised to stochastic processes in the fundamental work of Itô.

By combining the observation on Kwapień's theorem in the Notes of the previous lecture with Corollary 6.18 we obtain that the following assertions are equivalent for a Banach space E:

(1) the space of strongly measurable E-valued functions $f : (0,T) \to E$ which are stochastically integrable with respect to Brownian motion equals $L^2(0,T;E)$;

(2) the space E is isomorphic to a Hilbert space.

An explicit example of a uniformly bounded function $f : (0,1) \to l^p$ for $1 \leqslant p < 2$ which fails to be stochastically integrable was constructed in an early stage of the theory by Yor [110]. Further examples along this line were constructed Rosiński and Suchanecki [96] who also proved (for $H = \mathbb{R}$) the equivalence (1)⇔(2) of Theorem 6.17. Step 3 of the proof of (3)⇒(1) in Theorem 6.17 is a variation of their argument. In its present formulation, Theorem 6.17 can be found in [84]; a preliminary version was obtained in [16] by using different methods. The idea in Step 2 of the proof of (3)⇒(1) is taken from [84]. The implication (3)⇒(1) can alternatively be derived from variant of Theorem 5.24. This is the approach taken in [84], where also Theorems 6.14, 6.19, and the result of Exercise 5 were obtained.

In the Hilbert space literature, the series expansions of Example 6.12 and Exercise 5 are often taken as the starting point for defining the stochastic integral; see for instance the monograph of Da Prato and Zabczyk [27].

A more probabilistic approach to the theory of stochastic integration in Banach spaces is taken by Métivier and Pellaumail [76].