12

UMD-spaces

This lecture is devoted to the study of a class of Banach spaces, the so-called UMD-spaces, which share many of the good properties of Hilbert spaces and is sufficiently broad to include L^p-spaces for $1 < p < \infty$.

Experience has shown that the class of UMD-spaces is precisely the 'right' one for pursuing vector-valued stochastic analysis as well as vector-valued harmonic analysis. Indeed, many classical Hilbert space-valued results from both areas can be extended to the UMD-valued case, and often this fact characterises the UMD-property.

The relevant fact for our purposes is that the UMD-spaces are those Banach spaces E in which the Wiener integral of Lecture 6 can be extended from $\mathscr{L}(H,E)$-valued functions to $\mathscr{L}(H,E)$-valued stochastic processes. This is the subject matter of the next lecture. In the present lecture, we define UMD-spaces in terms of L^p-bounds for signed E-valued martingale difference sequences and study some of their elementary properties. At first sight, the definition of the UMD-property depends on the parameter $1 < p < \infty$. It is a deep result of Maurey and Pisier that the UMD-property is independent of $1 < p < \infty$. This theorem, which is proved in detail, enables us to prove that L^p-spaces are UMD-spaces for $1 < p < \infty$.

12.1 UMD$_p$-spaces

We begin with a definition.

Definition 12.1. *Let $(M_n)_{n=1}^N$ be an E-valued martingale. The sequence $(d_n)_{n=1}^N$ defined by $d_n := M_n - M_{n-1}$ (with the understanding that $M_0 = 0$) is called the* martingale difference sequence *associated with $(M_n)_{n=1}^N$.*

We call $(d_n)_{n=1}^N$ an *L^p-martingale difference sequence* if it is the difference sequence of an L^p-martingale.

If $(M_n)_{n=1}^N$ is a martingale with respect to the filtration $(\mathscr{F}_n)_{n=1}^N$, then $(d_n)_{n=1}^N$ is adapted to $(\mathscr{F}_n)_{n=1}^N$ and $\mathbb{E}(d_n|\mathscr{F}_m) = 0$ for $1 \leqslant m < n \leqslant N$.

It is easy to see that these two properties characterise martingale difference sequences.

Proposition 12.2. *Every L^2-martingale difference sequence with values in a Hilbert space H is orthogonal in $L^2(\Omega; H)$.*

Proof. We use the notations introduced above. For $1 \leqslant m < n \leqslant N$, from $\mathbb{E}d_n = \mathbb{E}(\mathbb{E}(d_n|\mathscr{F}_{n-1})) = 0$ we deduce that

$$\mathbb{E}[d_m, d_n] = \mathbb{E}(\mathbb{E}([d_m, d_n]|\mathscr{F}_{n-1})) = \mathbb{E}([d_m, \mathbb{E}(d_n|\mathscr{F}_{n-1})]) = 0.$$

The second identity follows from Proposition 11.6 (3) if d_m is replaced by a random variable in $g \in L^2(\Omega, \mathscr{F}_m) \otimes H$, and the general case follows from this since $L^2(\Omega, \mathscr{F}_m) \otimes H$ is dense in $L^2(\Omega, \mathscr{F}_m; H)$. $\square$

This suggests that in the context of stochastic analysis in Banach spaces, martingale difference sequences provide a substitute for orthogonal sequences. To formalise this idea we note that in the situation of Proposition 12.2, for any choice of signs $\varepsilon_n = \pm 1$ we have

$$\mathbb{E}\Big\| \sum_{n=1}^{N} \varepsilon_n d_n \Big\|^2 = \mathbb{E}\Big\| \sum_{n=1}^{N} d_n \Big\|^2. \tag{12.1}$$

It is this property that is generalised in the next definition. The exponent 2 has no special significance in the context of Banach spaces, and therefore we replace it by an exponent $1 < p < \infty$.

Definition 12.3. *Let $1 < p < \infty$. A Banach space E is said to be a UMD$_p$-space if there exists a constant β such that for all E-valued L^p-martingale difference sequences $(d_n)_{n=1}^N$ we have*

$$\mathbb{E}\Big\| \sum_{n=1}^{N} \varepsilon_n d_n \Big\|^p \leqslant \beta^p \mathbb{E}\Big\| \sum_{n=1}^{N} d_n \Big\|^p.$$

If $(d_n)_{n=1}^N$ is an E-valued martingale difference sequence, then the same is true for $(\varepsilon_n d_n)_{n=1}^N$. This gives the reverse inequality

$$\mathbb{E}\Big\| \sum_{n=1}^{N} d_n \Big\|^p \leqslant \beta^p \mathbb{E}\Big\| \sum_{n=1}^{N} \varepsilon_n d_n \Big\|^p.$$

The term 'UMD' is an abbreviation for 'unconditional martingale differences'. The least possible constant β in the above inequalities is called the *UMD$_p$-constant* of E, notation $\beta_p(E)$.

Every Hilbert space H is a UMD$_2$-space, with $\beta_2(H) = 1$; this is the content of (12.1). It is a trivial consequence of the definition that every closed subspace F of a UMD$_p$-space is a UMD$_p$-space, with $\beta_p(F) \leqslant \beta_p(E)$.

As we shall see in the next section, if a Banach space is UMD$_p$ for some $1 < p < \infty$, then it is UMD$_p$ for all $1 < p < \infty$. In particular, Hilbert spaces are UMD$_p$ for all $1 < p < \infty$. Taking this for granted for the moment, the next result implies that for $1 < p < \infty$ the spaces $L^p(A)$, and more generally $L^p(A;H)$ for Hilbert spaces H, are UMD$_p$-spaces.

Theorem 12.4. *Let $(A, \mathscr{A}, \mu)$ be a σ-finite measure space and let $1 < p < \infty$. If E is a UMD$_p$-space, then $L^p(A;E)$ is a UMD$_p$-space, with $\beta_p(L^p(A;E)) = \beta_p(E)$.*

Proof. Let $(d_n)_{n=1}^N$ be an L^p-martingale difference sequence with values in $L^p(A;E)$. With Fubini's theorem, for all choices of signs $\varepsilon_n = \pm 1$ we obtain

$$\begin{aligned}\mathbb{E}\Big\| \sum_{n=1}^N \varepsilon_n d_n \Big\|_{L^p(A)}^p &= \int_A \mathbb{E}\Big\| \sum_{n=1}^N \varepsilon_n d_n \Big\|^p \,d\mu \\ &\leqslant \beta_p(E)^p \int_A \mathbb{E}\Big\| \sum_{n=1}^N d_n \Big\|^p \,d\mu = \beta_p(E)^p \mathbb{E}\Big\| \sum_{n=1}^N d_n \Big\|_{L^p(A)}^p.\end{aligned}$$

In this computation we used that for μ-almost all $\xi \in A$ the sequences $(d_n(\xi))_{n=1}^N$ is an E-valued martingale difference sequence; this follows from the observation that under the identification $L^p(\Omega; L^p(A;E)) \simeq L^p(A; L^p(\Omega;E))$ we have $\mathbb{E}_{L^p(A;E)}(\cdot|\mathscr{F}_n) = I \otimes \mathbb{E}_{L^p(A)}(\cdot|\mathscr{F}_n)$. This proves that $L^p(A;E)$ is a UMD$_p$-space, with $\beta_p(L^p(A;E)) \leqslant \beta_p(E)$.

If $f \in L^p(A)$ has norm 1, then $x \mapsto f \otimes x$ defines an isometric embedding of E into $L^p(A;E)$; this gives the opposite inequality $\beta_p(E) \leqslant \beta_p(L^p(A;E))$. □

Duality provides another way to produce new UMD-spaces from old:

Proposition 12.5. *Let $1 < p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then E is a UMD$_p$-space if and only if E^* is a UMD$_q$-space, and in this situation we have $\beta_p(E) = \beta_q(E^*)$.*

Proof. Suppose E is a UMD$_p$-space and let $(d_n^*)_{n=1}^N$ be an E^*-valued L^q-martingale difference sequence with respect to $(\mathscr{F}_n)_{n=1}^N$. Fix an arbitrary $Y \in L^p(\Omega, \mathscr{F}_N; E)$ of norm 1, and define the E-valued L^p-martingale $(M_n)_{n=1}^N$ by $M_n := \mathbb{E}(Y|\mathscr{F}_n)$. Let $(d_n)_{n=1}^N$ be its difference sequence. Then $Y = \sum_{m=1}^N d_m$. If $1 \leqslant m < n \leqslant N$, then

$$\mathbb{E}\langle d_m, d_n^* \rangle = \mathbb{E}\mathbb{E}(\langle d_m, d_n^* \rangle|\mathscr{F}_{n-1}) = \mathbb{E}\langle d_m, \mathbb{E}(d_n^*|\mathscr{F}_{n-1})\rangle = 0.$$

The second identity is justified as in the proof of Proposition 12.2. A similar computation shows that $\mathbb{E}\langle d_m, d_n^* \rangle = 0$ if $1 \leqslant n < m \leqslant N$. Hence,

$$\begin{aligned}
\Big|\mathbb{E}\Big\langle Y, \sum_{n=1}^{N} \varepsilon_n d_n^*\Big\rangle\Big| &= \Big|\mathbb{E}\Big\langle \sum_{m=1}^{N} d_m, \sum_{n=1}^{N} \varepsilon_n d_n^*\Big\rangle\Big| \\
&= \Big|\mathbb{E}\Big\langle \sum_{m=1}^{N} \varepsilon_m d_m, \sum_{n=1}^{N} d_n^*\Big\rangle\Big| \\
&\leqslant \beta_p(E)\Big(\mathbb{E}\Big\| \sum_{m=1}^{N} d_m\Big\|^p\Big)^{\frac{1}{p}} \Big(\mathbb{E}\Big\| \sum_{n=1}^{N} d_n^*\Big\|^q\Big)^{\frac{1}{q}} \\
&= \beta_p(E)\Big(\mathbb{E}\Big\| \sum_{n=1}^{N} d_n^*\Big\|^q\Big)^{\frac{1}{q}}.
\end{aligned}$$

In the last identity we used the assumption that $\|Y\|_p = 1$. Since $L^p(\Omega, \mathscr{F}_N; E)$ is norming for $L^q(\Omega, \mathscr{F}_N; E^*)$ (see Exercise 1.5), by taking the supremum over all Y of norm 1 we obtain the estimate

$$\Big(\mathbb{E}\Big\| \sum_{n=1}^{N} \varepsilon_n d_n^*\Big\|^q\Big)^{\frac{1}{q}} \leqslant \beta_p(E)\Big(\mathbb{E}\Big\| \sum_{n=1}^{N} d_n^*\Big\|^q\Big)^{\frac{1}{q}}.$$

This proves that E^* is a UMD$_q$-space, with $\beta_q(E^*) \leqslant \beta_p(E)$.

If E^* is a UMD$_q$-space, the result just proved implies that E^{**} is a UMD$_p$-space, with $\beta_p(E^{**}) \leqslant \beta_q(E^*)$. Hence also E, being isometrically contained as a closed subspace in E^{**} (by the Hahn-Banach theorem each $x \in E$ defines a functional ϕ_x in E^{**} of norm $\|\phi_x\| = \|x\|$ by the formula $\langle x^*, \phi_x\rangle := \langle x, x^*\rangle$), is a UMD$_p$-space, with $\beta_p(E) \leqslant \beta_p(E^{**}) \leqslant \beta_q(E^*)$.

Combining both parts, we obtain the equality $\beta_p(E) = \beta_q(E^*)$. □

Remark 12.6. It can be shown that every UMD$_p$-space E is *reflexive*, that is, the canonical mapping $x \mapsto \phi_x$ from E to E^{**} is surjective. This fact will not be needed in what follows.

12.2 p-Independence of the UMD$_p$-property

This section is devoted to the proof of the highly non-trivial fact, already mentioned above, that the UMD$_p$-property is independent of the parameter $1 < p < \infty$. The work consists of two parts: a reduction of the problem to difference sequences of so-called Haar martingales, and then proving the p-independence for this class of martingales.

12.2.1 Reduction to Haar martingales

A probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is said to be *divisible* if for all $F \in \mathscr{F}$ and $0 < r < 1$ we have $F = F_1 \cup F_2$ with $F_1, F_2 \in \mathscr{F}$ and

$$\mathbb{P}(F_1) = r\mathbb{P}(F), \quad \mathbb{P}(F_2) = (1 - r)\mathbb{P}(F).$$

For $1 < p < \infty$, let us say that E has the *UMD$_p^{\rm div}$-property* if there exists a constant $\beta_p^{\rm div}(E)$ such that

$$\mathbb{E}\Big\|\sum_{n=1}^N \varepsilon_n d_n\Big\|^p \leqslant (\beta_p^{\rm div}(E))^p \mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p$$

for all E-valued L^p-martingale difference sequences $(d_n)_{n=1}^N$ defined on a divisible probability space. Trivially, if E has the UMD$_p$-property, then it has the UMD$_p^{\rm div}$-property and $\beta_p^{\rm div}(E) \leqslant \beta_p(E)$. The next lemma establishes the converse.

Lemma 12.7. *Let $1 < p < \infty$. If E has the UMD$_p^{\rm div}$-property, then it has the UMD$_p$-property and $\beta_p(E) = \beta_p^{\rm div}(E)$.*

Proof. Suppose that $(d_n)_{n=1}^N$ is an E-valued L^p-martingale difference sequence with respect to a filtration $(\mathscr{F}_n)_{n=1}^N$ on an arbitrary probability space $(\Omega, \mathscr{F}, \mathbb{P})$. The idea is to enlarge the probability space in such a way that it becomes divisible, without affecting the L^p-estimates for the martingale differences.

Consider $\widetilde{\Omega} := \Omega \times [0,1]$, $\widetilde{\mathscr{F}} := \mathscr{F} \times \mathscr{B}([0,1])$, and $\widetilde{P} := \mathbb{P} \times m$, where m is the Lebesgue measure on the Borel σ-algebra $\mathscr{B}([0,1])$. The probability space $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}})$ is divisible: this follows from the intermediate value theorem applied to the continuous function $t \mapsto \widetilde{\mathbb{P}}(\widetilde{F} \cap (\Omega \times [0,t]))$, where $\widetilde{F} \in \widetilde{\mathscr{F}}$.

Let $(M_n)_{n=1}^N$ be the martingale associated with $(d_n)_{n=1}^N$. Define $\widetilde{M}_n(\omega,t) := M_n(\omega)$ and $\widetilde{\mathscr{F}}_n := \mathscr{F}_n \times \mathscr{B}([0,1])$. It is easily checked that $(\widetilde{M}_n)_{n=1}^N$ is a martingale with respect to $(\widetilde{\mathscr{F}}_n)_{n=1}^N$ and, for every sequence of signs $(\varepsilon_n)_{n=1}^N$, its difference sequence $(\widetilde{d}_n)_{n=1}^N$ satisfies

$$\begin{aligned}\mathbb{E}\Big\|\sum_{n=1}^N \varepsilon_n d_n\Big\|^p &= \widetilde{\mathbb{E}}\Big\|\sum_{n=1}^N \varepsilon_n \widetilde{d}_n\Big\|^p \\ &\leqslant (\beta_p^{\rm div}(E))^p \widetilde{\mathbb{E}}\Big\|\sum_{n=1}^N \widetilde{d}_n\Big\|^p = (\beta_p^{\rm div}(E))^p \mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p\end{aligned}$$

using the UMD$_p^{\rm div}$-property of E. □

In the next step we restrict the class of probability spaces even further. If $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, we call a sub-σ-algebra $\mathscr{G}$ of $\mathscr{F}$ *dyadic* if it is generated by 2^m sets of measure 2^{-m} for some integer $m \geqslant 0$. We call a filtration in $(\Omega, \mathscr{F}, \mathbb{P})$ *dyadic* if each of its constituting σ-algebras is dyadic. For $1 < p < \infty$, let us say that E has the *UMD$_p^{\rm dyad}$-property* if there exists a constant $\beta_p^{\rm dyad}(E)$ such that

$$\mathbb{E}\Big\|\sum_{n=1}^N \varepsilon_n d_n\Big\|^p \leqslant (\beta_p^{\rm dyad}(E))^p \mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p$$

holds for all E-valued L^p-martingale difference sequences $(d_n)_{n=1}^N$ with respect to a dyadic filtration $(\mathscr{F}_n)_{n=1}^N$ on a divisible probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Trivially, if E has the UMD$_p$-property, then it has the UMD$_p^{\text{dyad}}$-property and $\beta_p^{\text{dyad}}(E) \leqslant \beta_p(E)$. In order to establish the converse we need a simple approximation result. The proof appears somewhat technical, but by drawing a picture one sees that it is nearly trivial.

Lemma 12.8. *Let $1 \leqslant p < \infty$ and $\varepsilon > 0$ be given. If f is a simple random variable on a divisible probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and $\mathscr{G}$ is a dyadic sub-σ-algebra of $\mathscr{F}$, there exists a dyadic sub-σ-algebra $\mathscr{G} \subseteq \mathscr{H} \subseteq \mathscr{F}$ and an $\mathscr{H}$-measurable simple random variable h such that $\|f - h\|_p < \varepsilon$.*

Proof. Suppose $\mathscr{G}$ is generated by 2^m sets of measure 2^{-m}.

It suffices to prove the lemma for indicator functions $f = 1_F$ with $F \in \mathscr{F}$. Considering $F \in \mathscr{F}$ as fixed, we write $1_F = \sum_G 1_{F \cap G}$ where the sum extends over the 2^m generating sets G of $\mathscr{G}$.

Take one such G and let $(b_j^G)_{j=1}^\infty$ denote the digits in the binary expansion of the real number $\mathbb{P}(F \cap G)$. Informally, we use the digits to write $F \cap G$ inductively as a union, up to a null set, of disjoint 'dyadic' subsets of maximal measure.

To be more precise, inductively define sets A_j^G and B_j^G by $A_0^G = F \cap G$ and $B_0^G = \varnothing$, and requiring, for $j \geqslant 1$, that $B_j^G \subseteq A_{j-1}^G$ satisfies $B_j^G \in \mathscr{F}$ and $\mathbb{P}(B_j^G) = b_j^G 2^{-j}$ (we may take $B_j^G := \varnothing$ if $b_j^G = 0$). Then put $A_j^G := A_{j-1}^G \setminus B_j^G$ and continue.

The sets $B_j^G \in \mathscr{F}$ thus constructed are disjoint, contained in G, and satisfy $\mathbb{P}\big((F \cap G) \setminus \bigcup_{j=1}^\infty B_j^G\big) = 0$. Let $n \geqslant 1$ be the first integer such that

$$\mathbb{P}\Big((F \cap G) \setminus \bigcup_{j=1}^n B_j^G\Big) < \frac{\varepsilon^p}{2^m}.$$

For each $1 \leqslant j \leqslant n$ such that $b_j^G = 1$ we have $\mathbb{P}(B_j^G) = 2^{-j}$. If follows that we can split G into disjoint subsets of measure 2^{-n} in such a way that each B_j^G, $1 \leqslant j \leqslant n$, is a finite union of these subsets.

We subdivide each of the 2^m generating sets G in this way. The number n varies over G, but by considering further subdivisions we may assume it to be independent of G. Let $\mathscr{H}$ be the σ-algebra generated by the 2^n sets of measure 2^{-n} thus obtained. This σ-algebra is dyadic, it contains $\mathscr{G}$, and the simple function

$$h := \sum_G \sum_{\substack{1 \leqslant j \leqslant n \\ b_j^G = 1}} 1_{B_j^G}$$

is $\mathscr{H}$-measurable and satisfies $\|f - h\|_p < \varepsilon$. $\square$

Lemma 12.9. *Let $1 < p < \infty$. If E has the UMD$_p^{\text{dyad}}$-property, then it has the UMD$_p$-property and $\beta_p(E) = \beta_p^{\text{dyad}}(E)$.*

Proof. Suppose that $(d_n)_{n=1}^N$ is an E-valued L^p-martingale difference sequence with respect to a filtration $(\mathscr{F}_n)_{n=1}^N$ on a divisible probability space $(\Omega, \mathscr{F}, \mathbb{P})$. The idea is approximate the d_n with simple functions as in the previous lemma.

Fixing $\varepsilon > 0$, we can find $\mathscr{F}_n$-measurable simple functions $s_n : \Omega \to E$ such that $\|d_n - s_n\|_p < \frac{\varepsilon}{N}$. By repeated application of Lemma 12.8 we find a sequence of dyadic σ-algebras $(\widetilde{\mathscr{F}_n})_{n=1}^N$ such that $\widetilde{\mathscr{F}}_{n-1} \subseteq \widetilde{\mathscr{F}_n} \subseteq \mathscr{F}_n$ (with the understanding that $\widetilde{\mathscr{F}_0} = \{\varnothing, \Omega\}$) and a sequence of step functions $(\widetilde{s}_n)_{n=1}^N$ such that each $\widetilde{s}_n$ is $\widetilde{\mathscr{F}_n}$-measurable and satisfies $\|s_n - \widetilde{s}_n\|_p < \frac{\varepsilon}{N}$.

Consider the sequence $(\widetilde{d}_n)_{n=1}^N$ defined by $\widetilde{d}_n := \mathbb{E}(d_n|\widetilde{\mathscr{F}_n})$. To see that this is a martingale difference sequence with respect to the filtration $(\widetilde{\mathscr{F}_n})_{n=1}^N$, note that for $1 < n \leqslant N$,

$$\begin{aligned}\mathbb{E}(\widetilde{d}_n|\widetilde{\mathscr{F}}_{n-1}) &= \mathbb{E}(\mathbb{E}(d_n|\widetilde{\mathscr{F}_n})|\widetilde{\mathscr{F}}_{n-1})\\ &= \mathbb{E}(d_n|\widetilde{\mathscr{F}}_{n-1}) = \mathbb{E}(\mathbb{E}(d_n|\mathscr{F}_{n-1})|\widetilde{\mathscr{F}}_{n-1}) = 0.\end{aligned}$$

Then, by the L^p-contractivity of conditional expectations,

$$\begin{aligned}\|d_n - \widetilde{d}_n\|_p &\leqslant \frac{2\varepsilon}{N} + \|\widetilde{s}_n - \widetilde{d}_n\|_p\\ &= \frac{2\varepsilon}{N} + \|\mathbb{E}(\widetilde{s}_n - \widetilde{d}_n|\widetilde{\mathscr{F}_n})\|_p\\ &= \frac{2\varepsilon}{N} + \|\mathbb{E}(\widetilde{s}_n - d_n|\widetilde{\mathscr{F}_n})\|_p \leqslant \frac{2\varepsilon}{N} + \|\widetilde{s}_n - d_n\|_p = \frac{4\varepsilon}{N}.\end{aligned}$$

Hence,

$$\begin{aligned}\Big\|\sum_{n=1}^N \varepsilon_n d_n\Big\|_p &\leqslant 4\varepsilon + \Big\|\sum_{n=1}^N \varepsilon_n \widetilde{d}_n\Big\|_p\\ &\leqslant 4\varepsilon + \beta_p^{\mathrm{dyad}}(E)\Big\|\sum_{n=1}^N \widetilde{d}_n\Big\|_p \leqslant 4\varepsilon(1 + \beta_p^{\mathrm{dyad}}(E))\Big\|\sum_{n=1}^N d_n\Big\|_p.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this shows that E has the UMD$_p^{\mathrm{div}}$-property with $\beta_p^{\mathrm{div}}(E) \leqslant \beta_p^{\mathrm{dyad}}(E)$. Together with Lemma 12.7 this proves the result. □

The final reduction consists of shrinking the class of difference sequences to *Haar martingale difference sequences*, which are defined as difference sequences of martingales with respect to a *Haar filtration*. This is a filtration $(\mathscr{F}_n)_{n=1}^N$, where $\mathscr{F}_1 = \{\varnothing, \Omega\}$ and each $\mathscr{F}_n$ (with $n \geqslant 1$) is obtained from $\mathscr{F}_{n-1}$ by dividing precisely one atom of $\mathscr{F}_{n-1}$ of maximal measure into two sets of equal measure (an *atom* of a σ-algebra $\mathscr{G}$ is a set $G \in \mathscr{G}$ such that $H \subseteq G$ with $H \in \mathscr{G}$ implies $H \in \{\varnothing, G\}$). By construction, each $\mathscr{F}_n$ is generated by n atoms, whose measures equal 2^{-k-1} or 2^{-k}, where k is the unique integer such that $2^{k-1} < n \leqslant 2^k$.

For $1 < p < \infty$, let us say that E has the *UMD*$_p^{\mathrm{Haar}}$*-property* if there exists a constant $\beta_p^{\mathrm{Haar}}(E)$ such that

$$\mathbb{E}\Big\|\sum_{n=1}^{N}\varepsilon_n d_n\Big\|^p \leqslant (\beta_p^{\mathrm{Haar}}(E))^p\mathbb{E}\Big\|\sum_{n=1}^{N} d_n\Big\|^p$$

holds for all E-valued Haar martingale difference sequences $(d_n)_{n=1}^N$ defined on a divisible probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Trivially, if E has the UMD$_p$-property, then it has the UMD$_p^{\mathrm{Haar}}$-property and $\beta_p^{\mathrm{Haar}}(E) \leqslant \beta_p(E)$.

Lemma 12.10. *Let* $1 < p < \infty$. *If* E *has the UMD*$_p^{\mathrm{Haar}}$*-property, then it has the UMD*$_p$*-property and* $\beta_p^{\mathrm{Haar}}(E) = \beta_p(E)$.

Proof. Suppose that $(d_n)_{n=1}^N$ is an E-valued L^p-martingale difference sequence with respect to a dyadic filtration $(\mathscr{F}_n)_{n=1}^N$ on a divisible probability space $(\Omega, \mathscr{F}, \mathbb{P})$. The idea is to 'embed' $(d_n)_{n=1}^N$ into a Haar martingale difference sequence. To be more precise, we shall construct an L^p-martingale difference sequence $(\widetilde{d}_k)_{k=1}^K$ with respect to a Haar filtration $(\widetilde{\mathscr{F}}_k)_{k=1}^K$ such that $M_n = \widetilde{M}_{k_n}$ and $\mathscr{F}_n = \widetilde{\mathscr{F}}_{k_n}$ for some subsequence $k_1 < \cdots < k_N$. Once this has been done, we note that $d_n = \sum_{j=k_{n-1}+1}^{k_n} \widetilde{d}_j$ and

$$\begin{aligned}\Big\|\sum_{n=1}^{N}\varepsilon_n d_n\Big\|_p &= \Big\|\sum_{n=1}^{N}\sum_{j=k_{n-1}+1}^{k_n}\varepsilon_n \widetilde{d}_j\Big\|_p = \Big\|\sum_{k=1}^{K}\widetilde{\varepsilon}_k\widetilde{d}_k\Big\|_p \\ &\leqslant \beta_p^{\mathrm{Haar}}(E)\Big\|\sum_{k=1}^{K}\widetilde{d}_k\Big\|_p = \beta_p^{\mathrm{Haar}}(E)\Big\|\sum_{n=1}^{N} d_n\Big\|_p,\end{aligned}$$

where $\widetilde{\varepsilon}_k = \varepsilon_{k_n}$ for $k = k_{n-1}+1, \dots, k_n$.

Each $\mathscr{F}_n$ is dyadic and therefore it is generated by $k_n := 2^{l_n}$ atoms of measure 2^{-l_n}. Since each atom of $\mathscr{F}_{n-1}$ is a finite union of atoms in $\mathscr{F}_n$ we have $k_1 < \cdots < k_N$. The σ-algebras $\widetilde{\mathscr{F}}_k$, with $k_{n-1} < k < k_n$ can now be constructed by splitting the atoms of $\widetilde{\mathscr{F}}_{k_{n-1}}$ one by one into two disjoint subsets of equal measure, so as to arrive at the atoms of $\widetilde{\mathscr{F}}_{k_n}$ by repeating this procedure $k_n - k_{n-1}$ times.

Now take $\widetilde{M}_{k_n} := M_n$ and $\widetilde{M}_k := E(\widetilde{M}_{k_n}|\widetilde{\mathscr{F}}_k)$ if $k_{n-1} < k \leqslant k_n$. □

12.2.2 *p*-Independence for Haar martingales

By the reductions of the previous subsection, in order to prove the p-independence of the UMD$_p$-property it suffices to consider Haar martingale difference sequences. Such sequences have a special property which is captured in the next lemma.

Lemma 12.11. *If* $(d_n)_{n=1}^N$ *is an* E*-valued Haar martingale difference sequence, then* $\|d_{n+1}\|$ *is* $\mathscr{F}_n$*-measurable for all* $n = 1, \dots, N-1$.

Proof. Suppose that $\mathscr{F}_{n+1}$ is obtained by splitting one of the $n+1$ generating atoms of $\mathscr{F}_n$, say A, into subsets A_1 and A_2 of equal measure. Then M_{n+1} and M_n only differ on A, so $d_{n+1} = 0$ on $\complement A$. Also, d_{n+1} is constant on A_1 and A_2, say with values x_1 and x_2. Then,

$$\mathbb{P}(A_1)x_1 + \mathbb{P}(A_2)x_2 = \int_A d_{n+1}\, d\mathbb{P} = \int_A \mathbb{E}(d_{n+1}|\mathscr{F}_n)\, d\mathbb{P} = 0,$$

and from $\mathbb{P}(A_1) = \mathbb{P}(A_2)$ we deduce that $x_1 + x_2 = 0$. Hence, $\|d_{n+1}\| = 1_{A_1}\|x_1\| + 1_{A_2}\|x_2\| = 1_A\|x_1\|$ is $\mathscr{F}_n$-measurable. □

In what follows we let $f = (f_n)_{n=1}^N$ be an E-valued Haar martingale with difference sequence $(d_n)_{n=1}^N$. By the lemma, the non-negative random variables $\|d_{n+1}\|$ are $\mathscr{F}_n$-measurable for $n = 1, \ldots, N-1$.

For a fixed sequence of signs $\varepsilon = (\varepsilon_n)_{n=1}^N$ we denote by $g = (g_n)_{n=1}^N$ the martingale transform $g_n = \sum_{j=1}^n \varepsilon_j d_j$. Further we let

$$f^*(\omega) := \max_{1\leqslant n\leqslant N} \|f_n(\omega)\|, \quad g^*(\omega) := \max_{1\leqslant n\leqslant N} \|g_n(\omega)\|.$$

In the proof of the next lemma we use the following notation: if $(X_n)_{n=1}^N$ is a sequence of E-valued random variables and $\tau : \Omega \to \{1, \ldots, N\}$ is another random variable, we define the random variable $X_\tau : \Omega \to E$ by

$$X_\tau(\omega) := X_{\tau(\omega)}(\omega).$$

Lemma 12.12. *Suppose that E is a UMD$_q$-space for some $1 < q < \infty$. For all $\delta > 0$ and $\beta > 2\delta + 1$ and all $\lambda > 0$ we have*

$$\mathbb{P}\{g^* > \beta\lambda,\ f^* \leqslant \delta\lambda\} \leqslant \alpha^q \mathbb{P}\{g^* > \lambda\},$$

where $\alpha = 4\delta\beta_q(E)/(\beta - 2\delta - 1)$.

Proof. Since $\mathscr{F}_1 = \{\varnothing, \Omega\}$, the random variable $f_1 = d_1$ is constant almost surely. If the constant value is greater than $\delta\lambda$, then the left hand side in the above inequality vanishes and there is nothing to prove. We may therefore assume that $f_1 \leqslant \delta\lambda$ almost surely.

Let

$$\begin{aligned}
\mu(\omega) &:= \min\{1 \leqslant n \leqslant N :\ \|g_n(\omega)\| > \lambda\},\\
\nu(\omega) &:= \min\{1 \leqslant n \leqslant N :\ \|g_n(\omega)\| > \beta\lambda\},\\
\sigma(\omega) &:= \min\{1 \leqslant n \leqslant N :\ \|f_n(\omega)\| > \delta\lambda \ \text{ or } \ \|d_{n+1}\| > 2\delta\lambda\}
\end{aligned}$$

with the convention that $\min\varnothing := N+1$. In the third definition we further use the convention that $d_{N+1} := 0$.

Let v_n be the indicator function of the set $\{\mu < n \leqslant \min\{\nu, \sigma\}\}$. Since $d = (d_n)_{n=1}^N$ is a Haar martingale difference sequence, the sequence $v = (v_n)_{n=1}^N$ is predictable by Lemma 12.11 and therefore

$$F_n := \sum_{j=1}^{n} v_j d_j$$

defines a martingale $F = (F_n)_{n=1}^N$ by the result of Example 11.19. On the set $\{\sigma \leqslant \mu\}$ we have $v_j \equiv 0$ for all j and therefore $F_N \equiv 0$ there. In particular this is the case on the set $\{\mu = N+1\} = \{g^* \leqslant \lambda\}$. On the set $\{\sigma > \mu\}$ we have

$$\|F_N\| = \Big\| \sum_{\mu < j \leqslant \min\{\nu,\sigma\}} d_j \Big\| = \|f_{\min\{\nu,\sigma\}} - f_\mu\| \leqslant 4\delta\lambda.$$

To see this, first note that $\mu(\omega) < \sigma(\omega)$ implies $\|f_\mu(\omega)\| \leqslant \delta\lambda$. Also, if $\min\{\nu(\omega), \sigma(\omega)\} = 1$, then by the assumption above $\|f_{\min\{\nu,\sigma\}}(\omega)\| = \|f_1(\omega)\| \leqslant \delta\lambda$; if $\min\{\nu(\omega), \sigma(\omega)\} > 1$, then from $\|f_{\min\{\nu,\sigma\}-1}(\omega)\| \leqslant \delta\lambda$ and $\|d_{\min\{\nu,\sigma\}}(\omega)\| \leqslant 2\delta\lambda$ it follows that $\|f_{\min\{\nu,\sigma\}}(\omega)\| \leqslant \|f_{\min\{\nu,\sigma\}-1}(\omega)\| + \|d_{\min\{\nu,\sigma\}}(\omega)\| \leqslant 3\delta\lambda$. This proves the claim.

We infer that

$$\mathbb{E}\|F_n\|^q \leqslant (4\delta\lambda)^q \mathbb{P}\{g^* > \lambda\}.$$

Now consider the martingale transform G of F by ε,

$$G_n := \sum_{j=1}^{n} \varepsilon_j v_j d_j.$$

On the set $\{\nu \leqslant N,\ \sigma = N+1\}$ we have $\min\{\nu, \sigma\} = \nu$ and

$$\|G_N\| = \Big\| \sum_{\mu < j \leqslant \nu} \varepsilon_j d_j \Big\| = \|g_\nu - g_\mu\| > \beta\lambda - 2\delta\lambda - \lambda,$$

where the last inequality uses that on the set $\{\nu \leqslant N,\ \sigma = N+1\}$ we have $\|g_\nu(\omega)\| > \beta\lambda$ and $\|g_\mu(\omega)\| \leqslant \|g_{\mu-1}(\omega)\| + \|d_\mu(\omega)\| \leqslant \lambda + 2\delta\lambda$.

By Chebyshev's inequality and the UMD$_q$-property,

$$\begin{aligned}
\mathbb{P}\{g^* > \beta\lambda,\ f^* \leqslant \delta\lambda\} &\leqslant \mathbb{P}\{\nu \leqslant N,\ \sigma = N+1\} \\
&\leqslant \mathbb{P}\{\|G_N\| > \beta\lambda - 2\delta\lambda - \lambda\} \\
&\leqslant \frac{1}{(\beta\lambda - 2\delta\lambda - \lambda)^q} \mathbb{E}\|G_N\|^q \\
&\leqslant \frac{(\beta_q(E))^q}{(\beta\lambda - 2\delta\lambda - \lambda)^q} \mathbb{E}\|F_N\|^q \\
&\leqslant \frac{(4\delta)^q (\beta_q(E))^q}{(\beta - 2\delta - 1)^q} \mathbb{P}\{g^* > \lambda\}.
\end{aligned}$$

In the first inequality we used that $f^*(\omega) \leqslant \delta\lambda$ implies that $\|d_j(\omega)\| \leqslant 2\delta\lambda$ for all j. This proves the lemma. $\square$

Theorem 12.13. *If E is a UMD$_q$-space for some $1 < q < \infty$, then it is a UMD$_p$-space for all $1 < p < \infty$.*

Proof. By the results of the previous subsection it suffices to show that E has the $\mathrm{UMD}_p^{\mathrm{Haar}}$-property for all $1 < p < \infty$. Thus we find ourselves in the situation of the previous lemma and need to prove the estimate

$$\mathbb{E}\|g_N\|^p \leqslant b^p \mathbb{E}\|f_N\|^p$$

with a constant $b \geqslant 0$ depending only on p, q, and E, but not on f, g and N.

Fix an arbitrary number $\beta > 1$. For $\delta > 0$ so small that $\beta > 2\delta + 1$, let $\alpha = \alpha_{\beta,\delta,q,E}$ be as in the lemma. Then, by an integration by parts and Doob's maximal inequality,

$$\begin{aligned}
\mathbb{E}\|g_N\|^p \leqslant \mathbb{E}\|g^*\|^p &= \beta^p \int_0^\infty p\lambda^{p-1}\mathbb{P}\{g^* > \beta\lambda\}\, d\lambda \\
&\leqslant \alpha^q \beta^p \int_0^\infty p\lambda^{p-1}\mathbb{P}\{g^* > \lambda\}\, d\lambda \\
&\qquad + \beta^p \int_0^\infty p\lambda^{p-1}\mathbb{P}\{f^* > \delta\lambda\}\, d\lambda \\
&\leqslant \alpha^q \beta^p \mathbb{E}\|g^*\|^p + \frac{\beta^p}{\delta^p}\mathbb{E}\|f^*\|^p \\
&\leqslant C_p^p \alpha^q \beta^p \mathbb{E}\|g_N\|^p + \frac{C_p^p \beta^p}{\delta^p}\mathbb{E}\|f_N\|^p,
\end{aligned}$$

where $C_p = p/(p-1)$. Since $\lim_{\delta\downarrow 0} \alpha_{\beta,\delta,q,E} = 0$, by taking $\delta > 0$ small enough we may arrange that $C_p^p \alpha^q \beta^p < 1$. Noting that $\mathbb{E}\|g_N\|^p < \infty$ since g_N is simple (recall that $\mathscr{F}_N$ is a finite σ-algebra) it follows that

$$\mathbb{E}\|g_N\|^p \leqslant \frac{C_p^p \beta^p}{(1 - C_p^p \alpha^q \beta^p)\delta^p}\mathbb{E}\|f_N\|^p.$$

This concludes the proof. ☐

This theorem justifies the following definition.

Definition 12.14. *A Banach space is called a* UMD-space *if it is a* UMD_p*-space for some (and hence, for all)* $1 < p < \infty$.

By combining Theorem 12.13 with the results of the previous section we see that all Hilbert spaces and all spaces $L^p(A)$ with $1 < p < \infty$ are UMD-spaces.

12.3 The vector-valued Stein inequality

In this final section we prove an extension, due to Bourgain, of a beautiful result of Stein which asserts that conditional expectation operators corresponding to the σ-algebras of a filtration form an R-bounded family.

Theorem 12.15 (Vector-valued Stein inequality). *Let E be a UMD-space and fix $1 < p < \infty$. If $(\mathscr{F}_t)_{t\in[0,T]}$ is a filtration on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, then the family of conditional expectation operators $\{\mathbb{E}(\cdot|\mathscr{F}_t) : t \in [0,T]\}$ is R-bounded (and hence γ-bounded) on $L^p(\Omega; E)$.*

Proof. Let $(\widetilde{r}_n)_{n=1}^N$ be a Rademacher sequence on a second probability space $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}})$ and define $\widetilde{\mathscr{F}}_n = \sigma(\widetilde{r}_1, \dots, \widetilde{r}_n)$, $n = 1, \dots, N$. Fix $t_1 < \dots < t_N$ in $[0,T]$. On the product space $\Omega \times \widetilde{\Omega}$ define the filtration $(\mathscr{G}_m)_{m=1}^{2N}$ by

$$\begin{aligned} \mathscr{G}_{2n-1} &:= \mathscr{F}_{t_n} \times \widetilde{\mathscr{F}}_{n-1}, \quad n = 1, \dots, N, \\ \mathscr{G}_{2n} &:= \mathscr{F}_{t_n} \times \widetilde{\mathscr{F}}_n, \qquad n = 1, \dots, N. \end{aligned}$$

For a random variable $X \in L^p(\Omega \times \widetilde{\Omega}; E)$ define the martingale $(M_m)_{m=1}^{2M}$ by

$$M_m := \mathbb{E}(X|\mathscr{G}_m), \quad m = 1, \dots, 2N.$$

Let $(d_m)_{m=1}^{2M}$ be the associated martingale difference sequence. Then by the UMD$_p$-property of E,

$$\Big\| \sum_{n=1}^N d_{2n} \Big\|_{L^p(\Omega;E)} \leqslant \beta_p(E) \Big\| \sum_{m=1}^{2N} d_m \Big\|_{L^p(\Omega;E)}. \tag{12.2}$$

Indeed, the sum on the left hand side equals $\frac{1}{2}\big(\sum_{m=1}^{2N} d_m + \sum_{m=1}^{2N} (-1)^m d_m\big)$.

Now fix $f_1, \dots, f_N \in L^p(\Omega; E)$ and put $X := \sum_{n=1}^N \widetilde{r}_n f_n$. For this choice of X we have

$$\begin{aligned} M_{2n-1} &= \sum_{j=1}^N \mathbb{E}(\widetilde{r}_j f_j | \mathscr{F}_{t_n} \times \widetilde{\mathscr{F}}_{n-1}) = \sum_{j=1}^{n-1} \widetilde{r}_j \mathbb{E}(f_j|\mathscr{F}_{t_n}), \\ M_{2n} &= \sum_{j=1}^N \mathbb{E}(\widetilde{r}_j f_j | \mathscr{F}_{t_n} \times \widetilde{\mathscr{F}}_n) = \sum_{j=1}^{n} \widetilde{r}_j \mathbb{E}(f_j|\mathscr{F}_{t_n}). \end{aligned}$$

Therefore $d_{2n-1} = 0$ and $d_{2n} = \widetilde{r}_n \mathbb{E}(f_j|\mathscr{F}_{t_n})$. It then follows from (12.2) that

$$\begin{aligned} \widetilde{\mathbb{E}} \Big\| \sum_{n=1}^N \widetilde{r}_n \mathbb{E}(f_n|\mathscr{F}_{t_n}) \Big\|_{L^p(\Omega;E)}^p &= \widetilde{\mathbb{E}} \Big\| \sum_{n=1}^N d_{2n} \Big\|_{L^p(\Omega;E)}^p \\ &\leqslant (\beta_p(E))^p \widetilde{\mathbb{E}} \Big\| \sum_{m=1}^{2N} d_m \Big\|_{L^p(\Omega;E)}^p \\ &= (\beta_p(E))^p \widetilde{\mathbb{E}} \Big\| \sum_{n=1}^N \widetilde{r}_n f_n \Big\|_{L^p(\Omega;E)}^p. \end{aligned}$$

$\square$

12.4 Exercises

1. Prove that a Banach space E is a UMD$_p$-space E if and only if for some (and hence, for all) $1 < p < \infty$ there exist constants $\beta_p^{\pm}(E)$ such that for all E-valued L^p-martingale difference sequences $(d_n)_{n=1}^N$ and all Rademacher sequences $(\widetilde{r}_n)_{n=1}^N$ independent of $(d_n)_{n=1}^N$ we have

$$\frac{1}{(\beta_p^-(E))^p}\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p \leqslant \mathbb{E}\Big\|\sum_{n=1}^N \widetilde{r}_n d_n\Big\|^p \leqslant (\beta_p^+(E))^p\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p.$$

2. Let $1 < p < \infty$. Prove that if H is a Hilbert space and $(d_n)_{n=1}^N$ is an H-valued L^p-martingale difference sequence, then

$$\frac{1}{c_p^p}\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p \leqslant \mathbb{E}\Big(\sum_{n=1}^N \|d_n\|^2\Big)^{\frac{p}{2}} \leqslant C_p^p\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p,$$

with constant depending only on p.
Hint: Combine Exercise 1 with the Kahane-Khintchine inequalities.

3. Prove that if X is a UMD-space and Y is a closed subspace, then X/Y is a UMD-space and give an estimate for its UMD constant.

4. A sequence $(x_n)_{n=1}^\infty$ in a Banach space E is called a *Schauder basis* if every $x \in E$ admits a unique representation $x = \sum_{n=1}^\infty a_n x_n$ with convergence in E. Using a closed graph argument one can show that the projections

$$D_N \sum_{n=1}^\infty a_n x_n := \sum_{n=1}^N a_n x_n,$$

are bounded. In fact, by the uniform boundedness theorem we even have $\sup_{N\geqslant 1}\|D_N\| < \infty$.
A Schauder basis is called *unconditional* if there exists a constant $0 < C < \infty$ such that for all $N \geqslant 1$, all scalars $a_1, \dots, a_N$, and all signs $\varepsilon_1, \dots, \varepsilon_N \in \{-1, +1\}$ we have

$$\frac{1}{C}\Big\|\sum_{n=1}^N a_n x_n\Big\| \leqslant \Big\|\sum_{n=1}^N \varepsilon_n a_n x_n\Big\| \leqslant C\Big\|\sum_{n=1}^N a_n x_n\Big\|.$$

The least admissible constant C is called the *unconditionality constant* of $(x_n)_{n=1}^\infty$.
Let $(x_n)_{n=1}^\infty$ be an unconditional Schauder basis of E with unconditionality constant C.
   a) Show that if $(r_n)_{n=1}^\infty$ is a Rademacher sequence, then for all $N \geqslant 1$ and all scalars $a_1, \dots, a_N$ we have

$$\frac{1}{C^2}\Big\|\sum_{n=1}^N a_n x_n\Big\|^2 \leqslant \mathbb{E}\Big\|\sum_{n=1}^N r_n a_n x_n\Big\|^2 \leqslant C^2\Big\|\sum_{n=1}^N a_n x_n\Big\|^2.$$

b) Show that $\sup_{N\geqslant 1}\|D_N\| \leqslant C$.

Assume next that E is a UMD-space.

c) Show that the sequence $(D_N)_{N=1}^\infty$ is R-bounded.
Hint: Use a) and the vector-valued Stein inequality.

5. In this exercise we prove a vector-valued version of a multiplier theorem due to MARCINKIEWICZ. Let $(x_n)_{n=1}^\infty$ be a Schauder basis of the UMD Banach space E which has an *unconditional blocking*, meaning that there is a sequence $0 = N_0 < N_1 < \dots$ and a constant $0 < C < \infty$ such that the corresponding block projections $\Delta_j := D_{N_j} - D_{N_{j-1}}$ (where $D_0 = 0$) satisfy
$$\frac{1}{C}\Big\|\sum_{j=1}^k \Delta_j x\Big\| \leqslant \Big\|\sum_{j=1}^k \varepsilon_j \Delta_j x\Big\| \leqslant C\Big\|\sum_{j=1}^k \Delta_j x\Big\|$$
for all choices $\varepsilon_n \in \{-1,1\}$. Suppose that $(\lambda_n)_{n=1}^N$ is a scalar sequence such that:
(i) $\sup\limits_{n\geqslant 1}|\lambda_n| < \infty$;
(ii) $\sup\limits_{j\geqslant 1} \sum\limits_{n=N_{j-1}+1}^{N_j-1} |\lambda_{n+1}-\lambda_n| < \infty$.
where $\lambda_0 = 0$. Prove that the multiplier
$$M\sum_{n=1}^\infty a_n x_n := \sum_{n=1}^\infty \lambda_n a_n x_n$$
is bounded.
Hint: Write
$$Mx = \sum_{j=1}^\infty \lambda_{N_j}\Delta_j x + \sum_{j=1}^\infty \sum_{n=N_{j-1}+1}^{N_j-1} (\lambda_n - \lambda_{n+1}) D_n \Delta_j x.$$
Now use a randomisation argument, the result of the previous exercise, and Proposition 9.6.
Remark. It can be shown that the trigonometric system $(e_n)_{n\in\mathbb{Z}}$, where $e_n(\theta) = e^{in\theta}$, is a Schauder basis in $L^p(\mathbb{T})$ for all $1 < p < \infty$, but this basis is unconditional only for $p = 2$. However, it is a classical result of LITTLEWOOD and PALEY that the dyadic blocking of $(e_n)_{n\in\mathbb{Z}}$ is unconditional in $L^p(\mathbb{T})$ for all $1 < p < \infty$ (in this blocking, the j-th block runs over the indices $2^{j-1} \leqslant |n| < 2^j$). In combination with the exercise, this gives the classical formulation of the Marcinkiewicz multiplier theorem.

Notes. The importance of UMD-spaces extends far beyond the domain of stochastic analysis. In fact, the subject was created in an effort to extend classical Fourier multiplier theorems to Banach-space valued functions. On the unit circle $\mathbb{T}$, an important Fourier multiplier is the Riesz projection

$$\sum_{n\in\mathbb{Z}} c_n e^{in\theta} \mapsto \sum_{n=0}^{\infty} c_n e^{in\theta}.$$

This projection, which corresponds to the multiplier $1_{\{n\geqslant 0\}}$, is bounded in $L^p(\mathbb{T})$ for all $1 < p < \infty$. On the real line, the Hilbert transform defined by the principle value integral

$$Hf(x) := \frac{1}{\pi}\mathrm{PV}\int_{-\infty}^{\infty} \frac{f(y)}{x-y}\, dy$$

is bounded on $L^p(\mathbb{R})$ for all $1 < p < \infty$; it can be shown that this operator corresponds to the multiplier $\frac{1}{i}(1_{\mathbb{R}_+} - 1_{\mathbb{R}_-})$. Both results are classical theorems of M. Riesz. In the Banach space-valued situation the validity of these results characterise the UMD-property:

Theorem 12.16. *Let $1 < p < \infty$. For a Banach space E the following assertions are equivalent:*

(1) *E is a UMD$_p$-space;*
(2) *The Riesz projection is bounded on $L^p(\mathbb{T}; E)$;*
(3) *The Hilbert transform is bounded on $L^p(\mathbb{R}; E)$.*

The implications (1)⇒(2) and (1)⇒(3) are due to Burkholder [18] and McConnell [74], and their converses to Bourgain [10]. We refer to the review papers [20, 97] for more details. Recently, far-reaching generalisations of Theorem 12.16 to the boundedness of Fourier multipliers and singular integral operators in vector-valued L^p-spaces have been proved by several authors. We refer to the excellent lecture notes by Kunstmann and Weis [61] for an overview and references to the literature.

The independence of the UMD$_p$-property of the parameter $1 < p < \infty$ (Theorem 12.13) was first proved by Maurey [73], who gives credit to Pisier. The proof via Lemma 12.12 presented here is adapted from Burkholder [19]. The reductions of Section 12.2.1 are a variation of those proposed in [73] and carried out in detail in the lecture notes of De Pagter [87] and the M.Sc. thesis of Hytönen [50].

Several alternative proofs of the p-independence exist; some of them characterise the UMD$_p$-property in terms of some other property not involving the parameter p. In order to state two such characterisations, due to Burkholder [17, 20], we need to introduce the following terminology.

A Banach space is called a *weak UMD-space* if there exists a constant β such that for all L^1-martingale difference sequences $(d_n)_{n=1}^N$, all sequences of signs $(\varepsilon_n)_{n=1}^N$, and all $r > 0$ we have

$$r\,\mathbb{P}\Big\{\Big\|\sum_{n=1}^{N}\varepsilon_n d_n\Big\| > r\Big\} \leqslant \beta\mathbb{E}\Big\|\sum_{n=1}^{N} d_n\Big\|.$$

A Banach space E is called ζ-*convex* if there exists a function ζ on $E \times E$, convex in both variables separately, satisfying $\zeta(0,0) > 0$ and $\zeta(x,y) \leqslant \|x+y\|$ if $\|x\| = \|y\| = 1$.

Theorem 12.17. *For a Banach space E the following assertions are equivalent:*

(1) *E is a UMD-space;*
(2) *E is a weak UMD-space;*
(3) *E is ζ-convex.*

For Hilbert spaces one may take $\zeta(x,y) := 1 + [x,y]$. For L^p-spaces an explicit expression for a function ζ appears to be unknown.

The scalar version of Theorem 12.15 is due to Stein [100]. Its extension to UMD-spaces is due to Bourgain, who stated the result without proof in [12]. The proof presented here is taken from [24].

The result of Exercise 4 is due to Clément, De Pagter, Sukochev, Witvliet [24] and Berkson and Gillespie [6]. Exercise 5 is an abstract version of Bourgain's version of the Marcinkiewicz multiplier theorem [12]. Other classical multiplier theorems, such as the Mihlin multiplier theorem, can be extended to UMD-spaces as well. As was first shown by Weis [108] it is even possible to consider operator-valued multipliers; typically one has to replace boundedness assumptions by suitable R-boundedness assumptions. We refer to Kunstmann and Weis [61] for an overview and further references.