8

Linear equations with additive noise I

Let A be the generator of a C_0-semigroup S on E. In the previous lecture we have seen that the inhomogeneous abstract Cauchy problem

$$u'(t) = Au(t) + f(t), \quad u(0) = x,$$

is solved by the convolution formula

$$u(t) = S(t)x + \int_0^t S(t-s)f(s)\,ds,$$

We now turn to the stochastic analogue of this equation,

$$\begin{cases} dU(t) = AU(t)\,dt + B\,dW_H, \\ \ \ u(0) = x, \end{cases}$$

where W_H is an H-cylindrical Brownian motion, and $B \in \mathscr{L}(H,E)$ is a bounded operator. In concrete examples, W_H models space-time white noise and B 'injects' this noise into the state space E. Reasoning by analogy, this equation should be solved by the stochastic convolution

$$U(t) = S(t)x + \int_0^t S(t-s)B\,dW_H.$$

We shall see that this is indeed correct, provided the $\mathscr{L}(H,E)$-valued function $S(\cdot)B$ is stochastically integrable with respect to W_H.

8.1 Stochastic preliminaries

In this section we collect several results which are needed in the proof of the main result of this lecture, Theorem 8.6. We begin with an integrations by parts formula.

Lemma 8.1 (Integration by parts). *For all $\phi \in C^1[0,T]$ and $h \in H$, almost surely the following identity holds:*

$$\int_0^T \phi'(t)W_H(t)h\,dt = \phi(T)W_H(T)h - \int_0^T \phi\otimes h\,dW_H.$$

Before we prove the lemma we clarify the meaning of the integral on the left hand side. Recalling that $W_H h$ is a Brownian motion, using Corollary 6.10 we select a version of $W_H h$ whose trajectories are continuous almost surely. Then the integral on the left hand side is well defined almost surely as a Lebesgue integral.

Proof. We may assume that $\phi'(0) = 0$; this somewhat simplifies the calculations below.

We begin by noting that $\int_0^T \phi\otimes h\,dW_H = \int_0^T \phi\,dW_H h$. For step functions this is clear from the definitions and the general case follows by approximation. Rescaling h to unit length, it is therefore enough to prove the almost sure identity

$$\int_0^T \phi'(t)W(t)\,dt = \phi(T)W(T) - \int_0^T \phi\,dW$$

for functions $\phi \in C^1[0,T]$, where W is a scalar Brownian motion. This identity is a special case of Itô's formula, but for those readers who are not familiar with it we shall give a self-contained argument (which is indeed nothing but the proof of Itô's formula in the special case considered here). Let

$$g := \sum_{n=1}^N c_n 1_{(t_{n-1},t_n]}, \quad G := \sum_{n=1}^N \sum_{m=1}^n c_m(t_m - t_{m-1})1_{(t_{n-1},t_n]}$$

with $c_1, \dots, c_N$ scalars and $0 = t_0 < \dots < t_N = T$. Then, almost surely,

$$\int_0^T g(t)W(t)\,dt = \sum_{n=1}^N c_n \int_{t_{n-1}}^{t_n} W(t)\,dt$$

and

$$\begin{aligned}
&G(T)W(T) - \int_0^T G\,dW\\
&= \sum_{m=1}^N c_m(t_m - t_{m-1})W(T) - \sum_{n=1}^N\sum_{m=1}^n c_m(t_m - t_{m-1})(W(t_n) - W(t_{n-1}))\\
&= \sum_{m=1}^N c_m(t_m - t_{m-1})W(T) - \sum_{m=1}^N\sum_{n=m}^N c_m(t_m - t_{m-1})(W(t_n) - W(t_{n-1}))\\
&= \sum_{m=1}^N c_m(t_m - t_{m-1})W(T) - \sum_{m=1}^N c_m(t_m - t_{m-1})(W(T) - W(t_{m-1}))\\
&= \sum_{m=1}^N c_m(t_m - t_{m-1})W(t_{m-1}).
\end{aligned}$$

Now let $\phi \in C^1[0,T]$ be given and put $g_k := \sum_{n=1}^{N_k} \phi'(t_{k,n-1}) 1_{(t_{k,n-1},t_{k,n}]}$, assuming that $\lim_{k\to\infty} \sup_{1\leqslant n\leqslant N_k} (t_{k,n} - t_{k,n-1}) = 0$. Then $\lim_{k\to\infty} g_k = \phi'$ uniformly on $(0,T]$. Defining the functions G_k in terms of the g_k as above, we have $\lim_{k\to\infty} G_k = \phi$ uniformly on $(0,T]$. The above computation gives the following identity, which almost surely holds for all k:

$$\begin{aligned}
&\Big| \int_0^T g_k(t) W(t)\, dt - \Big(G_k(T) W(T) - \int_0^T G_k \, dW \Big) \Big| \\
&= \Big| \sum_{n=1}^{N_k} \phi'(t_{k,n-1}) \int_{t_{k,n-1}}^{t_{k,n}} W(t)\, dt - \sum_{n=1}^{N_k} \phi'(t_{k,n-1})(t_{k,n} - t_{k,n-1}) W(t_{k,n-1}) \Big|.
\end{aligned}$$

As $k \to \infty$, the left hand side tends to $| \int_0^T \phi'(t) W(t)\, dt - (\phi(T) W(T) - \int_0^T \phi \, dW)|$ in $L^2(\Omega)$ and hence in measure, whereas the right hand side tends to 0 almost surely by path continuity. This proves the lemma. $\square$

We continue with a Fubini theorem for interchanging a Bochner integral and a stochastic integral of an H-valued function. In this context it is natural to impose an integrability condition which is L^1 with respect to the variable of Bochner integration and L^2 with respect to the variable of stochastic integration.

Lemma 8.2 (Stochastic Fubini theorem). *Let $\phi : (0,T) \times (0,T) \to H$ be a strongly measurable function satisfying*

$$\int_0^T \Big(\int_0^T \|\phi(s,t)\|_H^2 \, dt \Big)^{\frac{1}{2}} ds < \infty.$$

(1) *$t \mapsto \phi(s,t)$ belongs to $L^2(0,T;H)$ for almost all $s \in (0,T)$, and the $L^2(\Omega)$-valued function $s \mapsto \int_0^T \phi(s,t)\, dW_H(t)$ belongs to $L^1(0,T;L^2(\Omega))$;*
(2) *$s \mapsto \phi(s,t)$ belongs to $L^1(0,T;H)$ for almost all $t \in (0,T)$, and the H-valued function $t \mapsto \int_0^T \phi(s,t)\, ds$ belongs to $L^2(0,T;H)$;*
(3) *in $L^2(\Omega)$ we have*

$$\int_0^T \Big(\int_0^T \phi(s,t)\, dW_H(t) \Big) ds = \int_0^T \Big(\int_0^T \phi(s,t)\, ds \Big) dW_H(t).$$

Proof. (1): By assumption we have $\phi \in L^1(0,T;L^2(0,T;H))$, and therefore (1) is an immediate consequence of the Itô isometry (6.2).

(2): We claim that for a step function $\phi : (0,T) \times (0,T) \to H$ we have

$$\|\phi\|_{L^2(0,T;L^1(0,T;H))} \leqslant \|\phi\|_{L^1(0,T;L^2(0,T;H))}.$$

It suffices to prove this for $T = 1$. If $\phi = \sum_{j=1}^M \sum_{k=1}^N 1_{(s_{j-1},s_j)} 1_{(t_{k-1},t_k)} \otimes h_{jk}$, then

$$
\begin{aligned}
\|\phi\|^2_{L^2(0,T;L^1(0,T;H))} &= \sum_{k=1}^{N}(t_k - t_{k-1})\Big(\sum_{j=1}^{M}(s_j - s_{j-1})\|h_{jk}\|\Big)^2 \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}(s_i - s_{i-1})(s_j - s_{j-1})\sum_{k=1}^{N}(t_k - t_{k-1})\|h_{ik}\|\|h_{jk}\|
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\|\phi\|^2_{L^1(0,T;L^2(0,T;H))} &= \Big(\sum_{j=1}^{M}(s_j - s_{j-1})\Big(\sum_{k=1}^{N}(t_k - t_{k-1})\|h_{jk}\|^2\Big)^{\frac12}\Big)^2 \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M}(s_i - s_{i-1})(s_j - s_{j-1}) \\
&\quad \times \Big(\sum_{k=1}^{N}(t_k - t_{k-1})\|h_{ik}\|^2\Big)^{\frac12}\Big(\sum_{k=1}^{N}(t_k - t_{k-1})\|h_{jk}\|^2\Big)^{\frac12}.
\end{aligned}
$$

In view of the Cauchy-Schwarz inequality, this proves the claim. It follows that the identity mapping on step functions extends to a continuous embedding of $L^1(0,T;L^2(0,T;H))$ into $L^2(0,T;L^1(0,T;H))$. This gives (2).

(3): For step functions ϕ the identity follows by a trivial computation, and its extension to functions $\phi \in L^1(0,T;L^2(0,T;H))$ is obtained by approximation using (1) and (2). $\square$

8.2 Semigroup preliminaries

Let A be the generator of a C_0-semigroup S on E. Define

$$E^{\odot} := \overline{\mathscr{D}(A^*)},$$

the closure being taken with respect to the norm topology of E^*. Note that $E^{\odot}$ is a closed and weak*-dense subspace of E^*. We let $A^{\odot}$ be the *part* of A^* in $E^{\odot}$, that is,

$$
\begin{aligned}
\mathscr{D}(A^{\odot}) &:= \{x^* \in \mathscr{D}(A^*) :\ A^*x^* \in E^{\odot}\}, \\
A^{\odot}x^* &:= A^*x^*, \qquad x^* \in \mathscr{D}(A^{\odot}).
\end{aligned}
$$

Proposition 8.3. *Let A be the generator of a C_0-semigroup S on E. The adjoint semigroup S^* restricts to a C_0-semigroup $S^{\odot}$ on $E^{\odot}$ whose generator equals $A^{\odot}$.*

Proof. For $t \in [0,T]$, $x \in E$, and $x^* \in \mathscr{D}(A^*)$ we have

$$|\langle x, S^*(t)x^* - x^*\rangle| \leqslant \int_0^t |\langle x, S^*(s)A^*x^*\rangle|\,ds \leqslant t\|x\| \cdot \sup_{s\in[0,T]}\|S(s)\| \cdot \|A^*x^*\|.$$

Taking the supremum over all $x \in E$ of norm $\|x\| \leqslant 1$ gives

$$\limsup_{t \downarrow 0} \|S^*(t)x^* - x^*\| \leqslant \lim_{t \downarrow 0} t \cdot \sup_{s \in [0,T]} \|S(s)\| \cdot \|A^* x^*\| = 0.$$

Since $\mathscr{D}(A^*)$ is invariant under S^* (by duality we have $A^* S^*(t)x^* = S^*(t)A^*x^*$ for $x^* \in \mathscr{D}(A^*)$ and $t \geqslant 0$) and $S^*(t)$ is uniformly bounded on $[0,T]$, it follows that S^* restricts to a C_0-semigroup $S^\odot$ on $E^\odot$.

Let B denote the generator of $S^\odot$. If $x^\odot \in \mathscr{D}(B)$, then for all $x \in \mathscr{D}(A)$ we have

$$\langle x, Bx^\odot \rangle = \lim_{t \downarrow 0} \frac{1}{t} \langle x, S^\odot(t)x^\odot - x^\odot \rangle = \lim_{t \downarrow 0} \frac{1}{t} \langle S(t)x - x, x^\odot \rangle = \langle Ax, x^\odot \rangle.$$

Hence $x^\odot \in \mathscr{D}(A^*)$ and $A^* x^\odot = Bx^\odot$. Since $Bx^\odot \in E^\odot$ it follows that $x^\odot \in \mathscr{D}(A^\odot)$ and $A^\odot x^\odot = A^* x^\odot = Bx^\odot$. Conversely, if $x^\odot \in \mathscr{D}(A^\odot)$, then $A^\odot x^\odot \in E^\odot$ and $s \mapsto S^\odot(s)A^\odot x^\odot$ is strongly continuous and, for all $x \in E$,

$$\Big| \Big\langle x, A^\odot x^\odot - \frac{1}{t}(S^\odot(t)x^\odot - x^\odot) \Big\rangle \Big| \leqslant \|x\| \Big\| A^\odot x^\odot - \frac{1}{t} \int_0^t S^\odot(s) A^\odot x^\odot \, ds \Big\|.$$

Hence,

$$\Big\| A^\odot x^\odot - \frac{1}{t}(S^\odot(t)x^\odot - x^\odot) \Big\| \leqslant \Big\| A^\odot x^\odot - \frac{1}{t} \int_0^t S^\odot(s) A^\odot x^\odot \, ds \Big\|.$$

Since $A^\odot x^\odot \in E^\odot$, the right hand side tends to 0 as $t \downarrow 0$ by strong continuity. This proves that $x^\odot \in \mathscr{D}(B)$ and $Bx^\odot = A^\odot x^\odot$. $\square$

This proposition will be used in combination with the next approximation result.

Lemma 8.4. *For $k = 0, 1, 2, \dots$, linear combinations of the functions $\phi \otimes x$ with $\phi \in C^k[0,T]$ and $x \in E$ are dense in $C^k([0,T]; E)$.*

Proof. We begin with the case $k = 0$, which is proved by a standard partition of unity argument. Let $f \in C([0,T]; E)$ be arbitrary. Let $\varepsilon > 0$. Since f is uniformly continuous we may choose $\delta > 0$ such that $\|f(t) - f(s)\| < \varepsilon$ whenever $|t - s| < \delta$. Let $I_1, \dots, I_N$ be open intervals of length $< \delta$ covering $[0,T]$ and let $\phi_1, \dots, \phi_N$ be a partition of unity with respect to this cover, that is, $0 \leqslant \phi_n \leqslant 1$, ϕ_n is supported in I_n, and $\sum_{n=1}^N \phi_n = 1$. Choose points $t_n \in [0,T] \cap I_n$, let $x_n := f(t_n)$, and put $f_\varepsilon := \sum_{n=1}^N \phi_n \otimes x_n$. Fix $t \in [0,T]$. If $t \in I_n$, then $|t - t_n| < \delta$ and therefore $\|f(t) - x_n\| < \varepsilon$. If $t \notin I_n$, then $\phi_n(t) = 0$. Hence, using that $f = \sum_{n=1}^N \phi_n f$,

$$\|f(t) - f_\varepsilon(t)\| \leqslant \sum_{n=1}^N \phi_n(t) \|f(t) - x_n\| \leqslant \varepsilon \sum_{n:\ t \in I_n} \phi_n(t) \leqslant \varepsilon.$$

This proves that $\|f - f_\varepsilon\| \leqslant \varepsilon$.

The general case is proved with induction on k. Suppose the lemma has been proved for $k = 0, \ldots, l$ and let $f \in C^{l+1}([0,T]; E)$ be arbitrary. Then $f' \in C^l([0,T]; E)$ and therefore we can find functions $g_j \in C^l([0,T]; E)$ of the form $g_j = \sum_{n=1}^{N_j} \phi_{jn} \otimes x_{jn}$ with $\phi_{jn} \in C^l[0,T]$ and $x_{jn} \in E$ such that $\lim_{j\to\infty} g_j = f'$ in $C^l([0,T]; E)$. Let $\psi_{jn}(t) := \int_0^t \phi_{jn}(s)\, ds$, put $x_0 := f(0)$, and set

$$f_j := 1 \otimes x_0 + \sum_{n=1}^{N_j} \psi_{jn} \otimes x_{jn}.$$

Then $\lim_{j\to\infty} f_j = f$ in $C([0,T]; E)$ and $\lim_{n\to\infty} f_j' = f'$ in $C^l([0,T]; E)$, so $\lim_{j\to\infty} f_j = f$ in $C^{l+1}([0,T]; E)$. $\square$

8.3 Existence and uniqueness: cylindrical Brownian motion

We consider the stochastic abstract Cauchy problem

$$\begin{cases} dU(t) = AU(t)\, dt + B\, dW_H(t), & t \in [0,T], \\ U(0) = x. \end{cases} \tag{SACP}$$

Here A is the generator of a C_0-semigroup $\{S(t)\}_{t\geqslant 0}$ on E, W_H is an H-cylindrical Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P})$, and $B \in \mathscr{L}(H, E)$ is a given bounded operator.

An E-valued process $\{U(t)\}_{t\in[0,T]}$ will be called *strongly measurable* if it has a version which is strongly $\mathscr{B}([0,T]) \times \mathscr{F}$-measurable on $[0,T] \times \Omega$.

Definition 8.5. *A* weak solution *of the problem* (SACP) *is an E-valued process $\{U^x(t)\}_{t\in[0,T]}$ which has a strongly measurable version with the following properties:*

(i) *almost surely, the paths $t \mapsto U^x(t)$ are integrable;*
(ii) *for all $t \in [0,T]$ and $x^* \in \mathscr{D}(A^*)$ we have, almost surely,*

$$\langle U^x(t), x^* \rangle = \langle x, x^* \rangle + \int_0^t \langle U^x(s), A^* x^* \rangle\, ds + W_H(t) B^* x^*.$$

In order not to overburden notations, we do not distinguish notationally the process $\{U^x(t)\}_{t\in[0,T]}$ from its version with the properties (i) and (ii).

Theorem 8.6. *The following assertions are equivalent:*

(1) *the problem* (SACP) *has a weak solution $\{U^x(t)\}_{t\in[0,T]}$;*
(2) *$t \mapsto S(t)B$ is stochastically integrable on $(0,T)$ with respect to W_H.*

In this situation, for every $t \in (0,T)$ the function $s \mapsto S(t-s)B$ is stochastically integrable on $(0,t)$ with respect to W_H and almost surely we have

$$U^x(t) = S(t)x + \int_0^t S(t-s)B\, dW_H(s). \tag{8.1}$$

Proof. We start by noting that $t \mapsto U^x(t)$ is a weak solution corresponding to the initial value x if and only if $t \mapsto U^x(t) - S(t)x$ is a weak solution corresponding to the initial value 0. Without loss of generality we shall therefore assume that $x = 0$ and write $U(t) := U^0(t)$ for convenience.

(1) ⇒ (2): We will show first that for all $t \in [0,T]$ and $x^* \in \mathscr{D}(A^{\odot 2})$, almost surely we have

$$\langle U(t), x^* \rangle = \int_0^t B^* S^*(t-s)x^*\, dW_H(s). \tag{8.2}$$

Fix $t \in [0,T]$ and $x^\odot \in \mathscr{D}(A^\odot)$. By Fubini's theorem, almost surely the identity

$$\langle U(s), x^\odot \rangle = \int_0^s \langle U(r), A^\odot x^\odot \rangle\, dr + W_H(s)B^* x^\odot \tag{8.3}$$

holds for almost all $s \in (0,t)$; here we use that both terms on the right hand side are jointly measurable on $(0,t) \times \Omega$. In combination with Lemma 8.1 this gives, for any C^1-function $\phi : [0,t] \to \mathbb{R}$,

$$\begin{aligned}
&\int_0^t \phi'(s) \langle U(s), x^\odot \rangle\, ds \\
&\quad = \int_0^t \phi'(s) \Big(\int_0^s \langle U(r), A^\odot x^\odot \rangle\, dr \Big)\, ds + \int_0^t \phi'(s) W_H(s) B^* x^\odot\, ds \\
&\quad = \phi(t) \int_0^t \langle U(s), A^\odot x^\odot \rangle\, ds - \int_0^t \phi(s) \langle U(s), A^\odot x^\odot \rangle\, ds \\
&\qquad\quad + \phi(t) W_H(t) B^* x^\odot - \int_0^t \phi(s) B^* x^\odot\, dW_H(s)
\end{aligned}$$

almost surely. Multiplying both sides of (8.3) with $\phi(t)$, putting $f := \phi \otimes x^\odot$ and rewriting, we obtain

$$\langle U(t), f(t) \rangle = \int_0^t \langle U(s), f'(s) + A^\odot f(s) \rangle\, ds + \int_0^t B^* f(s)\, dW_H(s) \tag{8.4}$$

almost surely. By Lemma 8.4 applied to the Banach space $\mathscr{D}(A^\odot)$, this identity extends to arbitrary functions $f \in C^1([0,t]; \mathscr{D}(A^\odot))$. In particular we may take $f(s) = S^\odot(t-s)x^\odot$, with $x^\odot \in \mathscr{D}(A^{\odot 2})$. For this choice of f, the identity (8.4) reduces to (8.2).

So far we have proved that (8.2) holds for functionals $x^* \in \mathscr{D}(A^{\odot 2})$. We shall prove next that (8.2) holds for functionals $x^* \in E^*$. Then the stochastic integrability of $s \mapsto S(t-s)B$ on $(0,t)$ follows from Theorem 6.17.

The extension of (8.2) from functionals $x^* \in \mathscr{D}(A^{\odot 2})$ to functionals $x^* \in E^*$ is not entirely straightforward since in general $\mathscr{D}(A^{\odot 2})$ is only weak*-dense in E^*. Let $x^* \in E^*$ be arbitrary and fixed, and let weak*-$\lim_{n\to\infty} x_n^* = x^*$ with all $x_n^* \in \mathscr{D}(A^{\odot 2})$ (for instance, take $x_n^* = \lambda_n^3 R(\lambda_n, A^*)^3 x^*$ with suitable $\lambda_n \to \infty$). By dominated convergence, for all $f \in L^2(0,t;H)$ we have

$$\lim_{n\to\infty} [f, B^*S^*(t-\cdot)x_n^*]_{L^2(0,t;H)} = [f, B^*S^*(t-\cdot)x^*]_{L^2(0,t;H)}.$$

It follows that for all $N \geqslant 1$, $B^*S^*(t-\cdot)x^*$ belongs to the weak closure in $L^2(0,t;H)$ of the tail sequence $(B^*S^*(t-\cdot)x_n^*)_{n=N}^\infty$. By the Hahn-Banach theorem, $B^*S^*(t-\cdot)x^*$ belongs to the strong closure in $L^2(0,t;H)$ of the convex hull of this sequence. It follows that there exist vectors y_N^*, belonging to the convex hull of $(x_n^*)_{n=N}^\infty$, such that

$$\|B^*S^*(t-\cdot)y_N^* - B^*S^*(t-\cdot)x^*\|_{L^2(0,t;H)} < \frac{1}{N}.$$

The isometry (6.2) implies that

$$\lim_{N\to\infty} \int_0^t B^*S^*(t-s)y_N^*\, dW_H(s) = \int_0^t B^*S^*(t-s)x^*\, dW_H(s)$$

in $L^2(\Omega)$. By passing to a subsequence and using that weak*-$\lim_{N\to\infty} y_N^* = x^*$ (this follows from the fact that we used the tail sequence $(x_n^*)_{n=N}^\infty$ to define y_N^*), we obtain

$$\begin{aligned}\langle U(t), x^*\rangle = \lim_{j\to\infty} \langle U(t), y_{N_j}^*\rangle &= \lim_{j\to\infty} \int_0^t B^*S^*(t-s)y_{N_j}^*\, dW_H(s) \\ &= \int_0^t B^*S^*(t-s)x^*\, dW_H(s)\end{aligned}$$

almost surely.

(2) $\Rightarrow$ (1): Suppose now that the function $t \mapsto S(t)B$ is stochastically integrable on $(0,T)$. This implies the stochastic integrability of $s \mapsto S(t-s)B$ on $(0,t)$ for all $t \in (0,T]$. We check that the process U defined by the convolution (8.1) with $x = 0$ has a strongly measurable version which is a weak solution of the problem (SACP) with initial value $x = 0$.

To prove that U has a strongly measurable version we argue as follows. As in the proof of Step 1 of Theorem 6.17 (3)$\Rightarrow$(1) we may assume that H is separable. Then by Proposition 5.14 the $\gamma(L^2(0,T;H),E)$-valued function $t \mapsto R_t$ is strongly measurable, where R_t is the integral operator associated with $s \mapsto 1_{(0,t)}(s)S(t-s)B$. By covariance domination, $\|R_t\|_{\gamma(L^2(0,T;H),E)} \leqslant \|R_T\|_{\gamma(L^2(0,T;H),E)}$. Applying the Itô isometry of Theorem 6.14 we see that U defines an element of $L^\infty(0,T;L^2(\Omega;E))$. The existence of a strongly measurable version follows from this (cf. Example 1.21).

Fix $x^* \in \mathscr{D}(A^*)$ and $t \in [0,T]$. Then almost surely

$$\langle U(t), A^*x^* \rangle = \int_0^t B^*S^*(t-s)A^*x^* \, dW_H(s).$$

By the stochastic Fubini theorem applied to $\phi(s,t) := 1_{\{0 \leqslant s \leqslant t \leqslant T\}} B^*S^*(t-s)x^*$, the $L^2(\Omega)$-valued function $t \mapsto \langle U(t), A^*x^* \rangle$ is integrable on $(0,T)$ and

$$\begin{aligned}
\int_0^t \langle U(s), A^*x^* \rangle \, ds &= \int_0^t \int_0^s B^*S^*(s-r)A^*x^* \, dW_H(r) \, ds \\
&= \int_0^t \int_r^t B^*S^*(s-r)A^*x^* \, ds \, dW_H(r) \\
&= \int_0^t B^*S^*(t-r)x^* - B^*x^* \, dW_H(r) \\
&= \langle U(t), x^* \rangle - W_H(t)B^*x^*,
\end{aligned}$$

where all identities are understood in the sense of $L^2(\Omega)$. In particular the identities hold almost surely.

It remains to check that the trajectories of U are integrable almost surely. Let μ_t be the distribution of $U(t)$ and let Q_t be its covariance operator. We have

$$\langle Q_t \, x^*, x^* \rangle = \int_0^t \|B^*S^*(s)x^*\|_H^2 \, ds \leqslant \langle Q_T x^*, x^* \rangle = \langle Rx^*, x^* \rangle.$$

Hence by Fubini's theorem and covariance domination, for arbitrary but fixed $1 \leqslant p < \infty$ we obtain

$$\mathbb{E} \int_0^T \|U(t)\|^p \, dt = \int_0^T \int_E \|x\|^p \, d\mu_t(x) \, dt \leqslant T \int_E \|x\|^p \, d\mu_T(x) < \infty.$$

This implies that almost all trajectories $t \mapsto U(t,\omega)$ belong to $L^p(0,T;E)$. $\square$

Note that theorem 8.6 contains the following uniqueness assertion: if U^x and $\widetilde{U}^x$ are both weak solutions of (SACP), then U^x and $\widetilde{U}^x$ are versions of each other: both $U^x(t)$ and $\widetilde{U}^x(t)$ equal the right hand side of (8.1) almost surely. This justifies us to speak of 'the' solution of (SACP).

Comparing the proof of Theorem 8.6 with that of Theorem 7.17 we observe that the existence proofs are essentially identical, whereas the uniqueness parts are very different. The reason is that the exceptional sets in the definition of a weak solution of the stochastic problem (SACP) depend on t and x^*, which prevents us from applying Proposition 7.14 almost surely. Because of this it is no longer clear whether a weak solution is always a strong solution (cf. Proposition 7.16).

8.4 Existence and uniqueness: Brownian motion

Next we consider the problem (SACP) under the assumption that $B \in \gamma(H,E)$. In this situation the term '$B \, dW_H$' may be replaced by 'dW^B', where W^B is an E-valued Brownian motion canonically associated with B.

Definition 8.7. *An E-valued process $(W(t))_{t\in[0,T]}$ is called an E-*valued Brownian motion *if it enjoys the following properties:*

i) $W(0) = 0$ *almost surely;*
ii) $W(t-s)$ *and* $W(t) - W(s)$ *are identically distributed Gaussian random variables for all* $0 \leqslant s \leqslant t \leqslant T$*;*
iii) $W(t) - W(s)$ *is independent of* $\{W(r) :\ 0 \leqslant r \leqslant s\}$ *for all* $0 \leqslant s \leqslant t \leqslant T$*.*

Proposition 8.8. *Let $(W_H(t))_{t\in[0,T]}$ be an H-cylindrical Brownian motion and let $B \in \gamma(H,E)$. If $(h_n)_{n=1}^\infty$ is an orthonormal basis of $(\ker(B))^\perp$, then:*

(1) *the sum*

$$W^B(t) := \sum_{n=1}^\infty W_H(t)h_n \otimes Bh_n$$

converges almost surely and in $L^p(\Omega; E)$, $1 \leqslant p < \infty$, for all $t \in [0,T]$;
(2) *up to a null set, $W^B(t)$ is independent of the choice of the basis $(h_n)_{n=1}^\infty$;*
(3) *the process $(W^B(t))_{t\in[0,T]}$ defines an E-valued Brownian motion.*

The proof involves a straightforward application of Theorem 5.15, noting that for $0 \leqslant s \leqslant t \leqslant T$ the covariance operator of $W^B(t) - W^B(s)$ equals $(t-s)BB^*$.

This proposition shows that for operators $B \in \gamma(H,E)$ the problem (SACP) may be restated as

$$\begin{cases} dU(t) = AU(t)\,dt + dW^B(t), & t \in [0,T], \\ U(0) = x. \end{cases}$$

In the converse direction, every E-valued Brownian motion is of the form W^B for canonical choices of H and $B \in \gamma(H,E)$ (Exercise 2).

Definition 8.9. *Let $B \in \gamma(H,E)$. A* strong solution *of* (SACP) *is a strongly measurable E-valued process $(U^x(t))_{t\in[0,T]}$ with the following properties:*

i) the trajectories of U^x are integrable almost surely;
ii) for all $t \in [0,T]$, almost surely we have $\int_0^t U^x(s)\,ds \in \mathscr{D}(A)$ and

$$U^x(t) = x + A\int_0^t U^x(s)\,ds + W^B(t).$$

Theorem 8.10. *Let $B \in \gamma(H,E)$. The following assertions are equivalent:*

(1) *the problem* (SACP) *has a strong solution;*
(2) *the problem* (SACP) *has a weak solution.*

In this situation, the weak and strong solutions are versions of each other, and both are given by (8.1)*.*

Proof. We only need to prove that (2) implies (1). We may assume that $x = 0$. Let U be a weak solution of (SACP) with initial value $x = 0$. Fix $t \in [0,T]$. We claim that the function $\Psi_t : (0,t) \to \mathscr{L}(H,E)$,

$$\Psi_t(r)h := \int_r^t S(s-r)Bh\,ds,$$

is stochastically integrable with respect to W_H and

$$\int_0^t \Psi_t(r)\,dW_H(r) = \int_0^t U(s)\,ds. \tag{8.5}$$

To see this, note that for all $x^* \in E^*$ the stochastic Fubini theorem gives

$$\begin{aligned}\int_0^t \Psi_t^*(r)x^*\,dW_H(r) &= \int_0^t \int_r^t B^*S^*(s-r)x^*\,ds\,dW_H(r)\\ &= \int_0^t \int_0^s B^*S^*(s-r)x^*\,dW_H(r)\,ds = \int_0^t \langle U(s), x^*\rangle\,ds,\end{aligned}$$

where the last identity follows from the assumption that U is a weak solution and therefore satisfies (8.1). The claim now follows from Theorem 6.17.

Also, from $\Psi_t(r)h \in \mathscr{D}(A)$ and $A\Psi_t(r)h = S(t-r)Bh - Bh$ it follows that $A\Psi_t : (0,t) \to \mathscr{L}(H,E)$ is stochastically integrable with respect to W_H and

$$\int_0^t A\Psi_t(r)\,dW_H(r) = \int_0^t (S(t-r)B - B)\,dW_H(r) = U(t) - W^B(t),$$

where in the second identity we used that $W_H(t)B^*x^* = \langle W^B(t), x^*\rangle$.

Combining these facts it follows that Ψ_t is stochastically integrable as a function from $(0,t)$ to $\mathscr{L}(H,\mathscr{D}(A))$. It follows that the left hand side of (8.5) defines a $\mathscr{D}(A)$-valued Gaussian random variable. Moreover, as A is bounded from $\mathscr{D}(A)$ to E, almost surely we have

$$A\int_0^t U(s)\,ds = A\int_0^t \Psi_t(r)\,dW_H(r) = \int_0^t A\Psi_t(r)\,dW_H(r) = U(t) - W^B(t).$$

This shows that U is a strong solution. ◻

We may now apply the result of Exercise 5.4 as follows:

Corollary 8.11. *Let E have type 2 and assume that $B \in \gamma(H,E)$. Then the problem* (SACP) *has a unique strong solution, and this solution is given by the convolution* (8.1).

It can be shown that this solution has a version with continuous trajectories; this follows from the Da Prato-Kwapień-Zabczyk factorisation principle which will be discussed later on. It appears to be an open problem whether, in the more general situation of Theorems 8.6 and 8.10, a solution (if it exists) always has a continuous version.

8.5 Non-existence

In this section we present an example of a stochastic evolution equation driven by a rank one Brownian motion which has no (weak or strong) solution.

Example 8.12. Let $E = L^p(\mathbb{T})$, where $\mathbb{T}$ denotes the unit circle in the complex plane with its normalized Lebesgue measure. We let $A = d/d\theta$ denote the generator of the rotation (semi)group S on $L^p(\mathbb{T})$, $S(t)f(\theta) = f(\theta + t \bmod 2\pi)$. Consider the stochastic Cauchy problem

$$\begin{cases} dU(t) = AU(t) + \phi\, dW, \quad t \in [0, 2\pi], \\ U(0) = 0, \end{cases} \tag{8.6}$$

where W is a standard real Brownian motion and $\phi \in L^p(\mathbb{T})$ is a fixed element. This problem has a weak solution if and only if the operator $R := R_{2\pi} : L^2(\mathbb{T}) \to L^p(\mathbb{T})$ of Theorem 8.6 (with $T = 2\pi$) is γ-radonifying. Let $(h_n)_{n=1}^{\infty}$ be an orthonormal basis for $L^2(\mathbb{T})$. For all $N \geqslant M \geqslant 1$, by Fubini's theorem and the Khintchine inequality we have

$$\begin{aligned} \mathbb{E}\Big\| \sum_{n=M}^{N} \gamma_n R h_n \Big\|_{L^p(\mathbb{T})}^p &= \int_0^{2\pi} \mathbb{E}\Big| \sum_{n=M}^{N} \gamma_n R h_n(\theta) \Big|^p \, d\theta \\ &\eqsim_p \int_0^{2\pi} \Big(\sum_{n=M}^{N} |R h_n(\theta)|^2 \Big)^{\frac{p}{2}} d\theta = \Big\| \Big(\sum_{n=M}^{N} |R h_n|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbb{T})}^p. \end{aligned}$$

Now,

$$\sum_{n=M}^{N} |R h_n(\theta)|^2 = \sum_{n=M}^{N} \Big| \int_0^{2\pi} h_n(t) \phi(\theta + t \bmod 2\pi) \, dt \Big|^2 = \sum_{n=M}^{N} \big| [h_n, \phi_\theta]_{L^2(\mathbb{T})} \big|^2$$

where $\phi_\theta(t) := \phi(\theta + t \bmod 2\pi)$. Via an application of the Kahane-Khintchine inequality we deduce that $R \in \gamma(L^2(\mathbb{T}), L^p(\mathbb{T}))$ if and only if $\phi \in L^2(\mathbb{T})$. In particular, for $p \in [1, 2)$ and $\phi \in L^p(\mathbb{T}) \setminus L^2(\mathbb{T})$ the resulting initial value problem has no weak solution.

It is not a coincidence that a nonexistence is obtained in the range $p \in [1, 2)$ only. Indeed, for $p \in [2, \infty)$ the space $L^p(\mathbb{T})$ has type 2, and therefore Corollary 8.11 guarantees the existence of a strong solution for (8.6).

8.6 Exercises

1. This exercise offers an alternative approach to the integration by parts formula of Lemma 8.1. The starting point is the fact that if $\mathscr{H}$ is a real

Hilbert space, $\phi : [0,T] \to \mathbb{R}$ is of bounded variation, and $\psi : [0,T] \to \mathscr{H}$ is continuous, then

$$\int_0^T \psi(t)\, d\phi(t) = \phi(T)\psi(T) - \int_0^T \phi(t)\, d\psi(t),$$

where both integrals are interpreted as Riemann-Stieltjes integrals in $\mathscr{H}$. Let $(W(t))_{t\in[0,T]}$ be a standard Brownian motion.

a) Show that the function $\psi : [0,t] \to L^2(\Omega)$, $\psi(t) := W(t)$, is continuous.
b) Deduce Lemma 8.1 from the above integration by parts formula.

2. Let $(W(t))_{t\in[0,T]}$ be an E-valued Brownian motion. Show that there exists a unique Gaussian covariance operator $Q \in \mathscr{L}(E^*, E)$ such that

$$\mathbb{E}\langle W(s), x^*\rangle\langle W(t), y^*\rangle = \min\{s,t\}\langle Qx^*, y^*\rangle$$

for all $0 \leqslant s,t \leqslant T$ and $x^*, y^* \in E^*$.
Hint: Consider $Q := Q_T/T$, where Q_T is the covariance of $W(T)$.

3. We consider the problem (SACP) with initial value $x = 0$ and assume that it admits a weak solution U. Prove that U is a Gaussian process with covariance

$$\mathbb{E}\langle U(s), x^*\rangle\langle U(t), y^*\rangle = \int_0^{\min\{s,t\}} [B^*S^*(s-r)x^*, B^*S^*(t-r)y^*]\, ds$$

for all $0 \leqslant s,t \leqslant T$ and $x^*, y^* \in E^*$.

4. We consider the problem (SACP) with initial value x and assume that it admits a weak solution U^x.

a) Prove that the solvability of the problem (SACP) is independent of the time T. More precisely, show that if (SACP) has a weak (resp. strong) solution on some interval $[0,T]$, then it has a weak (resp. strong) solution on every interval $[0,T]$.
Hint: Use the semigroup property and Theorem 8.6.

By a) and uniqueness, U^x extends to a solution on $[0,\infty)$. For $f \in C_{\mathrm{b}}(E)$ and $t \geqslant 0$ we define the function $P(t)f : E \to \mathbb{R}$ by

$$P(t)f(x) := \mathbb{E}f(U^x(t)), \quad x \in E.$$

b) Explain why for all $f \in C_{\mathrm{b}}(E)$ and $t \geqslant 0$ we have the identity

$$\mathbb{E}f(U^x(t)) = \int_E f(S(t)x + y)\, d\mu_t(y),$$

where μ_t denotes the distribution of the random variable $U^0(t)$.
c) Deduce that $P(t)f \in C_{\mathrm{b}}(E)$.

d) Prove the identity
$$\mu_{t+s} = \mu_t * S(t)\mu_s,$$
where $*$ denotes convolution and $S(t)\mu_s$ is the image measure of μ_s under the operator $S(t)$.
Hint: Use Fourier transforms and observe that for the covariances Q_t of $U^0(t)$, $t \geqslant 0$, we have the identity
$$Q_{t+s} = Q_t + S(t)Q_sS^*(t).$$

e) Deduce that $P = (P(t))_{t\geqslant 0}$ is a semigroup of operators on $C_\mathrm{b}(E)$, in the sense that $P(0) = I$ and $P(t)P(s) = P(t+s)$ for all $t, s \geqslant 0$.

f) Prove that for all $x \in E$ and $f \in C_\mathrm{b}(E)$ we have
$$\lim_{t\to 0} P(t)f(x) = f(x)$$
uniformly on compact subsets K of E.
Hint: By the remark in Exercise 6.4, the process
$$V^x(t) := S(t)x + \int_0^t S(s)B\,dW_H(s), \quad t \in [0,T],$$
has a continuous version (a proof will be given later in this course). Now use b) together with the observation that for each fixed $t \in [0,T]$ the random variables $U^x(t)$ and $V^x(t)$ are identically distributed.

Remark: By considering (real and imaginary parts of) trigonometric polynomials of the form $x \mapsto \exp(i\langle x, x^*\rangle)$ it is not hard to show that P fails to be a C_0-semigroup on $C_\mathrm{b}(E)$ (and even on the closed subspace $UC_\mathrm{b}(E)$ of all bounded uniformly continuous functions) unless $A = 0$.

5. In addition to the assumptions of the previous exercise, let us assume that there exists a Borel probability measure μ_∞ on E such that $\lim_{t\to\infty} \mu_t = \mu_\infty$ in the sense that
$$\lim_{t\to\infty} \int_E f(x)\,d\mu_t(x) = \int_E f(x)\,d\mu_\infty(x)$$
for all $f \in C_\mathrm{b}(E)$.

a) Prove the identity
$$\mu_\infty = \mu_t * S(t)\mu_\infty.$$

b) Prove that μ_∞ is an *invariant measure* in the sense that for all $f \in C_\mathrm{b}(E)$ and $t \geqslant 0$ we have
$$\int_E P(t)f\,d\mu_\infty = \int_E f\,d\mu_\infty.$$

c) Prove that P extends to a C_0-semigroup of contractions on the space $L^p(E, \mu_\infty)$, $1 \leqslant p < \infty$.

Notes. The theory of (linear and non-linear) stochastic evolution equations in Hilbert spaces dates back to the 1970s and was developed extensively through the efforts of the Italian and Polish schools around Da Prato and Zabczyk. A comprehensive overview is given in the monographs [27, 28] by these two authors. Parts of the theory have been extended to (martingale-)type 2 spaces; we refer to the review paper by Brzeźniak [15] and the references given there.

The results of Section 8.3 are taken from [84] and generalise known Hilbert space results and improve the preliminary Banach space results of [16]. The proof of theorem Theorem 8.6 essentially follows the Hilbert space proof in [27]. The theory of adjoint semigroups was initiated by Phillips, who proved Proposition 8.3 and noted as a consequence that $E^{\odot} = E^*$ if E is reflexive.

The equivalence of weak and strong solutions in the case where B is γ-radonifying is taken from an unpublished note by Veraar.

The example in Section 8.5 is from [84]. Such examples cannot exist in Hilbert spaces, due to Corollary 8.11.

The semigroup P of Exercises 4 and 5 is called the *Ornstein-Uhlenbeck semigroup* associated with A and B. The literature on this class of semigroups is extensive, with contributions by many mathematicians. Using Itô's formula it can be shown that the infinitesimal generator L of P is given, on a suitable dense subspace of $\mathscr{D}(L)$ consisting of cylindrical functions, by

$$Lf(x) = \frac{1}{2}\mathrm{Tr}\,(BB^*D^2f(x)) + \langle Ax, Df(x)\rangle, \quad x \in \mathscr{D}(A),$$

where D denotes the Fréchet derivative and Tr the trace. The first term in the right hand side is the 'diffusion part' corresponding to BW_H and the second is the 'drift part' corresponding to A.

The clever argument in part f) of Exercise 4 is due to Veraar. A self-contained analytic proof can be found in see [42].

For a systematic account on invariant measures for stochastic evolution equations we refer to Da Prato and Zabczyk [28].