9

γ-Boundedness

In this lecture we address the second topic in the paradigm sketched in the introduction of Lecture 5: a 'Gaussian' generalisation to a Banach space setting of the notion of uniform boundedness of families of operators in Hilbert spaces. Roughly speaking, a family of operators $\mathscr{T}$ is said to be 'γ-bounded' if a Kahane contraction principle holds with scalars replaced by operators from $\mathscr{T}$. This makes γ-boundedness into a powerful tool for estimating Gaussian sums. Perhaps more important is the fact that there are numerous abstract methods to create γ-bounded families, which can be used to show that families of operators arising naturally in the context of parabolic PDEs (such as resolvents) and stochastic analysis (such as families of conditional expectation operators) are γ-bounded.

9.1 Randomised boundedness

Throughout this lecture $\varphi = (\varphi_n)_{n=1}^{\infty}$ denotes a sequence of independent symmetric real-valued random variables satisfying $\mathbb{E}\varphi_n^2 = 1$, $n \geqslant 1$. For instance, φ could be a Rademacher sequence or a Gaussian sequence.

We begin with a simple observation.

Proposition 9.1. *Let H_1 and H_2 be Hilbert spaces. For a subset $\mathscr{T} \subseteq \mathscr{L}(H_1, H_2)$ and a constant $M \geqslant 0$ the following assertions are equivalent:*

(1) *$\mathscr{T}$ is uniformly bounded and $\sup_{T\in\mathscr{T}} \|T\| \leqslant M$;*
(2) *for all $N \geqslant 1$, all $T_1, \dots, T_N \in \mathscr{T}$, and all $x_1, \dots, x_N \in H_1$,*

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^{N} \varphi_n T_n x_n\Big\|^2\Big)^{\frac12} \leqslant M\Big(\mathbb{E}\Big\|\sum_{n=1}^{N} \varphi_n x_n\Big\|^2\Big)^{\frac12}.$$

Proof. For the proof of (1)$\Rightarrow$(2), write $\|h\|^2 = [h, h]$ and use that $\mathbb{E}\varphi_j\varphi_k = \delta_{jk}$. For the proof of (2)$\Rightarrow$(1), consider the case $N = 1$ in (2) to obtain $\|Th\| \leqslant M\|h\|$ for all $T \in \mathscr{T}$ and $h \in H_1$. $\square$

With Hilbert spaces replaced by Banach spaces the implication (1)⇒(2) does not hold in general. This motivates the following definition.

Definition 9.2. *Let E_1 and E_2 be Banach spaces. An operator family $\mathscr{T} \subseteq \mathscr{L}(E_1, E_2)$ is said to be* φ-bounded *if there exists a constant $M \geqslant 0$ such that*

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^{N} \varphi_n T_n x_n\Big\|^2\Big)^{\frac{1}{2}} \leqslant M\Big(\mathbb{E}\Big\|\sum_{n=1}^{N} \varphi_n x_n\Big\|^2\Big)^{\frac{1}{2}},$$

for all $N \geqslant 1$, all $T_1, \dots, T_N \in \mathscr{T}$, and all $x_1, \dots, x_N \in E_1$. When φ is a Rademacher sequence, a φ-bounded family is called R-bounded*; when φ is a Gaussian sequence the family is called* γ-bounded.

The least admissible constant M is called the *φ-bound* of $\mathscr{T}$, notation: $\varphi(\mathscr{T})$. As in the Hilbert space case, every φ-bounded family $\mathscr{T}$ is uniformly bounded and we have

$$\sup_{T \in \mathscr{T}} \|T\| \leqslant \varphi(\mathscr{T}).$$

When φ is a Rademacher sequence or a Gaussian sequence, the bound $\varphi(\mathscr{T})$ is denoted by $R(\mathscr{T})$ and $\gamma(\mathscr{T})$, respectively. In these two cases, the Kahane-Khintchine inequality shows that the exponent 2 in the definition may be replaced by any exponent $1 \leqslant p < \infty$; this only affects the numerical value of the bounds. R-bounds and γ-bounds relative to the L^p-norm will be denoted by $R_p(\mathscr{T})$ and $\gamma_p(\mathscr{T})$. As a rule, we will state our results relative to the L^2-norm, but frequently the results carry over to L^p-norms if we make this modification.

Proposition 9.3 below shows that every R-bounded family is γ-bounded, and Corollary 3.6 and Theorem 3.7 imply that the converse holds if E_1 has finite cotype.

Proposition 9.3. *Any R-bounded family $\mathscr{T}$ is φ-bounded and $\varphi(\mathscr{T}) \leqslant R(\mathscr{T})$.*

Proof. Let $(r_n')_{n=1}^{\infty}$ be a Rademacher sequence on an independent probability space $(\Omega', \mathscr{F}', \mathbb{P}')$. Then for all $T_1, \dots, T_N \in \mathscr{T}$ and $x_1, \dots, x_N \in E_1$, by randomising we obtain

$$\begin{aligned}\mathbb{E}\Big\|\sum_{n=1}^{N} \varphi_n T_n x_n\Big\|^2 &= \mathbb{E}\mathbb{E}'\Big\|\sum_{n=1}^{N} r_n'\varphi_n T_n x_n\Big\|^2 \\ &\leqslant R(\mathscr{T})^2 \mathbb{E}\mathbb{E}'\Big\|\sum_{n=1}^{N} r_n'\varphi_n x_n\Big\|^2 = R(\mathscr{T})^2\mathbb{E}\Big\|\sum_{n=1}^{N} \varphi_n x_n\Big\|^2. \quad \square\end{aligned}$$

The proof of the next proposition is left as an exercise to the reader.

Proposition 9.4. *If $\mathscr{T} \subseteq \mathscr{L}(E_1, E_2)$ and $\mathscr{S} \subseteq \mathscr{L}(E_1, E_2)$ are φ-bounded, then the family $\mathscr{S} + \mathscr{T} = \{S + T : \ S \in \mathscr{S},\ T \in \mathscr{T}\}$ is φ-bounded in $\mathscr{L}(E_1, E_2)$ and*

$$\varphi(\mathscr{S}+\mathscr{T}) \leqslant \varphi(\mathscr{S})+\varphi(\mathscr{T}).$$

Likewise, if $\mathscr{T} \subseteq \mathscr{L}(E_1, E_2)$ *and* $\mathscr{S} \subseteq \mathscr{L}(E_2, E_3)$ *are* φ*-bounded, then the family* $\mathscr{S}\mathscr{T} = \{ST : S \in \mathscr{S},\ T \in \mathscr{T}\}$ *is* φ*-bounded in* $\mathscr{L}(E_1, E_3)$ *and*

$$\varphi(\mathscr{S}\mathscr{T}) \leqslant \varphi(\mathscr{S})\varphi(\mathscr{T}).$$

The *strong operator topology* of $\mathscr{L}(E_1, E_2)$ is the topology generated by all sets of the form

$$V(S, x, \varepsilon) := \{T \in \mathscr{L}(E_1, E_2) :\ \|Sx - Tx\| < \varepsilon\}$$

with given $S \in \mathscr{L}(E_1, E_2)$, $x \in E$, and $\varepsilon > 0$. Note that a set $O \subseteq \mathscr{L}(E_1, E_2)$ is open in this topology if and only if for all $S \in O$ there exist $x_1, \dots, x_k \in E_1$ and a number $\varepsilon > 0$ such that

$$\bigcap_{j=1}^{k} \{T \in \mathscr{L}(E_1, E_2) :\ \|Sx_j - Tx_j\| < \varepsilon\} \subseteq O.$$

It is an easy exercise to check that $\lim_{n\to\infty} T_n = T$ in the strong operator topology if and only if $\lim_{n\to\infty} T_n x = Tx$ for all $x \in E_1$.

Proposition 9.5 (Strong closure). *If* $\mathscr{T} \subseteq \mathscr{L}(E_1, E_2)$ *is* φ*-bounded, then its closure* $\overline{\mathscr{T}}$ *in the strong operator topology is* φ*-bounded and* $\varphi(\overline{\mathscr{T}}) = \varphi(\mathscr{T})$*.*

Proof. Let $\overline{T}_1, \dots, \overline{T}_N \in \overline{\mathscr{T}}$ and $x_1, \dots, x_N \in E_1$ be arbitrary. Given an $\varepsilon > 0$, choose operators $T_1, \dots, T_N \in \mathscr{T}$ such that $\|\overline{T}_n x_n - T_n x_n\| < 2^{-n}\varepsilon$, $n = 1, \dots, N$. Then, by the triangle inequality in $L^2(\Omega; E_2)$ applied twice,

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\sum_{n=1}^{N} \varphi_n \overline{T}_n x_n\Big\|^2\Big)^{\frac12} \\
&\quad\leqslant \Big(\mathbb{E}\Big\|\sum_{n=1}^{N} \varphi_n T_n x_n\Big\|^2\Big)^{\frac12} + \Big(\mathbb{E}\Big\|\sum_{n=1}^{N} \varphi_n (\overline{T}_n x_n - T_n x_n)\Big\|^2\Big)^{\frac12} \\
&\quad\leqslant \varphi(\mathscr{T})\Big(\mathbb{E}\Big\|\sum_{n=1}^{N} \varphi_n x_n\Big\|^2\Big)^{\frac12} + \sum_{n=1}^{N} \|\overline{T}_n x_n - T_n x_n\| \\
&\quad\leqslant \varphi(\mathscr{T})\Big(\mathbb{E}\Big\|\sum_{n=1}^{N} \varphi_n x_n\Big\|^2\Big)^{\frac12} + \varepsilon.
\end{aligned}$$

This proves that $\overline{\mathscr{T}}$ is φ-bounded with $\varphi(\overline{\mathscr{T}}) \leqslant \varphi(\mathscr{T})$. The converse inequality is trivial. □

The *absolute convex hull* of a set V, notation $\operatorname{abs\,conv}(V)$, is the set of all vectors of the form $\sum_{j=1}^{k} \lambda_j x_j$ with $\sum_{j=1}^{k} |\lambda_j| \leqslant 1$ and $x_j \in V$ for $j = 1, \dots, k$.

Proposition 9.6 (Convex hull). *If $\mathscr{T}$ is φ-bounded in $\mathscr{L}(E_1, E_2)$, then the convex hull and the absolute convex hull of $\mathscr{T}$ are φ-bounded in $\mathscr{L}(E_1, E_2)$ and $\varphi(\mathscr{T}) = \varphi(\mathrm{conv}(\mathscr{T})) = \varphi(\mathrm{abs\,conv}(\mathscr{T}))$.*

Proof. First we prove the statement for the convex hull. Choose $S_1, \dots S_n \in \mathrm{conv}(\mathscr{T})$ arbitrarily. Noting that

$$\mathrm{conv}(\mathscr{T}) \times \cdots \times \mathrm{conv}(\mathscr{T}) = \mathrm{conv}(\mathscr{T} \times \cdots \times \mathscr{T})$$

we can find $\lambda_1, \dots, \lambda_k \in [0, 1]$ with $\sum_{j=1}^k \lambda_j = 1$ such that $S_n = \sum_{j=1}^k \lambda_j T_{jn}$ with $T_{jn} \in \mathscr{T}$ for all $j = 1, \dots, k$ and $n = 1, \dots, N$. Then, for all $x_1, \dots, x_N \in E_1$,

$$\begin{aligned}\Big(\mathbb{E}\Big\|\sum_{n=1}^N \varphi_n S_n x_n\Big\|^2\Big)^{\frac12} &\leqslant \sum_{j=1}^k \lambda_j \Big(\mathbb{E}\Big\|\sum_{n=1}^N \varphi_n T_{jn} x_n\Big\|^2\Big)^{\frac12} \\ &\leqslant \varphi(\mathscr{T}) \sum_{j=1}^k \lambda_j \Big(\mathbb{E}\Big\|\sum_{n=1}^N \varphi_n x_n\Big\|^2\Big)^{\frac12} = \varphi(\mathscr{T}) \Big(\mathbb{E}\Big\|\sum_{n=1}^N \varphi_n x_n\Big\|^2\Big)^{\frac12}.\end{aligned}$$

This proves the φ-boundedness of $\mathrm{conv}(\mathscr{T})$ with the estimate $\varphi(\mathrm{conv}(\mathscr{T})) \leqslant \varphi(\mathscr{T})$. The opposite inequality $\varphi(\mathscr{T}) \leqslant \varphi(\mathrm{conv}(\mathscr{T}))$ is trivial.

The result for the absolute convex hull follows by noting that this hull is contained in the convex hull of $\mathscr{T} \cup \{0\} \cup -\mathscr{T}$; the set $\mathscr{T} \cup \{0\} \cup -\mathscr{T}$ is φ-bounded with the same φ-bound as $\mathscr{T}$ (use Proposition 2.16 to add the zero operator and replace some of the φ_n by $-\varphi_n$ in the random sums). $\square$

By combining Propositions 9.5 and 9.6 we obtain that the strongly closed absolutely convex hull of every φ-bounded set is φ-bounded. This may be used to show that φ-boundedness is preserved by taking integral means.

Theorem 9.7 (Integral means I). *Let $(A, \mathscr{A}, \mu)$ be a σ-finite measure space and let $\mathscr{T}$ be a φ-bounded subset of $\mathscr{L}(E_1, E_2)$. Suppose $f : A \to \mathscr{L}(E_1, E_2)$ is a function with the following properties:*

(i) *the function $\xi \mapsto f(\xi)x$ is strongly μ-measurable for all $x \in E_1$;*
(ii) *we have $f(\xi) \in \mathscr{T}$ for μ-almost all $\xi \in A$.*

For $\phi \in L^1(A)$ define $T_f^\phi \in \mathscr{L}(E_1, E_2)$ by

$$T_f^\phi x := \int_A \phi(\xi) f(\xi) x \, d\mu(\xi), \qquad x \in E_1,$$

The family $\mathscr{T}_f^\phi := \{T_f^\phi : \ \|\phi\|_1 \leqslant 1\}$ is φ-bounded and $\varphi(\mathscr{T}_f^\phi) \leqslant \varphi(\mathscr{T})$.

Proof. Since $\mathscr{T}$ is φ-bounded and therefore uniformly bounded, the integral defining $T_f^\phi x$ is well-defined as a Bochner integral in E_2 for every $x \in E_1$ and defines a bounded operator T_f^ϕ of norm $\|T_f^\phi\| \leqslant \|\phi\|_1 \sup_{T \in \mathscr{T}} \|T\|$.

To prove the φ-boundedness of the family $\mathscr{T}_f^\phi$ along with the estimate for its φ-bound it suffices to check that the family $\{T_f^\phi : \|\phi\|_1 = 1\}$ is contained in $\overline{\text{abs conv}}(\mathscr{T})$, where the bar denotes the closure in the strong operator topology of $\mathscr{L}(E_1, E_2)$.

Fix ϕ with $\|\phi\|_1 = 1$ and for $k = 1, 2, \ldots$ define $T^{(k)} \in \mathscr{L}(E_1^k, E_2^k)$ by

$$T^{(k)}(x_1, \ldots, x_k) := (T_f^\phi x_1, \ldots, T_f^\phi x_k)$$

and note that this operator is given by the Bochner integral

$$T^{(k)}(x_1, \ldots, x_k) = \int_A \phi(\xi) f^{(k)}(\xi)(x_1, \ldots, x_k)\, d\mu(\xi),$$

where $f^{(k)}(\xi)(x_1, \ldots, x_k) := (f(\xi)x_1, \ldots, f(\xi)x_k)$ is strongly μ-measurable as an E_2^k-valued function of the variable ξ.

Let us fix $x_1, \ldots, x_k \in E_1$. Let $N \in \mathscr{A}$ be a μ-null set such that (ii) holds in $A \setminus N$. Noting that

$$(|\phi|\mu)(B) := \int_B |\phi|\, d\mu = \int_B 1_{A\setminus N}|\phi|\, d\mu, \quad B \in \mathscr{A},$$

defines a probability measure on $(A, \mathscr{A})$ and writing

$$\phi(\xi)f(\xi) = \operatorname{sgn}(\phi(\xi))f(\xi) \cdot |\phi(\xi)|,$$

from Proposition 1.17 we deduce that

$$(T_f^\phi x_1, \ldots, T_f^\phi x_k) \in \overline{\text{abs conv}}\{(f(\xi)x_1, \ldots, f(\xi)x_k) : \ \xi \in A \setminus N\}.$$

In particular,

$$(T_f^\phi x_1, \ldots, T_f^\phi x_k) \in \overline{\text{abs conv}}\{(Tx_1, \ldots, Tx_k) : \ T \in \mathscr{T}\}.$$

This means that for every $\varepsilon > 0$ we can find $T \in \text{abs conv}(\mathscr{T})$ such that

$$\|T_f^\phi x_j - Tx_j\| < \varepsilon, \qquad j = 1, \ldots, k.$$

Since the choice of $x_1, \ldots, x_k \in E_1$ and $\varepsilon > 0$ were arbitrary, we have shown that every open set (in the strong operator topology) in $\mathscr{L}(E_1, E_2)$ containing T_f^ϕ intersects abs conv$(\mathscr{T})$. This is synonymous to saying that $T_f^\phi \in \overline{\text{abs conv}}(\mathscr{T})$. $\square$

So far we have been concerned with producing new φ-bounded families from old. We continue with two results which produce φ-bounded families 'from scratch'. In view of Proposition 9.3 it suffices to prove that such families are R-bounded. In both cases, however, the same argument already gives the φ-boundedness, and we prefer this route for the unity of presentation.

Theorem 9.8 (Integral means II). *Let $(A, \mathscr{A}, \mu)$ be a σ-finite measure space, E_1 and E_2 Banach spaces and let $f : A \to \mathscr{L}(E_1, E_2)$ be a function with the property that $\xi \mapsto f(\xi)x$ is strongly μ-measurable for all $x \in E_1$. Suppose that $g : A \to \mathbb{R}$ is a μ-integrable function such that for all $x \in E_1$ we have*

$$\|f(\xi)x\| \leqslant |g(\xi)|\|x\| \quad \mu\text{-almost everywhere.}$$

For $\phi \in L^\infty(A)$ define $T_f^\phi \in \mathscr{L}(E_1, E_2)$ by

$$T_f^\phi x := \int_A \phi(\xi) f(\xi) x \, d\mu(\xi), \quad x \in E_1.$$

The family $\mathscr{T}_f^\phi = \{T_f^\phi : \ \|\phi\|_\infty \leqslant 1\}$ is φ-bounded and $\varphi(\mathscr{T}_f^\phi) \leqslant \|g\|_1$.

Proof. For $\phi \in L^\infty(A)$, note that $\xi \mapsto \phi(\xi)f(\xi)x$ is μ-Bochner integrable in E_2 for all $x \in E_1$, so the operators T_f^ϕ are well-defined and bounded with $\|T_f^\phi\| \leqslant \|\phi\|_\infty \|g\|_1$.

Fix $\phi_1, \dots, \phi_N \in L^\infty(A)$ and $x_1, \dots, x_N \in E_1$. Using the Kahane contraction principle we estimate

$$\begin{aligned}
\Big\| \sum_{n=1}^N \varphi_n T_f^{\phi_n} x_n \Big\|_{L^2(\Omega;E_2)} &= \Big\| \int_A \sum_{n=1}^N \varphi_n \phi_n(\xi) f(\xi) x_n \, d\mu(\xi) \Big\|_{L^2(\Omega;E_2)} \\
&\leqslant \int_A \Big\| \sum_{n=1}^N \varphi_n \phi_n(\xi) f(\xi) x_n \Big\|_{L^2(\Omega;E_2)} d\mu(\xi) \\
&\leqslant \int_A \Big\| \sum_{n=1}^N \varphi_n f(\xi) x_n \Big\|_{L^2(\Omega;E_2)} d\mu(\xi) \\
&\leqslant \int_A |g(\xi)| \Big\| \sum_{n=1}^N \varphi_n x_n \Big\|_{L^2(\Omega;E_1)} d\mu(\xi) \\
&= \|g\|_1 \Big\| \sum_{n=1}^N \varphi_n x_n \Big\|_{L^2(\Omega;E_1)}.
\end{aligned}$$

□

Note that if E_1 is separable, we may apply the theorem to the function $g(\xi) := \|f(\xi)\|$, which is then μ-measurable (choose a dense sequence $(x_n)_{n=1}^\infty$ in E_1 and note that $\|f(\xi)\| = \sup_{n\geqslant 1} \|f(\xi)x_n\|$). A similar remark applies to the next theorem.

Theorem 9.9 (Functions with integrable derivative). *Let $f : (a, b) \to \mathscr{L}(E_1, E_2)$ and $g : (a, b) \to \mathbb{R}$ be such that the functions $t \mapsto f(t)x$ are continuously differentiable, g is integrable, and for all $x \in E_1$ we have*

$$\|f'(t)x\| \leqslant |g(t)|\|x\| \quad \mu\text{-almost everywhere.}$$

Then $\mathscr{T} := \{f(t) : \ t \in (a, b)\}$ is φ-bounded and $\varphi(\mathscr{T}) \leqslant \|f(a+)\| + \|g\|_1$.

Proof. Let us first prove that $f(a+) := \lim_{t\downarrow a} f(t)$ exists in the strong operator topology. For fixed $x \in E_1$, given $\varepsilon > 0$ choose $\delta > 0$ so small that $\int_a^{a+\delta} |g(t)|\,dt < \varepsilon$; then for all $a < a_1 < a_2 < a + \delta$ we have

$$\|f(a_2)x - f(a_1)x\| = \Big\| \int_{a_1}^{a_2} f'(t)x\,dt \Big\| \leqslant \int_{a_1}^{a_2} \|f'(t)x\|\,dt < \varepsilon \|x\|.$$

This gives the claim.

For all $a < t_1 \leqslant \cdots \leqslant t_N < b$ and $x_1, \ldots, x_N \in E$ we obtain, using Theorem 9.8,

$$\begin{aligned}
&\Big(\mathbb{E}\Big\| \sum_{n=1}^N \varphi_n f(t_n) x_n \Big\|^2\Big)^{\frac12} \\
&\quad = \Big(\mathbb{E}\Big\| \sum_{n=1}^N \varphi_n \Big[f(a+)x_n + \int_a^{t_n} f'(t)x_n\,dt\Big] \Big\|^2\Big)^{\frac12} \\
&\quad \leqslant \|f(a+)\| \Big(\mathbb{E}\Big\| \sum_{n=1}^N \varphi_n x_n \Big\|^2\Big)^{\frac12} + \Big(\mathbb{E}\Big\| \sum_{n=1}^N \varphi_n \int_a^b 1_{(a,t_n)}(t) f'(t) x_n\,dt \Big\|^2\Big)^{\frac12} \\
&\quad \leqslant (\|f(a+)\| + \|g\|_1) \Big(\mathbb{E}\Big\| \sum_{n=1}^N \varphi_n x_n \Big\|^2\Big)^{\frac12}.
\end{aligned}$$

□

9.2 Examples

We proceed with some important examples of φ-bounded families, where as before $(\varphi_n)_{n=1}^\infty$ is a sequence of independent symmetric real-valued random variables satisfying $\mathbb{E}\varphi_n^2 = 1$, $n \geqslant 1$. One example has already been recorded: a family of Hilbert space operators is φ-bounded if and only if it is uniformly bounded.

Example 9.10 (The contraction principle and φ-boundedness). Let E be a Banach space. Every real number a defines a bounded operator T_a on E by scalar multiplication: $T_a x = ax$. The Kahane contraction principle can be reformulated as saying that for every bounded set $A \subseteq \mathbb{R}$, the set $\mathscr{T}_A := \{T_a : a \in A\}$ is φ-bounded in $\mathscr{L}(E)$, with $\varphi(\mathscr{T}_A) = \sup\{|a| : a \in A\}$.

Example 9.11 (φ-Boundedness in L^p). Let $(A, \mathscr{A}, \mu)$ be a σ-finite measure space and let $1 \leqslant p < \infty$ be fixed. If S is a positive bounded operator on $E := L^p(A)$, i.e., $Sf \geqslant 0$ whenever $f \geqslant 0$ (we write $f_1 \geqslant f_2$ to mean that $f_1(\xi) \geqslant f_2(\xi)$ for μ-almost all $\xi \in A$), the set

$$\mathscr{T} := \{T \in \mathscr{L}(E) : |Tf| \leqslant S|f| \text{ for all } f \in E\}$$

is φ-bounded and we have $\varphi(\mathscr{T}) \leqslant K_p\|S\|$, where K_p is a universal constant depending only on p.

By Proposition 9.3 it suffices to prove this for Rademacher variables $(r_n)_{n=1}^\infty$. Using Fubini's theorem and the scalar Kahane-Khintchine inequality, we see that for all $T_1,\dots,T_N \in \mathscr{T}$ and $f_1,\dots,f_N \in E$,

$$\begin{aligned}
\mathbb{E}\Big\|\sum_{n=1}^N r_n T_n f_n\Big\|_E^p &= \int_A \mathbb{E}\Big|\sum_{n=1}^N r_n T_n f_n\Big|^p\,d\mu \lesssim_p \int_A \Big(\mathbb{E}\Big|\sum_{n=1}^N r_n T_n f_n\Big|^2\Big)^{\frac{p}{2}}\,d\mu \\
&= \int_A \Big(\sum_{n=1}^N |T_n f_n|^2\Big)^{\frac{p}{2}}\,d\mu \leqslant \int_A \Big(\sum_{n=1}^N (S|f_n|)^2\Big)^{\frac{p}{2}}\,d\mu \\
&= \int_A \Big(\mathbb{E}\Big|\sum_{n=1}^N r_n S|f_n|\Big|^2\Big)^{\frac{p}{2}}\,d\mu \lesssim_p \int_A \mathbb{E}\Big|\sum_{n=1}^N r_n S|f_n|\Big|^p\,d\mu \\
&= \mathbb{E}\Big\|\sum_{n=1}^N r_n S|f_n|\Big\|_E^p \leqslant \|S\|^p\mathbb{E}\Big\|\sum_{n=1}^N r_n |f_n|\Big\|_E^p \\
&= \|S\|^p\int_A \mathbb{E}\Big|\sum_{n=1}^N r_n |f_n|\Big|^p\,d\mu = \|S\|^p\int_A \mathbb{E}\Big|\sum_{n=1}^N r_n f_n\Big|^p\,d\mu \\
&= \|S\|^p\mathbb{E}\Big\|\sum_{n=1}^N r_n f_n\Big\|_E^p.
\end{aligned}$$

In this computation we used that $\mathbb{E}\big|\sum_{n=1}^N r_n a_n\big|^p = \mathbb{E}\big|\sum_{n=1}^N r_n |a_n|\big|^p$ for $a_1,\dots,a_N \in \mathbb{R}$; to see this, just replace r_n by $-r_n$ if $a_n < 0$. The result now follows from the Kahane-Khintchine inequality which permits us to replace the L^p-moments by L^2-moments.

9.3 A multiplier result

Let $(A,\mathscr{A},\mu)$ be a σ-finite measure space and let $\Phi : A \to \gamma(H,E)$ be uniformly bounded and strongly μ-measurable. For $f \in L^2(A;H)$ the integrals

$$R_\Phi f = \int_A \Phi(\xi)f(\xi)\,d\mu(\xi)$$

exist as Bochner integrals in E, and the resulting linear operator $R_\Phi : L^2(A;H) \to E$ is bounded. In the next lemma we consider the special case where Φ is a finite rank simple function.

Lemma 9.12. *Let $\Phi = \sum_{j=1}^k 1_{B_j} \otimes U_j$ be a finite rank simple function, where $U_j = \sum_{n=1}^N h_n \otimes x_{jn}$ with $h_1,\dots,h_N$ orthonormal in H and $B_1,\dots B_k \in \mathscr{A}$ disjoint and of finite μ-measure. Then R_Φ belongs to $\gamma(L^2(A;H),E)$ and*

$$\|R_\Phi\|^2_{\gamma(L^2(A;H),E)} = \mathbb{E}\Big\|\sum_{j=1}^{k}\sum_{n=1}^{N}\gamma_{jn}\sqrt{\mu(B_j)}U_j h_n\Big\|^2.$$

Proof. First we prove that $R_\Phi \in \gamma(L^2(A;H),E)$. By linearity it suffices to prove this for simple functions of the form $\Phi(t) = 1_B(t)U$, where $B \subseteq A$ satisfies $0 < \mu(B) < \infty$ and U of finite rank. But then we have $R_\Phi = U \circ i_B$, where $i_B : L^2(A;H) \to H$ is defined by $i_B f := \int_A 1_B(\xi) f(\xi)\, d\mu(\xi)$. Hence R_Φ is γ-radonifying by the right ideal property.

Let $\widetilde{H}$ denote the linear span of $\{h_1,\dots,h_N\}$ in H. The expression for the γ-norm of R_Φ is obtained from Corollary 5.5 and Theorem 5.15, taking any orthonormal basis of $L^2(A;\widetilde{H})$ containing the functions $f_j \otimes h_n$, where $f_j = 1_{B_j}/\sqrt{\mu(B_j)}$. □

Turning to the situation where $\Phi : A \to \gamma(H,E_1)$ is uniformly bounded and strongly μ-measurable, suppose next that E_2 is another Banach space and $M : A \to \mathscr{L}(E_1,E_2)$ is a uniformly bounded function with the property that $\xi \mapsto M(\xi)x$ is strongly μ-measurable for all $x \in E_1$ (in this situation, with a slight abuse of terminology we call M *strongly μ-measurable*). We put

$$(M\Phi)(\xi) := M(\xi)\Phi(\xi).$$

Let us check that the function $M\Phi$ is strongly μ-measurable. By strong μ-measurability, the range of Φ is μ-separably-valued in $\gamma(H,E_1)$. Therefore by Proposition 5.10 we may assume H is separable. Choose an orthonormal basis $(h_n)_{n=1}^\infty$ for H and let P_n denote the orthogonal projection onto the span of $\{h_1,\dots,h_n\}$. Then $\xi \mapsto (M\Phi P_n)(\xi) := M(\xi)\Phi(\xi)P_n$ is strongly μ-measurable, and the claim follows by noting that $\lim_{n\to\infty} M\Phi P_n = M\Phi$ pointwise in the norm of $\gamma(H,E_1)$ by Proposition 5.12.

As a result, the integral operator $R_{M\Phi}$ is well-defined as a bounded operator from $L^2(A;H)$ to E_2. Thus M induces a mapping

$$\widetilde{M} : R_\Phi \mapsto R_{M\Phi}.$$

We shall be interested in finding conditions that guarantee the boundedness of this mapping as an operator from $\gamma(L^2(A;H),E_1)$ to $\gamma(L^2(A;H),E_2)$.

First we check that the operators R_Φ, with Φ a finite rank step function, are dense in $\gamma(L^2(A;H),E_1)$. For simplicity we state the next lemma and the theorem following it for the Lebesgue interval $(0,T)$, and leave the simple extensions to general measure spaces to the interested reader.

Lemma 9.13. *The operators R_Φ, with $\Phi : (0,T) \to \gamma(H,E)$ a finite rank step function, are dense in $\gamma(L^2(0,T;H),E)$.*

Proof. Let $R \in \gamma(L^2(0,T;H),E)$ be given. By the same argument as in Step 4 of the proof of Theorem 6.17 we may assume that $R \in \gamma(L^2(0,T;\widetilde{H}),E)$, where $\widetilde{H}$ is a separable closed subspace of H. Then, by Proposition 5.12, we even may assume that $\widetilde{H}$ is finite-dimensional.

Let A_k denote the averaging operator on $L^2(0,T;H)$ with respect to the k-th dyadic partition of $(0,T)$ into 2^k subintervals of equal length. Then $\lim_{k\to\infty} R\circ A_k = R$ in $\gamma(L^2(0,T;\widetilde{H}),E)$ by Proposition 5.12. Every $R\circ A_k$ is of the form R_{Φ_k} for a step function $\Phi_k : (0,T)\to\gamma(\widetilde{H},E)$: indeed, take

$$\Phi_k(t)h = R(2^k 1_{(j2^{-k}T,(j+1)2^{-k}T)}\otimes h), \quad t\in(j2^{-k}T,(j+1)2^{-k}T).$$

As function with values in $\gamma(H,E)$, the Φ_k are finite rank step functions. □

The following multiplier theorem, due to Kalton and Weis in a slightly simpler setting, connects the notions of γ-boundedness and γ-radonification. It states that functions with γ-bounded range act as multipliers on spaces of γ-radonifying operators.

Theorem 9.14 (γ-Bounded functions as γ-multipliers). *Suppose that $M : (0,T)\to\mathscr{L}(E_1,E_2)$ is strongly measurable and has γ-bounded range $\{M(t):\ t\in(0,T)\} =: \mathscr{M}$. Then for every finite rank simple function $\Phi : (0,T)\to\gamma(H,E_1)$ the operator $R_{M\Phi}$ belongs to $\gamma(L^2(0,T;H),E_2)$ and*

$$\|R_{M\Phi}\|_{\gamma_p(L^2(0,T;H),E_2)} \leqslant \gamma_p(\mathscr{M})\,\|R_\Phi\|_{\gamma_p(L^2(0,T;H),E_1)}.$$

As a result, the map $\widetilde{M} : R_\Phi\mapsto R_{M\Phi}$ has a unique extension to a bounded operator

$$\widetilde{M} : \gamma_p(L^2(0,T;H),E_1)\to\gamma_p(L^2(0,T;H),E_2)$$

of norm $\|\widetilde{M}\|\leqslant\gamma(\mathscr{M})$.

Proof. The uniqueness part follows from Lemma 9.13. To prove existence we let Φ be a finite rank step function which is kept fixed throughout the proof. In order to show that $R_{M\Phi}$ belongs to $\gamma(L^2(0,T;H),E_2)$ and the above estimate holds we may assume that H is finite-dimensional. Let $(h_n)_{n=1}^N$ be an orthonormal basis of H.

Step 1 – In this step we consider the special case of the theorem where M is a simple function. By passing to a common refinement we may suppose that

$$\Phi(t) = \sum_{j=1}^k 1_{B_j}(t)U_j, \qquad M = \sum_{j=1}^k 1_{B_j}(t)V_j,$$

with disjoint intervals B_j of finite positive measure; the operators $U_j\in\gamma(H,E_1)$ are of finite rank and the $V_j\in\mathscr{L}(E_1,E_2)$ are bounded. Then,

$$(M\Phi)(t) = \sum_{j=1}^k 1_{B_j}(t)V_jU_j.$$

This is a finite rank simple function with values in $\gamma(H, E_2)$. Hence $R_{M\Phi} \in \gamma(L^2(0,T;H), E_2)$, and using Lemma 9.12 we find

$$\begin{aligned}
\|R_{M\Phi}\|^2_{\gamma(L^2(0,T;H),E_2)} &= \mathbb{E}\Big\| \sum_{j=1}^{k} \sum_{n=1}^{N} \gamma_{jn} \sqrt{|B_j|} V_j U_j h_n \Big\|^2 \\
&\leqslant (\gamma(\mathscr{M}))^2 \mathbb{E}\Big\| \sum_{j=1}^{k} \sum_{n=1}^{N} \gamma_{jn} \sqrt{|B_j|} U_j h_n \Big\|^2 \\
&= (\gamma(\mathscr{M}))^2 \|R_\Phi\|^2_{\gamma(L^2(0,T;H),E_1)}.
\end{aligned}$$

Step 2 – For the general case define the step functions $M_k : (0,T) \to \mathscr{L}(E_1, E_2)$ by the averaging procedure of Lemma 9.13, putting

$$M_k(t)x = \frac{2^k}{T} \int_{j2^{-k}T}^{(j+1)2^{-k}T} M(s)x\,ds, \quad t \in (j2^{-k}T, (j+1)2^{-k}T)$$

for $0 \leqslant j \leqslant 2^k - 1$. These integrals are well-defined by the strong measurability and boundedness of M. Moreover, $\lim_{k\to\infty} M_k x = Mx$ in $L^1(0,T;E_2)$ for all $x \in E_1$ and by passing to a subsequence we may assume that $\lim_{k\to\infty} M_k(t)x = M(t)x$ for almost all $t \in (0,T)$ (with an exceptional set depending on x). Since Φ is a finite rank simple function, this implies $\lim_{k\to\infty} R_{M_k\Phi} f = R_{M\Phi} f$ in E_2 for all $f \in L^2(0,T;H)$. Also note that the range of each M_k is γ-bounded with $\gamma(M_k) \leqslant \gamma(\mathscr{M})$ by Theorem 9.7.

Fix an orthonormal basis $(\phi_m)_{m=1}^\infty$ of $L^2(0,T)$ and fix indices $m_0 \leqslant m_1$. Let H_{m_0,m_1} denote the span in $L^2(0,T;H)$ of the functions $\phi_m \otimes h_n$ with $m_0 \leqslant m \leqslant m_1$ and $n = 1, \ldots, N$. By the Fatou lemma and Step 1,

$$\begin{aligned}
&\mathbb{E}\Big\| \sum_{m=m_0}^{m_1} \sum_{n=1}^{N} \gamma_n R_{M\Phi}(\phi_m \otimes h_n) \Big\|^2 \\
&\qquad \leqslant \liminf_{k\to\infty} \mathbb{E}\Big\| \sum_{m=m_0}^{m_1} \sum_{n=1}^{N} \gamma_n R_{M_k\Phi}(\phi_m \otimes h_n) \Big\|^2 \\
&\qquad = \liminf_{k\to\infty} \|R_{M_k\Phi}\|^2_{\gamma(H_{m_0,m_1}),E_2)} \\
&\qquad \leqslant \gamma(\mathscr{M})^2 \|R_\Phi\|^2_{\gamma(H_{m_0,m_1}),E_1)} \\
&\qquad = \gamma(\mathscr{M})^2 \mathbb{E}\Big\| \sum_{m=m_0}^{m_1} \sum_{n=1}^{N} \gamma_n R_\Phi(\phi_m \otimes h_n) \Big\|^2.
\end{aligned}$$

It follows that the sum $\sum_{m=1}^\infty \sum_{n=1}^N \gamma_n R_{M\Phi}(\phi_m \otimes h_n)$ converges in $L^2(\Omega;E_2)$. Hence $R_{M\Phi} \in \gamma(L^2(0,T;H), E_2)$ and the above estimate gives

$$\|R_{M\Phi}\|_{\gamma(L^2(0,T;H),E_2)} \leqslant \gamma(\mathscr{M}) \|R_\Phi\|_{\gamma(L^2(0,T;H),E_1)}. \qquad \square$$

9.4 Exercises

1. Let $(A, \mathscr{A}, \mu)$ be a σ-finite measure space and let $1 \leqslant p < \infty$. For a function $\phi \in L^\infty(A)$ define the multiplier $M_\phi \in \mathscr{L}(L^p(A; E))$ by
$$(M_\phi f)(\xi) := \phi(\xi) f(\xi), \quad \xi \in A.$$
Show that the set $M = \{M_\phi : \ \|\phi\|_\infty \leqslant 1\}$ is R-bounded and give an estimate for $R(M)$.

2. On l^p with $1 \leqslant p \leqslant \infty$, consider the left shift $S : (a_n)_{n\geqslant 1} \mapsto (a_{n+1})_{n\geqslant 1}$. For which values of p is the family $\{S^k : \ k \geqslant 1\}$ R-bounded in $\mathscr{L}(l^p)$?

3. In this exercise we prove that analyticity implies γ-boundedness on compact sets. Let $D \subseteq \mathbb{C}$ be open and let E be a Banach space. A function $f : D \to E$ is said to be *analytic* if the limit
$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$
exists in E for all $z_0 \in D$.
   a) Show that analytic functions are continuous.
   b) Use the Hahn-Banach theorem to show that Cauchy's formulas
   $$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0}\, dz, \quad f'(z_0) = -\frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^2}\, dz$$
   hold for an analytic function f, where Γ is a simple contour in D around z_0 (by (a), the integrals make sense as Bochner integrals).
   c) Let $f : D \to \mathscr{L}(E_1, E_2)$ be a function such that $z \mapsto f(z)x$ is analytic for all $x \in E_1$. Show that for every compact set $K \subseteq D$ the family
   $$\mathscr{T}_K := \{f(z) : \ z \in K\}$$
   is R-bounded.
   Hint: Use Theorem 9.9 and (c) to see that f is γ-bounded on every circle contained in D. Then use the first formula in (b) together with Theorem 9.7.

4. (!) Let $\Phi : (0, T) \to \mathscr{L}(H, E_1)$ be stochastically integrable with respect to an H-cylindrical Brownian motion W_H and suppose that $M : (0, T) \to \mathscr{L}(E_1, E_2)$ is strongly measurable and has γ-bounded range $\mathscr{M}$. Prove that $M\Phi : (0, T) \to \mathscr{L}(H, E_2)$ is stochastically integrable with respect to W_H and
$$\mathbb{E}\Big\| \int_0^T M\Phi \, dW_H \Big\|^p \leqslant (\gamma_p(\mathscr{M}))^p \mathbb{E}\Big\| \int_0^T \Phi \, dW_H \Big\|^p,$$
where $\gamma_p(\mathscr{M})$ is the γ-bound of $\mathscr{M}$ relative to the L^p-norm (see the discussion following Definition 9.2).

Hint: Use the norm of $\gamma_p(L^2(0,T;H),E_1)$ (see Lecture 5 for the notation used).

5. Let E_1 and E_2 be Banach spaces. Prove that the following assertions are equivalent:
   (1) E_1 has cotype 2 and E_2 has type 2;
   (2) every uniformly bounded subset of $\mathscr{L}(E_1,E_2)$ is R-bounded;
   (3) every uniformly bounded subset of $\mathscr{L}(E_1,E_2)$ is γ-bounded.

   Hint: For the proofs that (2) and (3) imply (1), consider suitable uniformly bounded families of rank one operators from E_1 to E_2. Recall that the notions of (co)type and Gaussian (co)type are equivalent (see Exercise 3.5).

   Remark: Via KWAPIEŃ's theorem (see the Notes of Lecture 5), from this exercise we infer that for a Banach space E the following assertions are equivalent:
   (1) E is isomorphic to a Hilbert space;
   (2) every uniformly bounded subset of $\mathscr{L}(E)$ is R-bounded;
   (3) every uniformly bounded subset of $\mathscr{L}(E)$ is γ-bounded.

Notes. The notion of R-boundedness has its origin in the work of BOURGAIN on vector-valued multiplier theorems and has since then been studied by many authors. The results presented here are taken from the fundamental papers by CLÉMENT, DE PAGTER, SUKOCHEV, WITVLIET [24] and WEIS [108]. We refer to DENK, HIEBER, PRÜSS [32] and KUNSTMANN and WEIS [61] for more on the history of this notion and bibliographical references. It is well established by now that a large class of operators associated with analytic semigroups are R-bounded in L^p, a fact which explains the importance of R-boundedness for the theory of parabolic PDEs. Profound R-boundedness results are also available in harmonic analysis (e.g., in connection with Fourier multipliers) and probability theory (in connection with conditional expectation operators). The examples presented in this lecture only give a glimpse of the rich body of results nowadays available.

The result of Exercise 3 is due to WEIS [108]. The result of Exercise 5 is due to LE MERDY and PISIER; see ARENDT and BU [4].