10

Linear equations with additive noise II

In this lecture we pick up the thread of Lecture 8 and continue our investigation of the stochastic abstract Cauchy problem with additive noise,

$$\begin{cases} dU(t) = AU(t)\,dt + B\,dW_H(t), \quad t \in [0,T], \\ U(0) = x. \end{cases}$$

The goal is to prove optimal Hölder regularity results for the solutions in the parabolic case, that is, for operators A generating an analytic C_0-semigroup. Since the problem is solved by

$$U(t) = S(t)x + \int_0^t S(t-s)B\,dW_H(s)$$

it suffices to concentrate on the case $x = 0$. Assuming that $x = 0$ and $B \in \gamma(H,E)$, we shall prove that U has a Hölder continuous version for any exponent $\alpha < \frac{1}{2}$. The main technical tool is the γ-boundedness of the family $\{t^\beta(-A)^\alpha S(t) : \ t \in (0,T)\}$ for $0 < \alpha < \beta < \frac{1}{2}$ (Lemma 10.17). Thus by the γ-multiplier theorem (Theorem 9.14) this family acts as a multiplier in $\gamma(L^2(0,T;H),E)$. This provides a powerful tool for estimating the above stochastic integral.

10.1 Analytic semigroups

We begin with a discussion of analytic semigroups. In this section, all Banach spaces are complex. In later sections we shall return to the setting of real Banach spaces and apply the results to their complexifications.

We begin with a definition (cf. Exercise 9.3).

Definition 10.1. *Let $D \subseteq \mathbb{C}$ be open. A function $f : D \to E$ is* analytic *if*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists in E for all $z_0 \in D$.

Clearly, if f is analytic, then $\langle f, x^* \rangle$ is analytic for all $x^* \in E^*$. In combination with the Hahn-Banach theorem, this fact may be used to show that many results on scalar-valued analytic functions extend to the vector-valued setting.

For $\eta \in (0, \pi]$ define the open sector

$$\Sigma_\eta = \{z \in \mathbb{C} \setminus \{0\} : \ |\arg(z)| < \eta\},$$

where the argument is taken in $(-\pi, \pi]$.

Definition 10.2. *A C_0-semigroup S on E is called* analytic on Σ_η *if for all $x \in E$ the function $t \mapsto S(t)x$ extends analytically to Σ_η and satisfies*

$$\lim_{z \in \Sigma_\eta,\, z \to 0} S(z)x = x.$$

We call S an analytic C_0-semigroup *if S is analytic on Σ_η for some $\eta \in (0, \pi]$.*

The supremum of all $\eta \in (0, \pi]$ such that S analytic on Σ_η is called the *angle of analyticity* of S.

If S is analytic on Σ_η, then for all $z_1, z_2 \in \Sigma_\eta$ we have

$$S(z_1)S(z_2) = S(z_1 + z_2).$$

Indeed, for each $x \in E$ the functions $z_1 \mapsto S(z_1)S(t)x$, $S(t)S(z_1)x$, and $S(z_1 + t)x$ are analytic extensions of $s \mapsto S(s+t)x$ and are therefore equal. Repeating this argument, the functions $z_2 \mapsto S(z_1)S(z_2)x$, $S(z_2)S(z_1)x$, and $S(z_1+z_2)x$ are analytic extensions of $t \mapsto S(z_1 + t)x$ and are therefore equal.

As in the proof of Proposition 7.3, the uniform boundedness theorem implies that if S is analytic on Σ_η, then S is uniformly bounded on $\Sigma_{\eta'} \cap \{z \in \mathbb{C} : \ |z| \leqslant r\}$ for all $0 < \eta' < \eta$ and $r \geqslant 0$. Thus it makes sense to call S a *uniformly bounded analytic C_0-semigroup* if S is uniformly bounded on Σ_η for some $\eta \in (0, \pi]$. Clearly, if A generates an analytic C_0-semigroup on Σ_η, then for any $0 < \eta' < \eta$ the operator $A - \mu$ generates a uniformly bounded analytic C_0-semigroup on $\Sigma_{\eta'}$ if μ (depending on η') is large enough.

Theorem 10.3. *For a closed and densely defined operator A the following assertions are equivalent:*

(1) *there exists $\eta \in (0, \frac{1}{2}\pi]$ such that A generates a uniformly bounded analytic C_0-semigroup on Σ_η;*
(2) *there exists $\theta \in (\frac{1}{2}\pi, \pi]$ such that $\Sigma_\theta \subseteq \varrho(A)$ and $\sup_{\lambda \in \Sigma_\theta} \|\lambda R(\lambda, A)\| < \infty$;*
(3) *$S(t)x \in \mathscr{D}(A)$ for all $x \in E$ and $t > 0$, and $\sup_{t>0} t\|AS(t)\| < \infty$.*

In this situation, the suprema $\widetilde{\eta}$ *and* $\widetilde{\theta}$ *for which* (1) *and* (2) *hold are related by* $\frac{1}{2}\pi + \widetilde{\eta} = \widetilde{\theta}$. *Furthermore we have the representation*

$$S(t)x = \frac{1}{2\pi i}\int_{\Gamma} e^{zt} R(z, A)x\, dz, \quad t > 0,\ x \in E,$$

where Γ *is the upwards oriented boundary of* $\Sigma_{\theta'} \setminus B$ *for some* $\theta' \in (\frac{1}{2}\pi, \theta)$ *and* B *is a closed ball centred at the origin.*

Proof. (1)⇒(2): By Proposition 7.8, if G is the generator of a uniformly bounded C_0-semigroup, then $\{\mathrm{Re}\lambda > 0\} \subseteq \varrho(G)$ and $\lambda R(\lambda, G)$ is uniformly bounded on every proper sub-sector Σ_ρ, $0 < \rho < \frac{1}{2}\pi$.

Let S be the C_0-semigroup generated by A and let it be uniformly bounded on the sector Σ_η. The implication (1)⇒(2) follows by applying the above observation to the uniformly bounded C_0-semigroups $(S(e^{i\eta'}t))_{t\geqslant 0}$ with $0 < \eta' < \eta$, whose generators are $e^{i\eta'}A$. This gives the uniform boundedness of $\lambda R(\lambda, A)$ on the union of all sectors $e^{i\eta'}\Sigma_\rho$ for $0 < \eta' < \eta$ and $0 < \rho < \frac{1}{2}\pi$, which equals $\Sigma_{\frac{1}{2}\pi+\eta'}$. This argument also proves the inequality $\widetilde{\theta} \geqslant \frac{1}{2}\pi + \widetilde{\eta}$.

(2)⇒(3): First we prove that the conditions of (2) imply the integral representation for $S(t)x$.

The integral converges absolutely for all $t > 0$ and $x \in E$, and as a function of t it extends to a bounded analytic function on the sector $\Sigma_{\eta'}$ for any $\eta' < \theta' - \frac{1}{2}\pi$. This proves the inequality $\widetilde{\theta} \leqslant \frac{1}{2}\pi + \widetilde{\eta}$.

Fix $t > 0$ and $x \in E$. For $\mu > 0$ such that $\mu \notin B$ define

$$v_\mu(t)x = \frac{1}{2\pi i}\int_{\Gamma} e^{zt}(\mu - z)^{-1} R(z, A)x\, dz.$$

Our aim is to show that $v_\mu(t)x = S(t)R(\mu, A)x$. Then,

$$S(t)x = \lim_{\mu\to\infty} S(t)\mu R(\mu, A)x = \lim_{\mu\to\infty} \mu v_\mu(t)x = \frac{1}{2\pi i}\int_{\Gamma} e^{zt} R(z, A)x\, dz,$$

where the first equality follows from Lemma 7.9 and the last is obtained by splitting $\Gamma = \Gamma_{r,1} \cup \Gamma_{r,2}$ with $\Gamma_{r,1} = \{z \in \Gamma : \|z\| \leqslant r\}$ and $\Gamma_{r,2} = \{z \in \Gamma : \|z\| \geqslant r\}$: for large fixed r, the integral over $\Gamma_{r,2}$ is less than ε, uniformly with respect to $\mu \geqslant 2r$, while the integral over $\Gamma_{r,1}$ tends to $\frac{1}{2\pi i}\int_{\Gamma_{r,1}} e^{zt}R(z, A)x\, dz$ by dominated convergence. Now pass to the limit $r \to \infty$.

The strategy is to prove that $t \mapsto v_\mu(t)x$ is a weak solution of the Cauchy problem

$$\begin{cases} u'(t) = Au(t), & t \in [0, T], \\ u(0) = R(\mu, A)x. \end{cases}$$

Then $t \mapsto v_\mu(t)x$ is a strong solution by Proposition 7.16 and by the uniqueness part of Theorem 7.17 it follows that $v_\mu(t)x = S(t)R(\mu, A)x$.

It is easily checked that $t \mapsto v_\mu(t)$ is integrable on $[0,T]$ (even continuous), and for all $x^* \in \mathscr{D}(A^*)$ we obtain

$$\begin{aligned}
\int_0^t \langle v_\mu(s), A^*x^*\rangle\,ds &= \int_0^t \frac{1}{2\pi i}\int_\Gamma e^{zs}(\mu - z)^{-1}\langle R(z,A)x, A^*x^*\rangle\,dz\,ds \\
&= \int_0^t \frac{1}{2\pi i}\int_\Gamma e^{zs}(\mu - z)^{-1}\langle zR(z,A)x - x, x^*\rangle\,dz\,ds \\
&\overset{(*)}{=} \int_0^t \frac{1}{2\pi i}\int_\Gamma e^{zs}(\mu - z)^{-1}\langle zR(z,A)x, x^*\rangle\,dz\,ds \\
&= \frac{1}{2\pi i}\int_\Gamma (e^{zt} - 1)(\mu - z)^{-1}\langle R(z,A)x, x^*\rangle\,dz \\
&\overset{(**)}{=} \frac{1}{2\pi i}\int_\Gamma e^{zt}(\mu - z)^{-1}\langle R(z,A)x, x^*\rangle\,dz - \langle R(\mu,A)x, x^*\rangle.
\end{aligned}$$

Here the equality $(*)$ follows from the observation that by Cauchy's theorem we have

$$\frac{1}{2\pi i}\int_\Gamma (\mu - z)^{-1}e^{zs}\,dz = 0,$$

since $\mu \notin B$ is on the right of Γ. The equality $(**)$ follows from

$$\frac{1}{2\pi i}\int_\Gamma (\mu - z)^{-1}\langle R(z,A)x, x^*\rangle\,dz = \langle R(\mu,A)x, x^*\rangle$$

by the analyticity of the resolvent (Exercise 7.2) and Cauchy's theorem.

Now we are ready for the proof that (2) implies (3). Fix $t > 0$ and $x \in E$. Since

$$M := \sup_{z\in\Gamma}\|AR(z,A)\| = \sup_{z\in\Gamma}\|zR(z,A) - I\|$$

is finite, the integral $\frac{1}{2\pi i}\int_\Gamma e^{tz}R(z,A)Ax\,dz$ converges absolutely. From Hille's theorem we deduce that $S(t)x \in \mathscr{D}(A)$ and

$$AS(t)x = \frac{1}{2\pi i}\int_\Gamma e^{tz}R(z,A)Ax\,dz.$$

By estimating this integral and letting the radius of the ball B in the definition of Γ tend to 0, it follows moreover that

$$\|AS(t)x\| \leqslant \frac{M}{\pi}\|x\|\int_0^\infty e^{\rho t\cos\theta'}\,d\rho = t^{-1}\frac{M}{\pi|\cos\theta'|}\|x\|.$$

(3)⇒(1): For all $x \in \mathscr{D}(A^n)$, $t \mapsto S(t)x$ is n times continuously differentiable and $S^{(n)}(t)x = A^nS(t)x = (AS(t/n))^nx$. Since $\mathscr{D}(A^n)$ is dense, the boundedness of $AS(t/n)$ and closedness of the nth derivative in $C([0,T];E)$ together imply that the same conclusion holds for $x \in E$. Moreover, $\|S^{(n)}(t)x\| \leqslant C^nn^n/t^n\|x\|$, where C is the supremum in (3). From $n! \geqslant n^n/e^n$ we obtain that for each $t > 0$ the series

$$S(z)x := \sum_{n=0}^{\infty} \frac{1}{n!}(z-t)^n S^{(n)}(t)x$$

converges absolutely on the ball $B(t, rt/eC)$ for all $0 < r < 1$ and defines an analytic function there. The union of these balls is the sector Σ_η with $\sin\eta = 1/eC$. We shall complete the proof by showing that $S(z)$ is uniformly bounded and satisfies $\lim_{z\to 0} S(z)x = x$ in $\Sigma_{\eta'}$ for each $0 < \eta' < \eta$. To this end we fix $0 < r < 1$. For $z \in B(t, rt/eC)$ we have

$$\|S(z)x\| \leqslant \sum_{n=0}^{\infty} \frac{1}{n!} r^n (t/eC)^n C^n n^n / t^n \|x\| \leqslant \sum_{n=0}^{\infty} r^n \|x\|.$$

This proves uniform boundedness on the sectors $\Sigma_{\eta'}$. To prove strong continuity it then suffices to consider $x \in \mathscr{D}(A)$, for which it follows from estimating the identity

$$S(z)x - x = e^{i\theta} \int_0^r S(se^{i\theta})Ax\,ds$$

where $z = re^{i\theta}$. □

Remark 10.4. We will use analyticity only through condition (3), which gives a 'real' characterisation of analyticity. In the context of semigroups on real Banach spaces this condition could be taken as the definition for analyticity, which has the advantage of avoiding the digressions through complexified spaces. In concrete examples, however, it is often easier to check analyticity using Definition 10.2 or condition (2) of Theorem 10.3.

By a rescaling argument we obtain:

Corollary 10.5. *If A generates an analytic C_0-semigroup S on E, then*

$$\limsup_{t\downarrow 0} t\|AS(t)\| < \infty.$$

From the fact that $S(t)x \in \mathscr{D}(A)$ for all $t > 0$ and $x \in E$ we deduce:

Corollary 10.6. *If A generates an analytic C_0-semigroup S on E, then for all initial values $x \in E$ the problem* (ACP) *has a unique classical solution, which is given by $u(t) = S(t)x$.*

10.2 Fractional powers

Throughout this section we assume that A is the generator of a C_0-semigroup S on E which is *uniformly exponentially stable* in the sense that there exist constants $M \geqslant 1$ and $\mu > 0$ such that $\|S(t)\| \leqslant Me^{-\mu t}$ for all $t \geqslant 0$.

The next definition is motivated by the trivial identity

$$c^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-ct}\, dt, \quad c > 0,$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$ is the Euler gamma function.

Definition 10.7. *For $0 < \alpha < 1$ we define the* fractional power $(-A)^{-\alpha}$ *of $-A$ by the formula*

$$(-A)^{-\alpha} x := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S(t)x\, dt, \quad x \in E.$$

Note that $(-A)^{-\alpha}$ is well-defined and bounded on E and commutes with $S(t)$ for all $t \geqslant 0$. Sometimes it is useful to extend the definition to the limiting values $\alpha \in \{0, 1\}$ by putting $(-A)^0 = I$ and $(-A)^{-1} = -A^{-1}$.

Lemma 10.8. *For all $0 < \alpha, \beta < 1$ satisfying $0 < \alpha + \beta < 1$ we have*

$$(-A)^{-\alpha}(-A)^{-\beta} = (-A)^{-\beta}(-A)^{-\alpha} = (-A)^{-\alpha-\beta}.$$

Proof. It suffices to prove that $(-A)^{-\alpha}(-A)^{-\beta} = (-A)^{-\alpha-\beta}$; the other identity follows upon interchanging α and β.

For all $x \in E$ we have

$$\begin{aligned}
(-A)^{-\alpha}(-A)^{-\beta} x &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} S(s+t)x\, ds\, dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \int_t^\infty t^{\alpha-1} (s-t)^{\beta-1} S(s)x\, ds\, dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \Big(\int_0^s t^{\alpha-1} (s-t)^{\beta-1}\, dt \Big) S(s)x\, ds \\
&\overset{(*)}{=} \frac{1}{\Gamma(\alpha+\beta)} \int_0^\infty s^{\alpha+\beta-1} S(s)x\, ds = (-A)^{-\alpha-\beta} x,
\end{aligned}$$

where the identity $(*)$ follows from

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^s t^{\alpha-1}(s-t)^{\beta-1} dt = \frac{s^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \tau^{\alpha-1}(1-\tau)^{\beta-1} d\tau = \frac{s^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}.$$

Indeed, computing as above,

$$\begin{aligned}
\Gamma(\alpha+\beta) \int_0^1 \tau^{\alpha-1}(1-\tau)^{\beta-1}\, d\tau &= \int_0^\infty \int_0^1 s^{\alpha+\beta-1} \tau^{\alpha-1} (1-\tau)^{\beta-1} e^{-s}\, d\tau\, ds \\
&= \int_0^\infty \int_0^s t^{\alpha-1}(s-t)^{\beta-1} e^{-s}\, dt\, ds \\
&= \int_0^\infty \int_t^\infty t^{\alpha-1}(s-t)^{\beta-1} e^{-s}\, ds\, dt \\
&= \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-s-t}\, ds\, dt \\
&= \Gamma(\alpha)\Gamma(\beta).
\end{aligned}$$

□

Lemma 10.9. *We have* $\sup_{0<\alpha<1} \|(-A)^{-\alpha}\| < \infty$.

Proof. We estimate $\|(-A)^{-\alpha}x\|$ by

$$\frac{1}{\Gamma(\alpha)}\Big\|\int_0^1 t^{\alpha-1}S(t)x\,dt\Big\| + \frac{1}{\Gamma(\alpha)}\Big\|\int_1^\infty t^{\alpha-1}S(t)x\,dt\Big\| =: \text{(I)} + \text{(II)}.$$

Now,

$$\text{(I)} \leqslant \frac{M\|x\|}{\alpha\Gamma(\alpha)} = \frac{M\|x\|}{\Gamma(\alpha+1)}, \quad \text{(II)} \leqslant \frac{M\|x\|}{\Gamma(\alpha)}\int_1^\infty t^{\alpha-1}e^{-\mu t}\,dt \leqslant \frac{M\|x\|}{\mu^\alpha},$$

and both right hand sides are uniformly bounded for $0 < \alpha < 1$. □

Lemma 10.10. *For all* $x \in E$, $\alpha \mapsto (-A)^{-\alpha}x$ *is continuous on* $[0,1]$.

Proof. First let $x \in \mathscr{D}(A)$ and put $Ax = y$. An integration by parts gives

$$\begin{aligned}(-A)^{-\alpha}x - x &= \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}S(t)x\,dt - x\\ &= -\frac{1}{\alpha\Gamma(\alpha)}\int_0^\infty t^{\alpha}S(t)y\,dt - x\\ &= -\int_0^\infty \Big(\frac{t^\alpha}{\Gamma(\alpha+1)} - 1\Big)S(t)y\,dt,\end{aligned}$$

where we used that $x = A^{-1}y = -\int_0^\infty S(t)y\,dt$. Hence, for any $r \geqslant 1$,

$$\|(-A)^{-\alpha}x - x\| \leqslant M\|y\|\int_0^r \Big|\frac{t^\alpha}{\Gamma(\alpha+1)} - 1\Big|\,dt + CM\|y\|\int_r^\infty t^\alpha e^{-\mu t}\,dt,$$

where $C = \sup\big\{\big|\frac{1}{\Gamma(\alpha+1)} - \frac{1}{t^\alpha}\big| :\ 0<\alpha<1,\ t \geqslant 1\big\}$. Choosing $r \geqslant 1$ so large that the second term is less than ε and then passing to the limit $\alpha \downarrow 0$ in the first, we obtain the continuity of $\alpha \mapsto (-A)^{-\alpha}x$ at $\alpha = 0$ for $x \in \mathscr{D}(A)$. In view of the previous lemma, continuity at $\alpha = 0$ for all $x \in E$ follows from this.

The continuity of $\alpha \mapsto (-A)^{-\alpha}x$ at $\alpha = 1$ is proved in the same way, this time noting that for all $x \in \mathscr{D}(A)$ we have

$$(-A)^{-\alpha}x - (-A)^{-1}x = \int_0^\infty \Big(\frac{t^{\alpha-1}}{\Gamma(\alpha)} - 1\Big)S(t)x\,ds.$$

Finally the continuity for $\alpha \in (0,1)$ follows from the continuity at $\alpha = 0$ and the 'semigroup' property of Lemma 10.8. □

Lemma 10.11. *For* $0 < \alpha < 1$ *the operator* $(-A)^{-\alpha}$ *is injective.*

Proof. Suppose $(-A)^{-\alpha}x = 0$. Then Lemma 10.8 implies $(-A)^{-\beta}x = 0$ for all $\alpha < \beta < 1$, and Lemma 10.10 gives $A^{-1}x = 0$. Hence $x = 0$. □

This lemma suggests the following definition.

Definition 10.12. *For* $0 < \alpha < 1$ *define* $(-A)^\alpha := ((-A)^{-\alpha})^{-1}$.

As an unbounded operator with the range of $(-A)^{-\alpha}$ as its natural domain, $(-A)^\alpha$ is a closed and injective operator in E. With respect to the norm

$$\|x\|_{\mathscr{D}((-A)^\alpha)} := \|(-A)^\alpha x\|, \tag{10.1}$$

$\mathscr{D}((-A)^\alpha)$ is a Banach space and $(-A)^\alpha : \mathscr{D}((-A)^\alpha) \to E$ is an isometric isomorphism. For later reference we note that $\mathscr{D}(A)$ is dense in $\mathscr{D}((-A)^\alpha)$. Indeed, for any $x \in \mathscr{D}((-A)^\alpha)$ we have $\lim_{\lambda\to\infty} \lambda R(\lambda, A)(-A)^\alpha x = (-A)^\alpha x$, and since $R(\lambda, A)$ and $(-A)^\alpha$ commute this means that $\lim_{\lambda\to\infty} \lambda R(\lambda, A)x = x$ in the norm of $\mathscr{D}((-A)^\alpha)$.

Lemma 10.13. *For* $0 < \alpha < 1$ *we have*

$$(-A)^{\alpha-1}(-A)^{-\alpha} = (-A)^{-\alpha}(-A)^{\alpha-1} = (-A)^{-1}.$$

Proof. This follows from Lemmas 10.8 and 10.10:

$$(-A)^{-1}x = \lim_{\beta\uparrow 1}(-A)^{-\beta}x = \lim_{\beta\uparrow 1}(-A)^{-\alpha}(-A)^{\alpha-\beta}x = (-A)^{-\alpha}(-A)^{\alpha-1}x. \quad \square$$

In the next two lemmas we assume that the C_0-semigroup S, in addition to being uniformly exponentially stable, is analytic.

Lemma 10.14. *For all* $0 < \alpha < 1$ *and* $t > 0$ *the operator* $(-A)^\alpha S(t)$ *is bounded and we have*

$$\sup_{t>0} t^\alpha \|(-A)^\alpha S(t)\| < \infty.$$

Proof. Since S is analytic, $S(t)$ maps E into $\mathscr{D}(A)$ and $\sup_{t>0} t\|AS(t)\| < \infty$. The boundedness of $(-A)^\alpha S(t)$ follows from the boundedness of $AS(t)$ by the identity $(-A)^\alpha S(t) = -(-A)^{\alpha-1}AS(t)$.

To prove the estimate, note that for all $x \in E$ we have

$$(-A)^\alpha S(t)x = \frac{-1}{\Gamma(1-\alpha)}\int_0^\infty s^{-\alpha} AS(t+s)x\,ds,$$

so, for $t > 0$,

$$\begin{aligned}\|(-A)^\alpha S(t)x\| &\leqslant \frac{C}{\Gamma(1-\alpha)}\int_0^\infty s^{-\alpha}(t+s)^{-1}\|x\|\,ds \\ &= \frac{Ct^{-\alpha}}{\Gamma(1-\alpha)}\int_0^\infty \tau^{-\alpha}(1+\tau)^{-1}\|x\|\,d\tau. \quad \square\end{aligned}$$

Lemma 10.15. *For all* $0 < \alpha < 1$ *we have*

$$\sup_{t>0} t^{-\alpha}\|S(t)(-A)^{-\alpha} - (-A)^{-\alpha}\| < \infty.$$

Proof. From the identity $(-A)^{-\alpha}(-A)x = (-A)^{1-\alpha}x$ for $x \in \mathscr{D}(A)$ and Lemma 10.14 we obtain, for $t > 0$,

$$\|S(t)(-A)^{-\alpha}x - (-A)^{-\alpha}x\| \leqslant \int_0^t \|(-A)^{-\alpha}AS(s)x\|\,ds \leqslant \frac{Ct^\alpha}{\alpha}\|x\|,$$

where $C = \sup_{t>0} t^{1-\alpha}\|(-A)^{1-\alpha}S(t)\|$. □

In the next section we shall consider generators A of analytic C_0-semigroups which are not necessarily uniformly exponentially stable. In this situation, fractional powers still can be defined for the shifted operators $A - \lambda$ with λ large enough. The next lemma states that the resulting domains $\mathscr{D}((\lambda - A)^\alpha)$ are independent of λ.

To make things more precise, let A be the generator of an arbitrary C_0-semigroup on E and suppose $M \geqslant 1$ and $\mu \in \mathbb{R}$ are such that $\|S(t)\| \leqslant Me^{\mu t}$ for all $t \geqslant 0$.

Lemma 10.16. *For all* $0 < \alpha < 1$ *and* $\lambda_1, \lambda_2 > \mu$ *we have*

$$\mathscr{D}((\lambda_1 - A)^\alpha) = \mathscr{D}((\lambda_2 - A)^\alpha)$$

with equivalent norms.

Proof. The linear operator $(\lambda_2 - A)^{-\alpha}(\lambda_1 - A)^\alpha$ is a bounded and injective mapping from $\mathscr{D}((\lambda_1 - A)^\alpha)$ onto $\mathscr{D}((\lambda_2 - A)^\alpha)$ with inverse $(\lambda_1 - A)^{-\alpha}(\lambda_2 - A)^\alpha$. Thus $\mathscr{D}((\lambda_1 - A)^\alpha)$ and $\mathscr{D}((\lambda_2 - A)^\alpha)$ are isomorphic as Banach spaces. It remains to prove that $\mathscr{D}((\lambda_1 - A)^\alpha) = \mathscr{D}((\lambda_2 - A)^\alpha)$ as linear subspaces of E. But this follows from the fact that these spaces are the completions of $\mathscr{D}(A)$ with respect to the equivalent norms $\|\cdot\|_{\mathscr{D}((\lambda_1 - A)^\alpha)}$ and $\|\cdot\|_{\mathscr{D}((\lambda_2 - A)^\alpha)}$. □

This proposition justifies the notation $E_\alpha := \mathscr{D}((\lambda - A)^\alpha)$); this defines E_α as a Banach space up to an equivalent norm.

10.3 Hölder regularity

We now turn to the stochastic abstract Cauchy problem

$$\begin{cases} dU(t) = AU(t)\,dt + B\,dW_H(t), & t \in [0,T], \\ U(0) = 0. \end{cases} \qquad \text{(SACP}_0\text{)}$$

We shall assume throughout this section that A generates an analytic C_0-semigroup S on E satisfying $\|S(t)\| \leqslant Me^{\mu t}$ for certain $M \geqslant 1$, $\mu \in \mathbb{R}$, and all $t \geqslant 0$.

The key lemma for proving Hölder regularity of the solutions of (SACP$_0$) reads as follows.

Lemma 10.17. *For all real numbers $\alpha, \beta, \eta \geqslant 0$ satisfying $0 \leqslant \alpha + \eta < \beta < 1$ and $\lambda > \mu$, the set*

$$\{t^\beta (\lambda - A)^\eta S(t) : \ t \in (0, T)\}$$

is R-bounded (and hence γ-bounded) in $\mathscr{L}(E, E_\alpha)$, with γ-bound $O(T^{\beta-\alpha-\eta})$.

Proof. For all $x \in E$ the $\mathscr{L}(E, E_\alpha)$-valued function $\Psi(t)x := t^\beta(\lambda - A)^\eta S(t)x$ is continuously differentiable on $(0, T)$ with derivative

$$\Psi'(t)x = \beta t^{\beta-1}(\lambda - A)^\eta S(t)x + t^\beta(\lambda - A)^\eta AS(t)x,$$

where the second expression on the right hand side is well-defined since we may write $AS(t) = S(t/2)AS(t/2)$. By Lemma 10.14,

$$\|\Psi'(t)\|_{\mathscr{L}(E,E_\alpha)} \leqslant Ct^{\beta-\alpha-\eta-1}, \quad t \in (0, T),$$

where C is a constant depending on T. Here we estimated the second term as

$$\begin{aligned}\|(\lambda - A)^\eta AS(t)\|_{\mathscr{L}(E,E_\alpha)} &= \|(\lambda - A)^\eta S(t/2)AS(t/2)\|_{\mathscr{L}(E,E_\alpha)} \\ &\leqslant Ct^{-\alpha-\eta}\|AS(t/2)\| \leqslant C't^{-\alpha-\eta-1}.\end{aligned}$$

Since $t^{\beta-\alpha-\eta-1}$ is integrable, the lemma follows from Theorem 9.9. □

After these preparations we are ready to state and prove the main results of this lecture. The first is an existence result.

Theorem 10.18. *If $B \in \gamma(H, E)$, then the $\mathscr{L}(H, E)$-valued function $t \mapsto S(t)B$ is stochastically integrable on $(0, T)$ with respect to W_H. As a consequence, the stochastic Cauchy problem* (SACP$_0$) *associated with A and B admits a unique strong solution.*

Proof. By Theorems 8.6 and 8.10 it suffices to check that the function $\Phi(t) = S(t)B$ is stochastically integrable with respect to W_H, or equivalently, that the operator

$$R_\Phi f := \int_0^T \Phi(t)f(t)\,dt, \quad f \in L^2(0, T; H),$$

is γ-radonifying from $L^2(0, T; H)$ to E.

Pick a number $\beta \in (0, \frac{1}{2})$ and write

$$\Phi(t) = t^\beta S(t)[t^{-\beta}B] := t^\beta S(t)\Psi(t),$$

where $\Psi(t) := t^{-\beta}B$. By Lemma 10.17 and the γ-multiplier theorem (Theorem 9.14), the operator R_Φ belongs to $\gamma(L^2(0, T; H), E)$ once we know that $R_\Psi \in \gamma(L^2(0, T; H), E)$. But this is immediate from the result of Exercise 5.3, since $t \mapsto t^{-\beta}$ belongs to $L^2(0, T)$ and B belongs to $\gamma(H, E)$. □

Under the assumptions of the theorem we define the E-valued process $(U(t))_{t\in[0,T]}$ by

$$U(t) := \int_0^t S(t-s)B\,dW_H(s).$$

In order to formulate the second main result, for a Banach space F and $0 \leqslant \beta < 1$ we define $C^\beta([0,T];F)$ as the space of all continuous functions $u : [0,T] \to F$ for which

$$\sup_{0\leqslant s<t\leqslant T} \frac{\|u(t)-u(s)\|}{(t-s)^\beta} < \infty.$$

The elements of $C^\beta([0,T];F)$ are said to be *Hölder continuous* of exponent β.

Theorem 10.19 (Hölder regularity). *Under the assumptions of the previous theorem, for all $\alpha \geqslant 0$ and $\beta \geqslant 0$ satisfying $\alpha+\beta < \frac12$ and $1 \leqslant p < \infty$ the solution U belongs to $L^p(\Omega;E_\alpha)$ and there exists a constant $C \geqslant 0$ such that for all $0 \leqslant s,t \leqslant T$,*

$$\big(\mathbb{E}\|U(t)-U(s)\|_{E_\alpha}^p\big)^{\frac1p} \leqslant C|t-s|^\beta.$$

As a consequence, for all $\alpha \geqslant 0$ and $\beta \geqslant 0$ satisfying $\alpha+\beta < \frac12$ the process $(U(t))_{t\in[0,T]}$ has a version with trajectories in $C^\beta([0,T];E_\alpha)$.

Proof. By the Kahane-Khintchine inequality it suffices to prove the L^p-estimate for $p=2$.

Fix $\alpha \geqslant 0$ and $\beta \geqslant 0$ such that $\alpha+\beta < \frac12$. Let us first prove that for all $t \in [0,T]$ the random $U(t)$ takes its values in E_α almost surely. We do so by showing that $S(\cdot)B$ is stochastically integrable as an $\mathscr{L}(H,E_\alpha)$-valued function. Fix $\alpha < \theta < \frac12$. Then $\{t^\theta S(t) :\ t \in (0,T)\}$ is γ-bounded in $\mathscr{L}(E,E_\alpha)$ by Lemma 10.17. As we have seen, the function $t \mapsto t^{-\theta}B$ defines an operator in $\gamma(L^2(0,T;H),E)$ of norm $\|t^{-\theta}\|_{L^2(0,T)}\|B\|_{\gamma(H,E)}$. Now Theorem 9.14 and the identity $S(t)B = t^\theta S(t)t^{-\theta}B$ imply that

$$\|R_{S(\cdot)B}\|_{\gamma(L^2(0,T;H),E_\alpha)} \leqslant C\|B\|_{\gamma(H,E)}.$$

Fix $0 \leqslant s \leqslant t \leqslant T$. By the triangle inequality in $L^2(\Omega;E_\alpha)$,

$$\begin{aligned}\big(\mathbb{E}\|U(t)-U(s)\|_{E_\alpha}^2\big)^{\frac12} &\leqslant \Big(\mathbb{E}\Big\|\int_0^s [S(t-r)-S(s-r)]B\,dW(r)\Big\|_{E_\alpha}^2\Big)^{\frac12}\\ &\quad + \Big(\mathbb{E}\Big\|\int_s^t S(t-r)B\,dW(r)\Big\|_{E_\alpha}^2\Big)^{\frac12}.\end{aligned}$$

Choose $\lambda \in \mathbb{R}$ sufficiently large in order that the fractional powers of $\lambda - A$ exist. For the first term we have, for any choice of $\varepsilon > 0$ such that $\alpha+\beta+\varepsilon < \frac12$, and using Lemmas 10.8 and 10.15,

$$
\begin{aligned}
&\mathbb{E}\Big\| \int_0^s [S(t-r) - S(s-r)]B\,dW(r)\Big\|_{E_\alpha}^2 \\
&\simeq \mathbb{E}\Big\| \int_0^s (s-r)^{\alpha+\beta+\varepsilon}(\lambda - A)^{\alpha+\beta} S(s-r) \\
&\qquad\qquad\qquad \times (s-r)^{-\alpha-\beta-\varepsilon}[S(t-s)-I](\lambda-A)^{-\beta}B\,dW(r)\Big\|^2 \\
&\leqslant C^2\mathbb{E}\Big\| \int_0^s (s-r)^{-\alpha-\beta-\varepsilon}[S(t-s)-I](\lambda-A)^{-\beta}B\,dW(r)\Big\|^2 \\
&= C^2\|[S(t-s)-I](\lambda-A)^{-\beta}B\|_{\gamma(H,E)}^2 \int_0^s (s-r)^{-2(\alpha+\beta+\varepsilon)}\,dr \\
&\leqslant C^2\|[S(t-s)-I](\lambda-A)^{-\beta}\|^2\|B\|_{\gamma(H,E)}^2 s^{1-2(\alpha+\beta+\varepsilon)} \\
&\leqslant C_T^2 (t-s)^{2\beta}\|B\|_{\gamma(H,E)}^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\mathbb{E}\Big\| \int_s^t S(t-r)B\,dW(r)\Big\|_{E_\alpha}^2 \\
&\qquad \eqsim \mathbb{E}\Big\| \int_s^t (t-r)^{\frac12-\beta}(\lambda-A)^{\alpha}S(t-r)(t-r)^{-\frac12+\beta}B\,dW(r)\Big\|^2 \\
&\qquad \leqslant C^2\mathbb{E}\Big\| \int_s^t (t-r)^{-\frac12+\beta}B\,dW(r)\Big\|^2 \\
&\qquad = C^2\|B\|_{\gamma(H,E)}^2 \int_s^t (t-r)^{-1+2\beta}\,dr \\
&\qquad \leqslant C_T^2\|B\|_{\gamma(H,E)}^2 (t-s)^{2\beta}.
\end{aligned}
$$

The first part of the theorem follows by combining these estimates.

For the second part, pick $\beta < \beta' < \frac12 - \alpha$. Given $p \geqslant 1$, by the above we find a constant C such that for $0 \leqslant s, t \leqslant T$,

$$\mathbb{E}\|U(t) - U(s)\|_{E_\alpha}^p \leqslant C^p|t-s|^{\beta' p}.$$

For p large enough the existence of a version with β-Hölder continuous trajectories now follows from Kolmogorov's theorem (Theorem 6.9). □

Example 10.20. For the stochastic heat equation in $L^2(0,1)$ with Dirichlet boundary conditions, Theorem 10.19 implies the existence of a solution U with trajectories in $\in C^\eta([0,T]; C^\theta[0,1])$ for all $\eta, \theta \geqslant 0$ satisfying $2\eta + \theta < \frac12$. This will be shown as a special case of a more general space-time regularity result in the last lecture.

10.4 Exercises

1. a) Show that the heat semigroup S on $L^p(\mathbb{R}^d)$ with $1 \leqslant p < \infty$ (Example 7.20) is analytic on the sector $\Sigma_{\frac12\pi}$, and uniformly bounded on every proper subsector of $\Sigma_{\frac12\pi}$.

Hint: Put $S(z)f = K_z * f$, where K_z is the analytic extension of the heat kernel. Use Young's inequality together with the estimate

$$\|K_z\|_1 = \left(\frac{|z|}{\mathrm{Re}(z)}\right)^{\frac{1}{2}d}.$$

b) Show that in $L^2(\mathbb{R}^d)$, S is contractive on $\Sigma_{\frac{1}{2}\pi}$.

Remark: Using interpolation theory, the above results imply the estimate

$$\|S(t)\|_p \leqslant \left(\frac{|z|}{\mathrm{Re}(z)}\right)^{|\frac{1}{2}-\frac{1}{p}|d}, \quad \mathrm{Re}(z) > 0.$$

2. This exercise gives a two-dimensional example of a bounded analytic C_0-semigroup which is uniformly exponentially stable, contractive on $\mathbb{R}_+$, and fails to be contractive on any open sector containing $\mathbb{R}_+$.

On $\mathbb{R}^2$ consider the norm $\|\cdot\|_Q$ induced by the inner product $[x,y]_Q := [Qx,y]$, where $[\cdot,\cdot]$ represents the standard inner product of $\mathbb{R}^2$ and

$$Q = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

a) Show that the symmetric matrix Q is positive and conclude that $[\cdot,\cdot]_Q$ does indeed define an inner product on $\mathbb{R}^2$.

On $(\mathbb{R}^2, \|\cdot\|_Q)$ we consider the C_0-semigroup S,

$$S(t) = e^{-t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

b) Show that $\|S(t)\|_Q = e^{-t}(t + \sqrt{t^2+1})$ and conclude that S is contractive on $\mathbb{R}_+$.

Hint: Use the fact that $\|S(t)\|_Q^2$ equals the largest eigenvalue of $S(t)S^*(t)$ (the adjoint refers to the inner product $[\cdot,\cdot]_Q$). For this, solve the equation $\det(S(t)Q^{-1}S'(t) - \lambda Q^{-1}) = 0$, where $S'(t)$ is the transpose of $S(t)$.

On the complexification $\mathbb{C}^2$ of $\mathbb{R}^2$ we consider the inner product

$$\langle x,y\rangle_Q := \langle Qx,y\rangle,$$

where this time $\langle\cdot,\cdot\rangle$ represents the standard inner product of $\mathbb{C}^2$. Prove that the complexified semigroup S has the following properties:

c) S extends to an entire C_0-semigroup which is uniformly bounded on the sector Σ_η for all $0 < \eta < \frac{1}{2}\pi$.

d) S fails to be contractive on any open sector Σ_η.

3. This exercise gives necessary and sufficient conditions for a closed densely defined operator A in E to generate an analytic C_0-semigroup which is contractive on a sector Σ_η. For $x \in E$ we define
$$\partial(x) = \{x^* \in E^* : \ \|x\| = \|x^*\|, \ \langle x, x^* \rangle = \|x\|\|x^*\|\}.$$
By the Hahn-Banach theorem, for all $x \in E$ we have $\partial(x) \neq \varnothing$. Let A be a closed densely defined operator in E and assume that $\varrho(A) \cap (0, \infty) \neq \varnothing$. Prove that the following assertions are equivalent:
(1) A generates an analytic C_0-semigroup on E which is contractive on an open sector Σ_η;
(2) There exists a constant $C \geqslant 0$ such that for all non-zero $x \in \mathscr{D}(A)$ and all $x^* \in \partial(x)$ we have
$$|\mathrm{Im}\langle Ax, x^* \rangle| \leqslant -C \,\mathrm{Re}\langle Ax, x^* \rangle;$$
(3) There exists a constant $C \geqslant 0$ such that for all non-zero $x \in \mathscr{D}(A)$ there exists $x^* \in \partial(x)$ such that
$$|\mathrm{Im}\langle Ax, x^* \rangle| \leqslant -C \,\mathrm{Re}\langle Ax, x^* \rangle.$$
Hint: For (1)⇒(2) differentiate the function $\mathrm{Re}\langle S(te^{i\eta'})x, x^* \rangle$ for $|\eta'| < \eta$ and $x^* \in \partial(x)$. For (3)⇒(1) observe that if $\cot \eta = C$, then for x and x^* as indicated and $\lambda = re^{i\eta'}$ with $|\eta'| < \eta$ we have $\|(\lambda - A)x\| \geqslant r\|x\| = |\lambda| \, \|x\|$.

4. Suppose that A is a closed linear operator with $(0, \infty) \subseteq \varrho(A)$ and $\sup_{\lambda > 0} (\lambda + 1)\|R(\lambda, A)\| < \infty$.
a) Show that $\Sigma_\eta \subseteq \varrho(A)$ and $\sup_{\lambda \in \Sigma_\eta} |\lambda + 1| \|R(\lambda, A)\| < \infty$ for some $\eta > 0$.
Define
$$(-A)^{-\alpha} := \frac{1}{2\pi i} \int_\Gamma (-z)^{-\alpha} R(z, A)\, dz,$$
where Γ is the upwards oriented boundary of $\Sigma_\eta \cup B$, where B is a closed ball centred at the origin.
b) Show, by using Cauchy's formula, that for all $x \in E$ we have
$$(-A)^{-\alpha} x = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{-\alpha} R(\lambda, A)x \, d\lambda.$$
c) Now assume that A generates a uniformly exponentially stable C_0-semigroup S and prove that the definition in b) agrees with Definition 10.7.

5. In this exercise we sketch an alternative approach to Theorem 10.19, whose notations and assumptions we use.
Let $U(t) = \int_0^t S(t-s)B \, dW_H(t)$. Being the weak solution of the problem $dU(t) = AU(t)\, dt + B\, dW_H$ with initial value $U(0) = 0$, the process U has a version with integrable trajectories. Using this version we define

$$V(t) := \int_0^t U(s)\, ds.$$

Let W_B be the Brownian motion canonically associated with B and W_H (see Proposition 8.8). Fixing $0 \leqslant \alpha < \frac{1}{2}$, we note that W_B has a version with trajectories in $C^\alpha([0,T];E)$ by Kolmogorov's theorem.

a) Show that almost surely, the following identity holds for all $t \in [0,T]$:

$$V(t) = \int_0^t S(t-s) W_B(s)\, ds.$$

b) Show that almost surely, U has trajectories in $C^\alpha([0,T];E)$.
Hint: Show that almost surely, the trajectories of V belong to $C^1([0,T];E)$ and have derivatives in $C^\alpha([0,T];E)$.

c) Refine this argument to obtain the result of Theorem 10.19.

Notes. The results of Sections 10.1 and 10.2 are standard.

Theorem 10.3 can be found in most textbooks on semigroups (see the Notes of Lecture 7). We have tried to shorten the proof as much as possible. Of course, much more is to be said about the representation of the operators $S(t)$ in terms of the resolvent $R(\lambda, A)$. Indeed, this formula is a special case of the complex inversion formula for the Laplace transform, and suitable generalisations can be given to arbitrary C_0-semigroups. We refer to Arendt, Batty, Hieber, Neubrander [3] for a thorough discussion of this topic. A systematic treatment of analytic semigroups and their applications to parabolic evolution equations is given in the monograph of Lunardi [72]. Exercise 2 is taken from [42].

Fractional powers of unbounded operators are discussed in Arendt, Batty, Hieber, Neubrander [3], Haase [45], Lunardi [72], and Pazy [89]. We followed the presentation of [89]. Our approach is rather *ad hoc* and was designed to keep the technicalities at a minimum. A more systematic approach starts from the Dunford integral along the lines of Exercise 4.

The results of Section 10.3 are taken from [34]. The proof of Theorem 10.19 presented here contains a simplification due to Veraar. Our results generalise the Hilbert space case which is due to Da Prato, Kwapień, Zabczyk [26]. The approach of [26] is based on a factorisation trick which is based on the identity

$$\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_r^t (t-s)^{\alpha-1}(s-r)^{-\alpha}\, ds = 1,$$

valid for $0 < \alpha < 1$ and $t > r \geqslant 0$. This identity allows one to write the solution process as a repeated integral. Hölder regularity is then obtained by applying the stochastic Fubini theorem and exploiting the regularising properties of fractional integrals. This method was extended to Banach spaces by Millet and Smoleński [77]. The idea of Exercise 5 is taken from Da Prato, Kwapień, Zabczyk [26].