7

Semigroups of linear operators

Having developed the probabilistic tools needed for our study of stochastic evolution equations, in this lecture we turn to the theory of C_0-semigroups. We review their basic properties and show how semigroups are used to solve the (deterministic) inhomogeneous abstract Cauchy problem

$$u'(t) = Au(t) + f(t).$$

Here A generates a C_0-semigroup on E and the forcing term f is a locally integrable E-valued function. As we shall see in the next lecture, the techniques for solving this problem by means of so-called weak and strong solutions can be extended to stochastic abstract Cauchy problems with additive noise, the main difference being that Bochner integrals are replaced by the stochastic integrals introduced in the previous lecture. Heuristically, the reason why this works is that the noise can be viewed as a 'random' forcing term.

7.1 C_0-semigroups

Linear equations of mathematical physics can often be cast in the abstract form

$$\begin{cases} u'(t) = Au(t), & t \in [0,T], \\ u(0) = x, \end{cases} \tag{ACP}$$

where A is a linear, usually unbounded, operator defined on a linear subspace $\mathscr{D}(A)$, the *domain* of A, of a Banach space E. Typically, E is a Banach space of functions suited for the particular problem and A is a partial differential operator. The abstract initial value problem (ACP) is referred to as the *abstract Cauchy problem* associated with A.

Example 7.1. Let D be an open domain in $\mathbb{R}^d$ with topological boundary ∂D. On D we consider the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,\xi) = \Delta u(t,\xi), & t \in [0,T],\ \xi \in D; \\ u(t,\xi) = 0, & t \in [0,T],\ \xi \in \partial D; \\ u(0,\xi) = u_0(\xi), & \xi \in D. \end{cases} \tag{7.1}$$

For initial values $x = u_0 \in L^p(D)$ with $1 \leqslant p < \infty$, this problem can be rewritten in the abstract form (ACP) by taking $E = L^p(D)$ and defining A by

$$\mathscr{D}(A) := \{f \in W^{2,p}(D) :\ f|_{\partial D} \equiv 0\} = W^{2,p}(D) \cap W_0^{1,p}(D),$$
$$Af := \Delta f, \quad f \in \mathscr{D}(A).$$

Here, $W^{k,p}(D)$ is the Sobolev space of all $f \in L^p(D)$ whose weak partial derivatives up to order k exist and belong to $L^p(D)$, $W_0^{k,p}(D)$ is the closure in $W^{k,p}(D)$ of all test functions $f \in C_{\mathrm{c}}^\infty(D)$, and $\Delta = \sum_{j=1}^d \frac{\partial^2}{\partial \xi_j^2}$ is the Laplacian. Note how the boundary condition is built into the definition of A by the specification of its domain.

The idea is now that instead of looking for a solution $u : [0,T] \times D \to \mathbb{R}$ of (7.1) one looks for a solution $u : [0,T] \to L^p(D)$ of (ACP). To get an idea how this may be done we first take a look at the much simpler case where $E = \mathbb{R}^d$ and $A : \mathscr{D}(A) = E \to E$ is represented by a $(d \times d)$-matrix. In that case, the unique solution of (ACP) is given by

$$u(t) = e^{tA} u_0, \qquad t \in [0,T],$$

where $e^{tA} = \sum_{n=0}^\infty \frac{t^n A^n}{n!}$. The matrices e^{tA} may be thought of as 'solution operators' mapping the initial value u_0 to the solution $e^{tA}u_0$ at time t. Clearly, $e^{0A} = I$, $e^{tA}e^{sA} = e^{(t+s)A}$, and $t \mapsto e^{tA}$ is continuous. We generalise these properties to infinite dimensions as follows.

Let E be a real or complex Banach space.

Definition 7.2. *A family $S = \{S(t)\}_{t\geqslant 0}$ of bounded linear operators acting on a Banach space E is called a C_0-*semigroup *if the following three properties are satisfied:*

S1. $S(0) = I$;
S2. $S(t)S(s) = S(t+s)$ *for all* $t, s \geqslant 0$;
S3. $\lim_{t\downarrow 0} \|S(t)x - x\| = 0$ *for all* $x \in E$.

The infinitesimal generator*, or briefly the* generator*, of S is the linear operator A with domain $\mathscr{D}(A)$ defined by*

$$\mathscr{D}(A) = \{x \in E :\ \lim_{t\downarrow 0} \frac{1}{t}(S(t)x - x) \ \text{exists}\},$$
$$Ax = \lim_{t\downarrow 0} \frac{1}{t}(S(t)x - x), \quad x \in \mathscr{D}(A).$$

We shall frequently use the trivial observation that if A generates the C_0-semigroup $(S(t))_{t\geqslant 0}$, then $A-\mu$ generates the C_0-semigroup $(e^{-\mu t}S(t))_{t\geqslant 0}$.

The next two propositions collect some elementary properties of C_0-semigroups and their generators.

Proposition 7.3. *Let S be a C_0-semigroup on E. There exist constants $M \geqslant 1$ and $\mu \in \mathbb{R}$ such that $\|S(t)\| \leqslant Me^{\mu t}$ for all $t \geqslant 0$.*

Proof. There exists a number $\delta > 0$ such that $\sup_{t\in[0,\delta]} \|S(t)\| =: \sigma < \infty$. Indeed, otherwise we could find a sequence $t_n \downarrow 0$ such that $\lim_{n\to\infty} \|S(t_n)\| = \infty$. By the uniform boundedness theorem, this implies the existence of an $x \in E$ such that $\sup_{n\geqslant 1} \|S(t_n)x\| = \infty$, contradicting the strong continuity assumption (S3). This proves the claim. By the semigroup property (S2), for $t \in [(k-1)\delta, k\delta]$ it follows that $\|S(t)\| \leqslant \sigma^k \leqslant \sigma^{(t+1)/\delta}$, where the second inequality uses that $\sigma \geqslant 1$ by (S1). This proves the proposition, with $M = \sigma^{\frac{1}{d}}$ and $\mu = \frac{1}{d}\ln\sigma$. $\square$

Proposition 7.4. *Let S be a C_0-semigroup on E with generator A.*

(1) *For all $x \in E$ the orbit $t \mapsto S(t)x$ is continuous for $t \geqslant 0$.*
(2) *For all $x \in \mathscr{D}(A)$ and $t \geqslant 0$ we have $S(t)x \in \mathscr{D}(A)$ and $AS(t)x = S(t)Ax$.*
(3) *For all $x \in E$ we have $\int_0^t S(s)x\,ds \in \mathscr{D}(A)$ and*
$$A\int_0^t S(s)x\,ds = S(t)x - x.$$
If $x \in \mathscr{D}(A)$, then both sides are equal to $\int_0^t S(s)Ax\,ds$.
(4) *The generator A is a closed and densely defined operator.*
(5) *For all $x \in \mathscr{D}(A)$ the orbit $t \mapsto S(t)x$ is continuously differentiable for $t \geqslant 0$ and*
$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax, \quad t \geqslant 0.$$

Proof. (1): The right continuity of $t \mapsto S(t)x$ follows from the right continuity at $t = 0$ (S3) and the semigroup property (S2). For the left continuity, observe that
$$\|S(t)x - S(t-h)x\| \leqslant \|S(t-h)\|\|S(h)x - x\| \leqslant \sup_{s\in[0,t]} \|S(s)\|\|S(h)x - x\|,$$
where the supremum is finite by Proposition 7.3.

(2): This follows from the semigroup property:
$$\lim_{h\downarrow 0} \frac{1}{h}(S(t+h)x - S(t)x) = S(t)\lim_{h\downarrow 0}\frac{1}{h}(S(h)x - x) = S(t)Ax.$$

(3): The first identity follows from

$$\begin{aligned}\lim_{h\downarrow 0}\frac{1}{h}(S(h)-I)\int_0^t S(s)x\,ds &= \lim_{h\downarrow 0}\frac{1}{h}\Big(\int_0^t S(s+h)x\,ds - \int_0^t S(s)x\,ds\Big)\\ &= \lim_{h\downarrow 0}\frac{1}{h}\Big(\int_t^{t+h} S(s)x\,ds - \int_0^h S(s)x\,ds\Big)\\ &= S(t)x - x,\end{aligned}$$

where we used the continuity of $t \mapsto S(t)x$. The identity for $x \in \mathscr{D}(A)$ will follow from the second part of the proof of (4).

(4): Denseness of $\mathscr{D}(A)$ follows from the first part of (3), since by (1) we have $\lim_{t\downarrow 0}\frac{1}{t}\int_0^t S(s)x\,ds = x$.

To prove that A is closed we must check that the graph $\mathscr{G}(A) = \{(x, Ax) : x \in \mathscr{D}(A)\}$ is closed in $E \times E$. Suppose that $(x_n)_{n=1}^\infty$ is a sequence in $\mathscr{D}(A)$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Ax_n = y$ in E. We must show that $x \in \mathscr{D}(A)$ and $Ax = y$. Using that $\lim_{t\downarrow 0}\frac{1}{t}(S(t)-I)S(s)x_n = S(s)Ax_n$ uniformly for $s \in [0,h]$, we obtain

$$\begin{aligned}\frac{1}{h}(S(h)x - x) &= \lim_{n\to\infty}\frac{1}{h}(S(h)x_n - x_n)\\ &= \lim_{n\to\infty}\frac{1}{h}\Big(A\int_0^h S(s)x_n\,ds\Big)\\ &= \lim_{n\to\infty}\frac{1}{h}\lim_{t\downarrow 0}\frac{1}{t}(S(t)-I)\int_0^h S(s)x_n\,ds\\ &= \lim_{n\to\infty}\frac{1}{h}\lim_{t\downarrow 0}\int_0^h \frac{1}{t}(S(t)-I)S(s)x_n\,ds\\ &= \lim_{n\to\infty}\frac{1}{h}\int_0^h S(s)Ax_n\,ds\\ &= \frac{1}{h}\int_0^h S(s)y\,ds.\end{aligned}$$

Passing to the limit for $h \downarrow 0$ this gives $x \in \mathscr{D}(A)$ and $Ax = y$. The above identities also prove the second part of (3).

(5): This follows from (1), (2), and the definition of A. □

In hindsight, the second part of (3) is a special case of Hille's theorem. However, our proof of the closedness of A already gave the result in this particular case.

Definition 7.5. *A* classical solution *of (ACP) is a continuous function $u : [0,T] \to E$ which belongs to $C^1((0,T];E) \cap C((0,T];\mathscr{D}(A))$ and satisfies $u(0) = x$ and $u'(t) = Au(t)$ for all $t \in (0,T]$.*

Here $\mathscr{D}(A)$ is regarded as a Banach space endowed with the graph norm.

Corollary 7.6. *For initial values $x \in \mathscr{D}(A)$ the problem (ACP) has a unique classical solution, which is given by $u(t) = S(t)x$.*

Proof. Part (5) of the proposition proves that $t \mapsto u(t) = S(t)x$ is a classical solution. Suppose that $t \mapsto v(t)$ is another classical solution. It is easy to check that the function $s \mapsto S(t-s)v(s)$ is continuous on $[0,t]$ and continuously differentiable on $(0,t)$ with derivative

$$\frac{d}{ds}S(t-s)v(s) = -AS(t-s)v(s) + S(t-s)v'(s) = 0$$

where we used that v is a classical solution. Thus, $s \mapsto S(t-s)v(s)$ is constant on every interval $[0,t]$. Since $v(0) = x$ it follows that $v(t) = S(t-t)v(t) = S(t-0)v(0) = S(t)x = u(t)$. $\square$

Note that for $x \in \mathscr{D}(A)$ the orbit $t \mapsto S(t)x$ even belongs to $C^1([0,T];E) \cap C([0,T];\mathscr{D}(A))$. The reason for defining classical solutions as we did above is that there exist important classes of C_0-semigroups which have the property that $t \mapsto S(t)x$ is a classical solution not only for $x \in \mathscr{D}(A)$, but for all $x \in E$. An example is the class of analytic C_0-semigroups which will be studied later on in this course.

Definition 7.7. *Let T be a linear operator with domain $\mathscr{D}(T)$ on a complex Banach space E. The* resolvent set *of T is the set $\varrho(T)$ consisting of all $\lambda \in \mathbb{C}$ for which there exists a (necessarily unique) bounded linear operator $R(\lambda,T)$ on E such that*

(i) *$R(\lambda,T)(\lambda - T)x = x$ for all $x \in \mathscr{D}(T)$;*
(ii) *$R(\lambda,T)x \in \mathscr{D}(T)$ and $(\lambda - T)R(\lambda,T)x = x$ for all $x \in E$.*

The spectrum *of T is the complement $\sigma(T) := \mathbb{C} \setminus \varrho(T)$.*

We call $R(\lambda,T) = (\lambda - T)^{-1}$ the *resolvent* of T at λ. It is routine to check the *resolvent identity*: for all $\lambda_1, \lambda_2 \in \varrho(T)$ we have

$$R(\lambda_1,T) - R(\lambda_2,T) = (\lambda_2 - \lambda_1)R(\lambda_1,T)R(\lambda_2,T).$$

When T is an operator on a real Banach space we put $\varrho(T) := \varrho(T_{\mathbb{C}})$ and $\sigma(T) := \sigma(T_{\mathbb{C}})$, where $T_{\mathbb{C}}$ is the complexification of T (see Exercise 1).

In the next two lemmas, A is the generator of a C_0-semigroup S on a Banach space E (in the case of a real Banach space, all formulas involving complex numbers should be interpreted in terms of complexifications). We fix constants $M \geqslant 1$ and $\mu \in \mathbb{R}$ such that $\|S(t)\| \leqslant Me^{\mu t}$ for all $t \geqslant 0$.

Proposition 7.8. *We have $\{\lambda \in \mathbb{C} : \ \mathrm{Re}\lambda > \mu\} \subseteq \varrho(A)$ and on this set the resolvent of A is given by*

$$R(\lambda,A)x = \int_0^\infty e^{-\lambda t}S(t)x\,dt, \quad x \in E.$$

As a consequence, for $\mathrm{Re}\lambda > \mu$ we have

$$\|R(\lambda,A)\| \leqslant \frac{M}{\mathrm{Re}\lambda - \mu}.$$

Proof. Fix $x \in E$ and define $R_\lambda x := \int_0^\infty e^{-\lambda t} S(t)x\,dt$. From a straightforward computation using the semigroup property we obtain the identity

$$\lim_{h \downarrow 0} \frac{1}{h}(S(h) - I)R_\lambda x = \lambda R_\lambda x - x$$

from which it follows that $R_\lambda x \in \mathscr{D}(A)$ and $AR_\lambda x = \lambda R_\lambda x - x$. This shows that the bounded operator R_λ is a right inverse for $\lambda - A$.

Integrating by parts and using that $\frac{d}{dt}S(t)x = S(t)Ax$ for $x \in \mathscr{D}(A)$ we obtain

$$\lambda \int_0^T e^{-\lambda t} S(t)x\,dt = -e^{-\lambda T} S(T)x + x + \int_0^T e^{-\lambda t} S(t)Ax\,dt.$$

Since $\mathrm{Re}\lambda > \mu$, sending $T \to \infty$ gives $\lambda R_\lambda x = x + R_\lambda Ax$. This shows that R_λ is also a left inverse. □

Lemma 7.9. *For all $x \in E$ we have* $\lim_{\lambda\to\infty} \lambda R(\lambda, A)x = x$.

Proof. First we prove this for $x \in \mathscr{D}(A)$ by using the resolvent identity. Pick $\lambda' > \mu$. Writing $(\lambda' - A)x =: y$ we have

$$\lambda R(\lambda, A)x - x = \frac{\lambda}{\lambda - \lambda'}(R(\lambda', A)y - R(\lambda, A)y) - R(\lambda', A)y$$

Passing to the limit $\lambda \to \infty$ the right hand side tends to 0. This gives the result for $x \in \mathscr{D}(A)$. By the estimate of Proposition 7.8, the operators $\lambda R(\lambda, A)$ are uniformly bounded for $\lambda \geqslant \mu_0 > \mu$. Therefore the result for $x \in E$ follows by density. □

This lemma self-improves to $\lim_{\lambda\to\infty} \lambda^n R(\lambda, A)^n x = x$, which shows that $\mathscr{D}(A^n)$ is dense in E for all $n \geqslant 1$.

7.2 Duality

For the discussion of the inhomogeneous Cauchy problem in the next section we need some preliminary material on duality of densely defined linear operators.

Let E_1 and E_2 be Banach spaces. To keep track of domains it will be useful to *define* a linear operator A with domain $\mathscr{D}(A)$ from E_1 to E_2 as a pair $(A, \mathscr{D}(A))$, where $\mathscr{D}(A)$ is a linear subspace of E_1 and $A : \mathscr{D}(A) \to E_2$ is a linear mapping. If $(A, \mathscr{D}(A))$ is densely defined, that is, if $\mathscr{D}(A)$ is dense in E_1, we may define a linear operator $(A^*, \mathscr{D}(A^*))$ from E_2^* to E_1^* in the following way. Define $\mathscr{D}(A^*)$ to be the set of all $x_2^* \in E_2^*$ with the property that there exists an element $x_1^* \in E_1^*$ such that

$$\langle x, x_1^* \rangle = \langle Ax, x_2^* \rangle, \qquad \forall x \in \mathscr{D}(A).$$

Since $\mathscr{D}(A)$ is dense in E_1, the element $x_1^* \in E_1^*$ (if it exists) is unique and we set

$$A^* x_2^* := x_1^*, \quad x_2^* \in \mathscr{D}(A^*).$$

Definition 7.10. *Let $(A, \mathscr{D}(A))$ be a densely defined linear operator. The operator $(A^*, \mathscr{D}(A^*))$ is called the* adjoint *of $(A, \mathscr{D}(A))$.*

In order to discuss the properties of A^* in a systematic way it is helpful to consider the topology on the dual space E^* induced by the elements of a Banach space E, the so-called weak*-topology.

Definition 7.11. *The* weak*-topology *on E^* is the topology generated by all sets of the form*

$$\{x^* \in E^* : \ |\langle x, y^* - x^* \rangle| < \varepsilon\}$$

where $x \in E$, $y^* \in E^*$, and $\varepsilon > 0$.

It is easily checked that the mappings $x^* \mapsto \langle x, y^* - x^* \rangle$ are continuous with respect to the weak*-topology, and that the weak*-topology is the coarsest topology on E^* with this property.

Lemma 7.12. *Let V be a non-empty subset of E. The* annihilator

$$V^\perp := \{x^* \in E^* : \ \langle v, x^* \rangle = 0 \ \text{ for all } \ v \in V\}$$

is weak*-closed.

Proof. Let $y^* \notin V^\perp$ be arbitrary. By assumption there exists $v \in V$ such that $\langle v, y^* \rangle \neq 0$. The set

$$U := \Big\{x^* \in E^* : \ |\langle v, y^* - x^* \rangle| < \frac{1}{2} |\langle v, y^* \rangle| \Big\}$$

is weak*-open, contains y^*, and is disjoint from $V^\perp$. It follows that the complement of $V^\perp$ is weak*-open. □

It is an exercise in linear algebra to check that a linear subspace F of E^* is weak*-dense if and only if it separates the points of E, that is, whenever $x \neq y$ in E there is an $x^* \in F$ such that $\langle x, x^* \rangle \neq \langle y, x^* \rangle$. This fact is not really needed however. Whenever we say that a subspace F of E^* is weak*-dense, what we shall actually use is that F separates the points of E and all formulations could be adapted accordingly.

Proposition 7.13. *Let E_1 and E_2 be Banach spaces and let $(A, \mathscr{D}(A))$ be a densely defined linear operator from E_1 to E_2.*

(1) *The adjoint $(A^*, \mathscr{D}(A^*))$ is weak*-closed from E_2^* to E_1^*, that is, the graph of A^* is weak*-closed in $E_2^* \times E_1^*$.*
(2) *If $(A, \mathscr{D}(A))$ is also closed, then $(A^*, \mathscr{D}(A^*))$ is weak*-densely defined, that is, the domain of A^* is weak*-dense in E_2^*.*

Proof. We start with the preliminary remark that if E and F are Banach spaces, then the pairing

$$\langle (x,y),(x^*,y^*)\rangle := \langle x,x^*\rangle + \langle y,y^*\rangle$$

allows us to identify $E^* \times F^*$ with the dual of $E \times F$.

(1): Let $\mathscr{G}(A^*) = \{(x_1^*, A^*x_1^*) : \ x_1^* \in \mathscr{D}(A^*)\}$ be the graph of A^* in $E_2^* \times E_1^*$. By definition of $\mathscr{D}(A^*)$ we have $(x_2^*, x_1^*) \in \mathscr{G}(A^*)$ if and only if

$$\langle (-Ax_1, x_1), (x_2^*, x_1^*)\rangle = 0, \quad \forall x_1 \in \mathscr{D}(A).$$

In other words, $\mathscr{G}(A^*)$ is the annihilator of $\rho(\mathscr{G}(A))$, where $\rho : E_1 \times E_2 \to E_2 \times E_1$ is defined by $\rho(x_1, x_2) = (-x_2, x_1)$. By Lemma 7.12, $\mathscr{G}(A^*)$ is weak*-closed. This proves that A^* is weak*-closed.

(2): Now assume that $(A, \mathscr{D}(A))$ is also closed. We will show that $\mathscr{D}(A^*)$ separates the points of E_2. Suppose $x_2 \neq y_2$ in E_2. Then $(0, x_2 - y_2)$ is a non-zero element of $E_1 \times E_2$ which does not belong to $\mathscr{G}(A)$. Since $\mathscr{G}(A)$ is closed, by the Hahn-Banach theorem there exists an element $(x_1^*, x_2^*) \in (\mathscr{G}(A))^\perp$ such that

$$\langle (0, x_2 - y_2), (x_1^*, x_2^*)\rangle = \langle x_2 - y_2, x_2^*\rangle \neq 0.$$

To finish the proof we check that $x_2^* \in \mathscr{D}(A^*)$. For all $x_1 \in \mathscr{D}(A)$ we have $(x_1, Ax_1) \in \mathscr{G}(A)$ and therefore

$$0 = \langle (x_1, Ax_1), (x_1^*, x_2^*)\rangle = \langle x_1, x_1^*\rangle + \langle Ax_1, x_2^*\rangle.$$

But this means that $x_2^* \in \mathscr{D}(A^*)$ and $A^*x_2^* = -x_1^*$. $\square$

The following simple result 'dualises' the definition of $\mathscr{D}(A^*)$.

Proposition 7.14. *Let $(A, \mathscr{D}(A))$ be a closed and densely defined linear operator from E_1 to E_2. If $x_1 \in E_1$ and $x_2 \in E_2$ are such that $\langle x_2, x_2^*\rangle = \langle x_1, A^*x_2^*\rangle$ for all $x_2^* \in \mathscr{D}(A^*)$, then $x_1 \in \mathscr{D}(A)$ and $Ax_1 = x_2$.*

Proof. We must prove that $(x_1, x_2) \in \mathscr{G}(A)$. Since $\mathscr{G}(A)$ is closed in $E_1 \times E_2$, by the Hahn-Banach theorem it suffices to check that $\langle (x_1, x_2), (x_1^*, x_2^*)\rangle = 0$ for all $(x_1^*, x_2^*) \in (\mathscr{G}(A))^\perp$.

Fix an arbitrary $(x_1^*, x_2^*) \in (\mathscr{G}(A))^\perp$. As in the second part of the previous proof we have $x_2^* \in \mathscr{D}(A^*)$ and $A^*x_2^* = -x_1^*$. Hence,

$$\langle (x_1, x_2), (x_1^*, x_2^*)\rangle = \langle x_1, -A^*x_2^*\rangle + \langle x_2, x_2^*\rangle = 0. \qquad \square$$

7.3 The abstract Cauchy problem

We now take a look at the inhomogeneous abstract Cauchy problem

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in [0,T], \\ u(0) = x \end{cases} \tag{IACP}$$

with initial value $x \in E$. We assume that A generates a C_0-semigroup S on E and take $f \in L^1(0,T;E)$.

Adapting the notion of a classical solution to the problem (IACP) leads to the so-called problem of maximal regularity. Instead of going into this, we refer to the Notes for more information and introduce here two alternative notions of solutions in terms of the integrated equation.

Definition 7.15. *A* strong solution *of* (IACP) *is a function $u \in L^1(0,T;E)$ such that for all $t \in [0,T]$ we have $\int_0^t u(s)\,ds \in \mathscr{D}(A)$ and*

$$u(t) = x + A\int_0^t u(s)\,ds + \int_0^t f(s)\,ds.$$

A weak solution *of* (IACP) *is a function $u \in L^1(0,T;E)$ such that for all $t \in [0,T]$ and $x^* \in \mathscr{D}(A^*)$ we have*

$$\langle u(t), x^*\rangle = \langle x, x^*\rangle + \int_0^t \langle u(s), A^*x^*\rangle\,ds + \int_0^t \langle f(s), x^*\rangle\,ds.$$

As an immediate consequence of Proposition 7.14 we make the following observation:

Proposition 7.16. *Every weak solution of* (IACP) *is a strong solution.*

Of course the converse holds trivially. We proceed with an existence and uniqueness result for strong solutions of (IACP).

Theorem 7.17. *For all $x \in E$ and $f \in L^1(0,T;E)$ the problem* (IACP) *admits a unique strong solution u, which is given by the convolution formula*

$$u(t) = S(t)x + \int_0^t S(t-s)f(s)\,ds. \tag{7.2}$$

If $f \in L^p(0,T;E)$ with $1 \leqslant p < \infty$, then $u \in L^p(0,T;E)$.

Proof. For the existence part, by Proposition 7.16 it suffices to show that (IACP) admits a weak solution. It is an easy consequence of Proposition 7.4 (3) that u is a weak solution corresponding to the initial value x if and only if $t \mapsto u(t) - S(t)x$ is a weak solution corresponding to the initial value 0. Therefore, without loss of generality we may assume that $x = 0$.

Let u be given by (7.2). Then $u \in L^1(0,T;E)$; if $f \in L^p(0,T;E)$, then $u \in L^p(0,T;E)$. By Fubini's theorem and Proposition 7.4 (3), for all $t \in [0,T]$ and $x^* \in \mathscr{D}(A^*)$ we have

$$\begin{aligned}\int_0^t \langle u(s), A^* x^* \rangle \, ds &= \int_0^t \int_0^s \langle f(r), S^*(s-r) A^* x^* \rangle \, dr \, ds \\ &= \int_0^t \int_r^t \langle f(r), S^*(s-r) A^* x^* \rangle \, ds \, dr \\ &= \int_0^t \langle f(r), S^*(t-r) x^* - x^* \rangle \, dr \\ &= \langle u(t), x^* \rangle - \int_0^t \langle f(r), x^* \rangle \, dr.\end{aligned}$$

To prove uniqueness, suppose that u and $\widetilde{u}$ are strong solutions of (IACP). Then $v := u - \widetilde{u}$ is integrable and satisfies $v(t) = A \int_0^t v(s) \, ds$ for all $t \in [0, T]$. Put

$$w(t) := \int_0^t \int_0^s v(r) \, dr \, ds.$$

By the fundamental theorem of calculus, w is continuously differentiable on $[0, T]$, and using Hille's theorem we see that $w(t) \in \mathscr{D}(A)$ and

$$w'(t) = \int_0^t v(s) \, ds = \int_0^t A \int_0^s v(r) \, dr \, ds = Aw(t).$$

Fix $t \in [0, T]$ and put $g(s) := S(t-s)w(s)$. Then g is continuously differentiable on $[0, t]$ with derivative

$$g'(s) = -AS(t-s)w(s) + S(t-s)w'(s) = 0.$$

It follows that g is constant on $[0, t]$. Hence

$$w(t) = g(t) = g(0) = S(t)w(0) = 0.$$

We have shown that $\int_0^t \int_0^s v(r) \, dr \, ds = 0$ for all $t \in [0, T]$. It follows that $v = 0$ almost everywhere. $\square$

7.4 Examples of C_0-semigroups

In this section we collect, without proofs, a number of important examples of C_0-semigroups. We encourage the reader to formulate the corresponding initial value problems; cf. Example 7.1. References to the literature are given in the Notes.

Example 7.18 (Multiplication semigroup). Let $(A, \mathscr{A}, \mu)$ be a σ-finite measure space and let $m : A \to \mathbb{R}$ be μ-measurable. If $\operatorname{ess\,sup}_{\xi \in A} f(\xi) < \infty$, then the formula

$$S(t)f(\xi) := e^{tm(\xi)} f(\xi)$$

defines a C_0-semigroup on $L^p(A)$ for $1 \leqslant p < \infty$. The domain of its generator A consists of all $f \in L^p(A)$ such that $mf \in L^p(A)$, and for $f \in \mathscr{D}(A)$ we have $Af = mf$.

Example 7.19 (Translation semigroup). On the space $L^p(\mathbb{R}_+)$, $1 \leqslant p < \infty$, the formula

$$(S(t)f)(\xi) := f(\xi + t)$$

defines a C_0-semigroup S. The domain of its generator A consists of all $f \in L^p(\mathbb{R})$ whose weak derivative f' exists and belongs to $L^p(\mathbb{R})$, and for $f \in \mathscr{D}(A)$ we have $Af = f'$.

These two examples represent perhaps the simplest constructions of C_0-semigroups and can be extended in various ways. We continue with two examples involving the Laplace operator.

Example 7.20 (Heat semigroup). On $L^p(\mathbb{R}^d)$, $1 \leqslant p < \infty$, the formula

$$(S(t)f)(\xi) := \frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^d} f(\eta) \exp\Big(-\frac{|\xi - \eta|^2}{4t}\Big)\, d\eta$$

defines a C_0-semigroup. Its generator A is given by $\mathscr{D}(A) = W^{2,p}(\mathbb{R}^d)$ and $Af = \Delta f$.

Example 7.21 (Heat semigroup on bounded domains with Dirichlet boundary conditions). Let D be a bounded domain in $\mathbb{R}^d$ with C^2-boundary ∂D. On the space $L^p(D)$ with $1 \leqslant p < \infty$, the *Dirichlet Laplacian* is the operator A defined by

$$\mathscr{D}(A) := W^{2,p}(D) \cap W_0^{1,p}(D),$$
$$Af := \Delta f \quad \text{for} \quad f \in \mathscr{D}(A).$$

See Example 7.1. This operator is the generator of a C_0-semigroup on $L^p(D)$.

The previous two examples admit far-reaching generalisations to more general second order elliptic operators, and also different kinds of boundary conditions can be allowed.

We continue with two examples of operators generating a C_0*-group.* These are defined in the same way as C_0-semigroups, except that the index set is now the whole real line.

Example 7.22 (Wave group). On the space $W^{1,2}(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ we consider the operator A defined by

$$\mathscr{D}(A) := W^{2,2}(\mathbb{R}^d) \times W^{1,2}(\mathbb{R}^d),$$
$$A(f_1, f_2) := (f_2, \Delta f_1) \quad \text{for} \quad (f_1, f_2) \in \mathscr{D}(A).$$

This operator is the generator of a C_0-group on $W^{1,2}(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ which is associated with the wave equation $u''(t) = \Delta u$, written as a system $u' = v$, $v' = \Delta u$.

Example 7.23 (Unitary C_0-groups on Hilbert spaces). If A is a self-adjoint operator on a complex Hilbert space H, then iA is the generator of a C_0-group S of unitary operators on H. This classical result of Stone is of fundamental importance in quantum mechanics. By the spectral theorem for self-adjoint operators, this example can be viewed as a special case of Example 7.18.

7.5 Exercises

1. Suppose that E is a real Banach space. The product $E \times E$ can be given the structure of a complex vector space by introducing a complex scalar multiplication as follows:

$$(a + ib)(x, y) := (ax - by, bx + ay).$$

The idea is, of course, to think of the pair $(x, y) \in E \times E$ as if it were $x + iy$. The resulting complex vector space is denoted by $E_{\mathbb{C}}$.

a) Prove that the formula

$$\|(x, y)\| := \sup_{\theta \in [0,2\pi]} \|(\cos\theta)x + (\sin\theta)y\|$$

defines a norm on $E_{\mathbb{C}}$ which turns $E_{\mathbb{C}}$ into a complex Banach space.

b) Check that this norm on $E_{\mathbb{C}}$ extends the norm of E in the sense that for all $x \in E$,

$$\|(x, 0)\| = \|(0, x)\| = \|x\|.$$

c) Check that for all $x, y \in E$ we have $\|(x, y)\| = \|(x, -y)\|$.

d) Show that if T is a (real-)linear bounded operator on E, then T extends to a bounded (complex-)linear operator $T_{\mathbb{C}}$ on $E_{\mathbb{C}}$ by putting

$$T_{\mathbb{C}}(x, y) := (Tx, Ty),$$

and check that $\|T_{\mathbb{C}}\| = \|T\|$.

A norm on $E_{\mathbb{C}}$ with the properties b), c), d) is called a *complexification* of the norm of E. The norm introduced in a) is by no means the unique complexification of the norm of E, and in concrete examples there is often a more natural choice.

e) Show that any two complex norms on $E_{\mathbb{C}}$ which satisfy b) and c) are equivalent.

By e), the spectrum of $T_{\mathbb{C}}$ is independent of the particular complexification chosen.

2. In this exercise we prove some properties of resolvents. We assume that $(T, \mathscr{D}(T))$ is a linear operator from E to E with resolvent set $\varrho(T)$.

a) Prove that if $\varrho(T) \neq \varnothing$, then T is closed.

b) Prove the resolvent identity: for all $\lambda_1, \lambda_2 \in \varrho(T)$ we have

$$R(\lambda_1, T) - R(\lambda_2, T) = (\lambda_2 - \lambda_1)R(\lambda_1, T)R(\lambda_2, T).$$

c) Prove that $\varrho(T)$ is an open subset of $\mathbb{C}$.

d) Prove that

$$\lim_{\lambda \to \mu} \frac{R(\lambda, T) - R(\mu, T)}{\lambda - \mu} = -R(\mu, T)^2$$

with convergence in the operator norm.

e) Prove that if T is closed and densely defined, then $\varrho(T^*) = \varrho(T)$ and

$$R(\lambda, T^*) = R(\lambda, T)^*, \quad \lambda \in \varrho(T) = \varrho(T^*).$$

f) Show that every closed subset of $\mathbb{C}$ is the spectrum of a suitable closed operator T.

3. Let S be a C_0-semigroup on E which is *uniformly bounded*, that is, $\sup_{t\geqslant 0}\|S(t)\| < \infty$. We show that there exists an equivalent norm $|\!|\!|\cdot|\!|\!|$ on E such that S is a *contraction semigroup* with respect to $|\!|\!|\cdot|\!|\!|$, that is, $|\!|\!|S(t)|\!|\!| \leqslant 1$ for all $t \geqslant 0$.
   a) Show that $|\!|\!|x|\!|\!| := \sup_{t\geqslant 0}\|S(t)x\|$ defines an equivalent norm on E.
   b) Show that S is a contraction semigroup with respect to $|\!|\!|\cdot|\!|\!|$.

4. Let S be a C_0-semigroup on E with generator A, and suppose that $\|S(t)\| \leqslant Me^{\mu t}$ for all $t \geqslant 0$. Prove that

$$\|(R(\lambda, A))^k\| \leqslant M/(\mathrm{Re}\lambda - \mu)^k, \quad \mathrm{Re}\lambda > \mu,\ k = 1, 2, \ldots.$$

   Hint: By considering $A - \mu$ instead of A we may assume that $\mu = 0$. In that situation observe that $|\!|\!|R(\lambda, A)|\!|\!| \leqslant 1/\mathrm{Re}\lambda$.

   Remark: A celebrated theorem of Hille and Yosida asserts that the converse holds as well. We refer to the Notes for more information.

5. Let $(A, \mathscr{A}, \mu)$ be a σ-finite measure space. Suppose $f : A \to E^*$ is a function such that $\xi \mapsto \langle x, f(\xi)\rangle$ belongs to $L^1(A)$ for all $x \in E$.
   a) Show that the map $S : E \to L^1(A)$ defined by $Sx := \langle x, f\rangle$ is closed.
   b) Conclude from this that the formula

$$\langle x, x^*\rangle := \int_A \langle x, f\rangle\, d\mu$$

   defines a bounded linear functional $x^* \in E^*$.

   The functional x^* is called the *weak*-integral* of f with respect to μ, notation:

$$x^* =: \mathrm{weak}^* \int_A f\, d\mu.$$

   c) Show that the weak*-integral commutes with adjoints of bounded operators on E.
   d) Show that if f is an E^*-valued Bochner integrable function, then the Bochner integral and the weak*-integral of f agree.

   Now suppose that A generates a C_0-semigroup on E and put $S^*(t) := (S(t))^*$ for $t \geqslant 0$.

   e) Prove the following dual version of the identities in Proposition 7.4 (3): for all $x^* \in E^*$ and $t \geqslant 0$ we have $\mathrm{weak}^* \int_0^t S^*(s)x^*\, ds \in \mathscr{D}(A^*)$ and

$$A^*\Big(\mathrm{weak}^* \int_0^t S^*(s)x^*\, ds\Big) = S^*(t)x^* - x^*.$$

   If $x^* \in \mathscr{D}(A^*)$, then both sides are equal to $\mathrm{weak}^* \int_0^t S^*(s)A^*x^*\, ds$.

Notes. Excellent recent introductions to the theory of C_0-semigroups include the monographs by ARENDT, BATTY, HIEBER, NEUBRANDER [3], DAVIES [29], ENGEL and NAGEL [38], GOLDSTEIN [41], PAZY [89]. For a discussion of the examples in Section 7.4 we refer to these sources. Their monumental 1957 treatise of HILLE and PHILLIPS[48] is freely available on-line (`http://www.ams.org/online_bks/coll31/`).

Due to limitations of space and time we have chosen not to discuss the two basic generation theorems of semigroup theorem. The first of these, the *Hille-Yosida theorem*, reads as follows.

Theorem 7.24 (Hille-Yosida theorem). *For a densely defined operator A on a Banach space E and constants $M \geqslant 1$ and $\mu \in \mathbb{R}$, the following assertions are equivalent:*

(1) *A generates a C_0-semigroup on E satisfying $\|S(t)\| \leqslant Me^{\mu t}$ for all $t \geqslant 0$;*
(2) *$\{\lambda \in \mathbb{C} : \ \lambda > \mu\} \subseteq \varrho(A)$ and $\|(R(\lambda, A))^k\| \leqslant M/(\mathrm{Re}\lambda - \mu)^k$ for all $\lambda > \mu$ and $k = 1, 2, \ldots$;*
(3) *$\{\lambda \in \mathbb{C} : \ \mathrm{Re}\lambda > \mu\} \subseteq \varrho(A)$ and $\|(R(\lambda, A))^k\| \leqslant M/(\mathrm{Re}\lambda - \mu)^k$ for all $\mathrm{Re}\lambda > \mu$ and $k = 1, 2, \ldots$.*

For C_0-contraction semigroups, Theorem 7.24 was obtained independently and simultaneously by HILLE [47] and YOSIDA [111]; the extension to arbitrary C_0-semigroups is due to FELLER, MIYADERA, PHILLIPS. The easy implication (1)⇒(3) has been discussed in Exercise 4 and (3)⇒(2) is trivial; the difficult implication is (2)⇒(1).

In order to state the second generation theorem, the *Lumer-Phillips theorem*, for $x \in E$ define $\partial(x) := \{x^* \in E^* : \ \|x^*\| = \|x\|, \ \langle x, x^*\rangle = \|x\|\|x^*\|\}$. By the Hahn-Banach theorem, $\partial(x) \neq \varnothing$.

Theorem 7.25 (Lumer-Phillips theorem). *For a densely defined operator A on a Banach space E with $\varrho(A) \cap (0, \infty) \neq \varnothing$ the following assertions are equivalent:*

(1) *A generates a C_0-contraction semigroup on E;*
(2) *For all $x \in \mathscr{D}(A)$ and $\lambda > 0$ we have $\|(\lambda - A)x\| \geqslant \lambda\|x\|$;*
(2) *For all $x \in \mathscr{D}(A)$ and all $x^* \in \partial(x)$ we have $\mathrm{Re}\langle Ax, x^*\rangle \leqslant 0$;*
(3) *For all $x \in \mathscr{D}(A)$ there exists $x^* \in \partial(x)$ such that $\mathrm{Re}\langle Ax, x^*\rangle \leqslant 0$.*

This theorem, as its name suggests, is due to LUMER and PHILLIPS [71]. We shall return to it later in the context of analytic C_0-semigroups. A detailed account of Theorems 7.24 and 7.25 and their history is given in [38].

The terminology for the various notions of solutions is not entirely standard. Ours is suggested by that of DA PRATO and ZABCZYK [27] for solutions of stochastic evolution equations.

The results of Section 7.2 can be found in any introductory text on functional analysis.

Theorem 7.17 is due to Ball [5], who also proved the following converse: if (IACP) admits a unique weak solution for all $f \in L^1(0,T;E)$ and initial values $x \in E$, then A is the generator of a C_0-semigroup on E.

The convolution formula (7.2) is often taken as the definition of a *mild solution*. Typical questions then revolve around proving regularity properties of mild solutions in terms of properties of the forcing function f and the semigroup S. We refer to [89, Chapter 4] for some elementary results in this direction. For the treatment of certain classes of non-linear Cauchy problems it is of particular importance to know whether the mild solutions have *maximal L^p-regularity*, meaning that for all $f \in L^p(0,T;E)$ the solution u belongs to $W^{1,p}(0,T;E) \cap L^p(0,T;\mathscr{D}(A))$. A necessary condition for this is that S be analytic; it is a classical result that this condition is also sufficient in Hilbert spaces. For analytic C_0-semigroups on Banach spaces the maximal regularity problem has recently be settled by Kalton and Lancien [57] (who gave a counterexample in L^p-spaces E) and Weis [108] (who obtained necessary and sufficient conditions for maximal L^p-regularity in UMD Banach spaces E). We refer to the lectures by Kunstmann and Weis [61] for a detailed account of this problem and its history, as well as a number of non-trivial examples.

A systematic discussion of complexifications is given in Muñoz, Sarantopoulos, Tonge [79]. The reader is warned that not every complex Banach space is the complexification of some underlying real Banach space. The first (non-constructive) proof of this fact was given by Bourgain [11]. An explicit counterexample was found subsequently by Kalton [56].