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Stochastic integration II: the Itô integral

We have seen in Lecture 6 how to integrate functions $\Phi : (0, T) \to \mathscr{L}(H, E)$ with respect to an H-cylindrical Brownian motion W_H. In this lecture we address the problem of extending the theory of stochastic integration to processes $\Phi : (0, T) \times \Omega \to \mathscr{L}(H, E)$. As it turns out, very satisfactory results can be obtained in the setting of UMD Banach spaces E. The reason for this is that in these spaces we can prove a decoupling theorem for certain martingale difference sequence which, in the context of stochastic integrals, enables us to replace W_H by an independent copy $\widetilde{W}_H$. The stochastic integral of Φ with respect to $\widetilde{W}_H$ can be defined path by path using the results of Lecture 6, and the decoupling inequality allows us to translate integrability criteria for this integral to the integral with respect to W_H.

13.1 Decoupling

We begin with an abstract decoupling result for a suitable class of martingale difference sequences.

Let $1 < p < \infty$ be fixed and suppose that $(\xi_n)_{n=1}^N$ is a sequence of centred integrable random variables in $L^p(\Omega)$. We assume that a filtration $(\mathscr{F}_n)_{n=1}^N$ is given such that the following conditions are satisfied for $n = 1, \dots, N$:

(1) ξ_n is $\mathscr{F}_n$-measurable for all $1 \leqslant n \leqslant N$;
(2) ξ_n is independent of $\mathscr{F}_m$ for all $1 \leqslant m < n \leqslant N$.

Note that $\mathbb{E}(\xi_n | \mathscr{F}_m) = \mathbb{E}\xi_n = 0$ for $1 \leqslant m < n \leqslant N$, so $(\xi_n)_{n=1}^N$ is a martingale difference sequence with respect to $(\mathscr{F}_n)_{n=1}^N$.

On the product space $(\Omega \times \Omega, \mathscr{F} \times \mathscr{F}, \mathbb{P} \times \mathbb{P})$ we define, with a slight abuse of notation,

$$\xi_n(\omega, \widetilde{\omega}) := \xi_n(\omega), \quad \widetilde{\xi}_n(\omega, \widetilde{\omega}) := \xi_n(\widetilde{\omega}). \tag{13.1}$$

The sequences $(\xi_n)_{n=1}^N$ and $(\widetilde{\xi}_n)_{n=1}^N$ are independent and identically distributed. The point here is that we identify each ξ_n with a random variable on

$\Omega \times \Omega$ which depends only on the first coordinate and introduce an independent copy $\widetilde{\xi}_n$ which depends only on the second coordinate. Clearly, $(\xi_n)_{n=1}^N$ and $(\widetilde{\xi}_n)_{n=1}^N$ are martingale difference sequences on $\Omega \times \Omega$ with respect to the filtrations $(\mathscr{F}_n)_{n=1}^N$ and $(\widetilde{\mathscr{F}}_n)_{n=1}^N$ defined by

$$\mathscr{F}_n := \mathscr{F}_n \times \{\varnothing, \Omega\}, \quad \widetilde{\mathscr{F}}_n := \{\varnothing, \Omega\} \times \mathscr{F}_n, \tag{13.2}$$

where again there is a slight abuse of notation in the first definition.

Let $(v_n)_{n=1}^N$ be a predictable sequence of E-valued random variables on Ω. Recall that this means that v_n is $\mathscr{F}_{n-1}$-measurable for $n = 1, \ldots, N$, with the understanding that $\mathscr{F}_0 = \{\varnothing, \Omega\}$ (so that v_1 is constant almost surely). We identify $(v_n)_{n=1}^N$ with a predictable sequence $(v_n)_{n=1}^N$ on $\Omega \times \Omega$ in the same way as above by putting $v_n(\omega, \widetilde{\omega}) := v_n(\omega)$.

Theorem 13.1 (Decoupling). *If, in addition to the above assumptions, E is a UMD-space, then*

$$\mathbb{E}\Big\|\sum_{n=1}^N \xi_n v_n\Big\|^p \eqsim_{p,E} \mathbb{E}\Big\|\sum_{n=1}^N \widetilde{\xi}_n v_n\Big\|^p$$

with constants depending on p and E only.

Proof. The proof uses a trick similar to that of Theorem 12.15.

For $n = 1, \ldots, N$ define

$$d_{2n-1} := \tfrac{1}{2}(\xi_n + \widetilde{\xi}_n)v_n \quad \text{and} \quad d_{2n} := \tfrac{1}{2}(\xi_n - \widetilde{\xi}_n)v_n.$$

We claim that $(d_j)_{j=1}^{2N}$ is a martingale difference sequence with respect to the filtration $(\mathscr{D}_j)_{j=1}^{2N}$, where

$$\mathscr{D}_{2n-1} = \sigma(\mathscr{F}_{n-1}, \widetilde{\mathscr{F}}_{n-1}, \xi_n + \widetilde{\xi}_n), \quad \mathscr{D}_{2n} = \sigma(\mathscr{F}_n, \widetilde{\mathscr{F}}_n).$$

In view of

$$\sum_{n=1}^N \xi_n v_n = \sum_{j=1}^{2N} d_j \quad \text{and} \quad \sum_{n=1}^N \widetilde{\xi}_n v_n = \sum_{j=1}^{2N} (-1)^{j+1} d_j,$$

the result then follows from the definition of the UMD$_p$-property.

It remains to prove the claim. We begin by observing that $(d_n)_{n=1}^{2N}$ is $(\mathscr{D}_n)_{n=1}^{2N}$-adapted. Moreover,

$$\begin{aligned}\mathbb{E}(d_{2n}|\mathscr{D}_{2n-1}) &\overset{\text{(i)}}{=} \tfrac{1}{2}v_n \mathbb{E}(\xi_n - \widetilde{\xi}_n | \mathscr{F}_{n-1}, \widetilde{\mathscr{F}}_{n-1}, \xi_n + \widetilde{\xi}_n) \\ &\overset{\text{(ii)}}{=} \tfrac{1}{2}v_n \mathbb{E}(\xi_n - \widetilde{\xi}_n | \xi_n + \widetilde{\xi}_n) \overset{\text{(iii)}}{=} 0.\end{aligned}$$

Here (i) follows from the $\mathscr{F}_{n-1}$-measurability of v_n, (ii) from Proposition 11.7 and the independence of $\sigma(\xi_n, \widetilde{\xi}_n)$ and $\sigma(\mathscr{F}_{n-1}, \widetilde{\mathscr{F}}_{n-1})$ (which follows from the independence of ξ_n and $\mathscr{F}_{n-1}$), and (iii) uses that ξ_n and $\widetilde{\xi}_n$ are independent and identically distributed (Exercise 11.2). Similarly,

$$\mathbb{E}(d_{2n-1}|\mathscr{D}_{2n-2}) = \tfrac{1}{2}v_n\mathbb{E}(\xi_n + \widetilde{\xi}_n|\mathscr{F}_{n-1}, \widetilde{\mathscr{F}}_{n-1}) = \tfrac{1}{2}v_n\mathbb{E}(\xi_n + \widetilde{\xi}_n) = 0$$

since $\xi_n + \widetilde{\xi}_n$ is independent of $\sigma(\mathscr{F}_{n-1}, \widetilde{\mathscr{F}}_{n-1})$ and ξ_n, $\widetilde{\xi}_n$ are centred. □

13.2 Stochastic integration

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. A function $\Phi : (0,T)\times\Omega \to \mathscr{L}(H,E)$ is said to be a *finite rank adapted step process* with respect to a given filtration $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}$ if it is of the form

$$\Phi(t,\omega) = \sum_{m=1}^{M}\sum_{n=1}^{N} 1_{(t_{n-1},t_n)}(t)1_{A_{mn}}(\omega)\sum_{j=1}^{k} h_j \otimes x_{jmn}, \tag{13.3}$$

where $0 \leqslant t_0 < \cdots < t_N \leqslant T$, for each $n = 1,\dots,N$ the sets $A_{1n},\dots,A_{Mn}$ are disjoint and belong to $\mathscr{F}_{t_{n-1}}$, the vectors $h_1,\dots,h_k \in H$ are orthonormal, and the vectors x_{jmn} belong to E.

In what follows we assume that W_H is an H-cylindrical Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P})$, adapted to $\mathbb{F}$ in the sense that the random variables $W_H(t)h$ are $\mathscr{F}_t$-measurable and the increments $W_H(t)h - W_H(s)h$ are independent of $\mathscr{F}_s$ for $t > s$. It follows from Exercise 11.4 that the filtration $\mathbb{F}^{W_H}$ generated by W_H has these properties.

The *stochastic integral* with respect to W_H of a finite rank adapted step process Φ of the form (13.3) is defined as

$$\int_0^T \Phi(t)\,dW_H(t) := \sum_{m=1}^{M}\sum_{n=1}^{N} 1_{A_{mn}}\sum_{j=1}^{k}(W_H(t_n)h_j - W_H(t_{n-1})h_j)x_{jmn}.$$

We leave it to the reader to check that this definition does not depend on the particular representation of Φ in (13.3). Note that $\int_0^T \Phi(t)\,dW_H(t)$ belongs to $L^p(\Omega, \mathscr{F}_T; E)$ for all $1 \leqslant p < \infty$, and satisfies

$$\mathbb{E}\int_0^T \Phi(t)\,dW_H(t) = 0.$$

The latter follows by linearity from

$$\begin{aligned}
&\mathbb{E}\big(1_{A_{mn}}(W_H(t_n)h_j - W_H(t_{n-1})h_j)\big)\\
&\qquad = \mathbb{E}\big(\mathbb{E}(1_{A_{mn}}(W_H(t_n)h_j - W_H(t_{n-1})h_j)|\mathscr{F}_{t_{n-1}})\big)\\
&\qquad = \mathbb{E}(1_{A_{mn}}\mathbb{E}(W_H(t_n)h_j - W_H(t_{n-1})h_j)|\mathscr{F}_{t_{n-1}}) = 0.
\end{aligned}$$

For each $\omega \in \Omega$ the trajectory $t \mapsto \Phi_\omega(t) := \Phi(t,\omega)$ is a finite rank step function and therewith defines an element R_{Φ_ω} of $\gamma(L^2(0,T;H),E)$. This results in a simple random variable

$$R_\Phi : \Omega \to \gamma(L^2(0,T;H),E).$$

In order to extend the above stochastic integral to a more general class of $\mathscr{L}(H,E)$-valued processes we shall proceed as in Lecture 6 by estimating the $L^p(\Omega;E)$-norm of the stochastic integral in terms of R_Φ. Due to the presence of the random variables $1_{A_{mn}}$, however, the Gaussian computation of Theorem 6.14 breaks down. In the proof of the next theorem we circumvent this problem by replacing W_H by an independent copy $\widetilde{W}_H$ and use the decoupling estimate of Theorem 13.1.

Theorem 13.2 (Itô isomorphism). *Let E be a UMD space and fix $1 < p < \infty$. For all finite rank adapted step processes $\Phi : (0,T) \times \Omega \to \mathscr{L}(H,E)$ we have*

$$\mathbb{E}\Big\| \int_0^T \Phi(t)\,dW_H(t)\Big\|^p \eqsim_{p,E} \mathbb{E}\|R_\Phi\|^p_{\gamma(L^2(0,T;H),E)},$$

with constants depending only on p and E.

Proof. As in (13.1) we identify W_H with an H-cylindrical Brownian motion on the product $\Omega \times \Omega$ and define an independent copy on $\widetilde{W}_H$ on $\Omega \times \Omega$ by putting

$$W_H(t)h(\omega,\widetilde{\omega}) := W_H(t)h(\omega), \quad \widetilde{W}_H(t)h(\omega,\widetilde{\omega}) := W_H(t)h(\widetilde{\omega}).$$

If $\Phi : (0,T) \times \Omega \to \mathscr{L}(H,E)$ is a finite rank adapted step process of the form (13.3), we define the decoupled stochastic integral

$$\int_0^T \Phi(t)\,d\widetilde{W}_H(t) := \sum_{n=1}^N \sum_{m=1}^M 1_{A_{mn}} \sum_{j=1}^k \big(\widetilde{W}_H(t_n)h_j - \widetilde{W}_H(t_{n-1})h_j\big) x_{jmn}.$$

The plan of the proof is to apply Theorem 13.1 to the real-valued sequence $(\xi_{jn})_{\substack{1\leqslant j\leqslant k\\ 1\leqslant n\leqslant N}}$ and the E-valued sequence $(v_{jn})_{\substack{1\leqslant j\leqslant k\\ 1\leqslant n\leqslant N}}$,

$$\xi_{jn} := W_H(t_n)h_j - W_H(t_{n-1})h_j, \quad v_{jn} := \sum_{m=1}^M 1_{A_{mn}} \otimes x_{jmn}.$$

With these notations,

$$\int_0^T \Phi(t)\,dW_H(t) = \sum_{n=1}^N \sum_{j=1}^k \xi_{jn} v_{jn}, \quad \int_0^T \Phi(t)\,d\widetilde{W}_H(t) = \sum_{n=1}^N \sum_{j=1}^k \widetilde{\xi}_{jn} v_{jn}.$$

We consider the filtration $(\mathscr{F}_{jn})_{\substack{1\leqslant j\leqslant k\\ 1\leqslant n\leqslant N}}$, where $\mathscr{F}_{jn}$ is the σ-algebra generated by all $\xi_{j'n'}$ with $(j',n')\leqslant (j,n)$; the pairs are ordered lexicographically according to the rule $(j',n')\leqslant (j,n) \Longleftrightarrow n'<n$ or $[n'=n \ \&\ j'\leqslant j]$.

With respect to this filtration, the sequence $(\xi_{jn})_{\substack{1\leqslant j\leqslant k\\ 1\leqslant n\leqslant N}}$ is centred and has the properties (1) and (2) stated at the beginning of Section 13.1 and $(v_{jn})^N_{\substack{1\leqslant j\leqslant k\\ 1\leqslant n\leqslant N}}$ is predictable.

Let us denote by $\mathbb{E}_1$ and $\mathbb{E}_2$ the expectations with respect to the first and second coordinate of $\Omega\times\Omega$. Applying successively Theorem 13.1, the Kahane-Khintchine inequality, and Theorem 6.14 (pointwise with respect to Ω_1), we obtain

$$\begin{aligned}\mathbb{E}_1\mathbb{E}_2\Big\|\int_0^T \Phi(t)\,dW_H(t)\Big\|^p &\eqsim_{p,E} \mathbb{E}_1\mathbb{E}_2\Big\|\int_0^T \Phi(t)\,d\widetilde{W}_H(t)\Big\|^p\\ &\eqsim_{p,E} \mathbb{E}_1\Big(\mathbb{E}_2\Big\|\int_0^T \Phi(t)\,d\widetilde{W}_H(t)\Big\|^2\Big)^{\frac{p}{2}}\\ &\eqsim_{p,E} \mathbb{E}_1\|R_\Phi\|^p_{\gamma(L^2(0,T;H),E)}. \qquad \square\end{aligned}$$

Definition 13.3. *A random variable $R\in L^p(\Omega;\gamma(L^2(0,T;H),E))$ is called* adapted *if it belongs to the closure in $L^p(\Omega;\gamma(L^2(0,T;H),E))$ of the finite rank adapted step processes.*

The closed subspace in $L^p(\Omega;\gamma(L^2(0,T;H),E))$ of all adapted elements with be denoted by $L^p_{\mathbb{F}}(\Omega;\gamma(L^2(0,T;H),E))$. Theorem 13.2 shows that the stochastic integral extends uniquely to an isomorphic embedding

$$J_T^{W_H} : L^p_{\mathbb{F}}(\Omega;\gamma(L^2(0,T;H),E)) \to L^p(\Omega;E).$$

Definition 13.4. *Let E be a Banach space and fix $1<p<\infty$. A process $\Phi : (0,T)\times\Omega\to\mathscr{L}(H,E)$ is said to be* L^p-stochastically integrable *with respect to the H-cylindrical Brownian motion W_H if there exists a sequence of finite rank adapted step processes $\Phi_n : (0,T)\times\Omega\to\mathscr{L}(H,E)$ such that:*

(1) *for all $h\in H$ we have $\lim_{n\to\infty}\Phi_n h=\Phi h$ in measure;*

(2) *there exists a random variable $X\in L^p(\Omega;E)$ such that $\lim_{n\to\infty}\int_0^T \Phi_n\,dW_H = X$ in $L^p(\Omega;E)$.*

The L^p-stochastic integral *of Φ is then defined as*

$$\int_0^T \Phi\,dW_H := \lim_{n\to\infty}\int_0^T \Phi_n\,dW_H.$$

The remarks (a) and (b) following Definition 6.15 extend to the present situation, but (c) is no longer automatic since stochastic integrals of step processes are no longer Gaussian. This is the reason why the adjective 'L^p-' has been built into the definition.

Theorem 13.5. *Let $1 < p < \infty$. If $\Phi : (0,T) \times \Omega \to \mathscr{L}(H,E)$ is L^p-stochastically integrable with respect to W_H, then the stochastic integral process $\big(\int_0^t \Phi \, dW_H\big)_{t\in[0,T]}$ is an E-valued L^p-martingale which has a continuous version satisfying the maximal inequality*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \Big\| \int_0^t \Phi \, dW_H \Big\|^p\Big) \leqslant \big(\frac{p}{p-1}\big)^p \, \mathbb{E}\Big\| \int_0^T \Phi \, dW_H \Big\|^p.$$

Proof. Choose a sequence $(\Phi_n)_{n\geqslant 1}$ of finite rank adapted step processes such that the conditions of Definition 13.4 are satisfied and put $X_n(t) := \int_0^t \Phi_n \, dW_H$. Clearly, there exists a continuous version $\widetilde{X}_n$ of each X_n, and by the Pettis measurability theorem we have $\widetilde{X}_n \in L^p(\Omega; C([0,T];E))$. To see that this theorem can be applied in the present situation, first note that there exists a separable closed subspace E_0 of E such that each X_n has trajectories in $C([0,T];E_0)$. The space $C([0,T];E_0)$ is separable, and the linear span of the functionals $\delta_t \otimes x^*$ is norming in its dual; moreover, $\langle X_n, \delta_t \otimes x^* \rangle = \int_0^t \Phi_n^* x^* \, dW_H$ almost surely and the right hand side is measurable as a function on Ω.

By Doob's maximal inequality (we use that the stochastic integral process is a martingale; see Exercise 3), for every choice of $0 \leqslant t_1 < \cdots < t_N \leqslant T$ we have

$$\mathbb{E}\big(\sup_{j=1,\dots,N} \|\widetilde{X}_n(t_j) - \widetilde{X}_m(t_j)\|^p\big) \leqslant \big(\frac{p}{p-1}\big)^p \, \mathbb{E}\|X_n(T) - X_m(T)\|^p.$$

Hence, by path continuity and Fatou's lemma,

$$\mathbb{E}\big(\sup_{t\in[0,T]} \|\widetilde{X}_n(t) - \widetilde{X}_m(t)\|^p\big) \leqslant \big(\frac{p}{p-1}\big)^p \, \mathbb{E}\|X_n(T) - X_m(T)\|^p.$$

This inequality shows that the sequence $(\widetilde{X}_n)_{n\geqslant 1}$ is a Cauchy sequence in $L^p(\Omega; C([0,T];E))$. Since for all $t \in [0,T]$ we have $\lim_{n\to\infty} X_n(t) = X(t)$ in $L^p(\Omega;E)$, the limit $\widetilde{X} = \lim_{n\to\infty} \widetilde{X}_n$ defines a continuous version of X.

The final inequality follows from Doob's maximal inequality in the same way as above (replace $\widetilde{X}_n - \widetilde{X}_m$ by $\widetilde{X}$). $\square$

As in Lecture 6, in the special case $E = \mathbb{R}$ we may identify $\mathscr{L}(H,\mathbb{R})$ with H and Theorem 13.2 reduces to the statement that the L^p-stochastic integral of an adapted step process $\phi : (0,T) \times \Omega \to H$ satisfies

$$\mathbb{E}\Big\| \int_0^T \phi \, dW_H \Big\|^p \eqsim_p \mathbb{E}\|\phi\|^p_{L^2(0,T;H)} \tag{13.4}$$

The constants depend only on p since the UMD_p-constant of Hilbert spaces only depend on p. From this it is not hard to see (Exercise 2) that a strongly adapted measurable process $\phi : (0,T)\times\Omega \to H$ is L^p-stochastically integrable with respect to W_H if and only if $\phi \in L^p(\Omega; L^2(0,T;H))$, and the isomorphism (13.4) extends to this situation.

Definition 13.6. *A process $\Phi : (0,T)\times\Omega \to \mathscr{L}(H,E)$ is called H-*strongly measurable *if for each $h \in H$ the process $\Phi h : (0,T)\times\Omega \to E$ is strongly measurable. Such a process Φ is called* adapted *if for each $h \in H$ the process Φh is adapted.*

We are now in a position to state the main result of this section, which extends Theorem 6.17 to $\mathscr{L}(H,E)$-valued processes.

Theorem 13.7. *Let E be a UMD space and fix $1 < p < \infty$. For an H-strongly measurable adapted process $\Phi : (0,T)\times\Omega \to \mathscr{L}(H,E)$ the following assertions are equivalent:*

(1) *Φ is L^p-stochastically integrable with respect to W_H;*
(2) *$\Phi^* x^* \in L^p(\Omega; L^2(0,T;H))$ for all $x^* \in E^*$, and there exists a random variable $X \in L^p(\Omega;E)$ such that for all $x^* \in E^*$,*
$$\langle X, x^*\rangle = \int_0^T \Phi^* x^* \, dW_H(t) \quad \text{in } L^p(\Omega).$$
(3) *$\Phi^* x^* \in L^p(\Omega; L^2(0,T;H))$ for all $x^* \in E^*$, and there exists a random variable $R \in L^p(\Omega; \gamma(L^2(0,T;H),E))$ such that for all $f \in L^2(0,T;H)$ and $x^* \in E^*$,*
$$\langle Rf, x^*\rangle = \int_0^T \langle \Phi(t)f(t), x^*\rangle \, dt \quad \text{in } L^p(\Omega).$$

If these equivalent conditions are satisfied, the random variables X and R are uniquely determined, we have $X = \int_0^T \Phi \, dW_H$ in $L^p(\Omega;E)$, and
$$\mathbb{E}\Big\| \int_0^T \Phi \, dW_H \Big\|^p \eqsim_{p,E} \mathbb{E}\|R\|^p_{\gamma(L^2(0,T;H),E)}.$$
Moreover, $R \in L^p_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;H),E))$, that is, R is adapted.

Proof. We sketch the main steps and refer to the Notes for more information.

(1)⇒(2): This is proved in the same way as in Theorem 6.17. Note that the stochastic integrals $\int_0^T \Phi^* x^* \, dW_H$ are well-defined by the above remarks.

(2)⇒(3): For the special case where $\mathbb{F}$ is the filtration generated by W_H, a proof will be outlined below.

(1)⇒(3): This is an immediate consequence of Theorem 13.2: if $(\Phi_n)_{n=1}^\infty$ is an approximating sequence for Φ, then the operators $(R_{\Phi_n})_{n=1}^\infty$ form a Cauchy sequence in $L^p_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;H),E))$ and its limit has the desired properties.

(3)⇒(1): First one shows that R is adapted (see Exercise 1). Knowing this, the proof can be finished in the same way as the corresponding implication of Theorem 6.17. □

Unfortunately we are not able to give a fully self-contained proof of the implication (2)⇒(3). In the sequel we shall not need this implication; we only use the equivalence (1)⇔(3) which is the most useful part of the theorem. In spite of this we want to sketch a proof of (2)⇒(3) under the simplifying assumption that the filtration is the one generated by W_H. In this situation we can apply a version of the so-called martingale representation theorem for H-cylindrical Brownian motions W_H. In most textbook proofs, the integrator is a Brownian motion (or a more general martingale); the extension to cylindrical Brownian motions is obtained from it by an approximation argument as in the proof of the martingale convergence theorem (Theorem 11.21).

Recall that the filtration $\mathbb{F}^{W_H}$ has been defined in Exercise 11.4.

Lemma 13.8. *Let $1 < p < \infty$ and $\xi \in L^p(\Omega, \mathscr{F}_T^{W_H})$. There exists unique $\phi \in L^p_{\mathbb{F}^{W_H}}(\Omega; L^2(0,T;H))$ such that*

$$\xi = \mathbb{E}\xi + \int_0^T \phi \, dW_H.$$

The proof of this lemma is beyond the scope of these lectures. Roughly speaking it proceeds like this. First, we may assume that $\mathbb{E}\xi = 0$. By approximation we may further assume that H is finite-dimensional. From

$$W_H(t)h = \int_0^T 1_{(0,t)} \otimes h \, dW_H(t)$$

we see that every X in the linear span of the random variables $W_H(t)h$ can be represented by a stochastic integral. Since the stochastic integral defines an isomorphic embedding, it remains to show that this span is dense in the closed subspace of $L^p(\Omega, \mathscr{F}_T^{W_H}; H)$ consisting of all mean 0 elements.

The next result extends the lemma to UMD spaces. Recall that

$$J_T^{W_H} : L^p_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;H),E)) \to L^p(\Omega; E)$$

is the isomorphic embedding of Theorem 13.2.

Theorem 13.9. *Let E be a UMD space, let $1 < p < \infty$, and let $X \in L^p(\Omega, \mathscr{F}_T^{W_H}; E)$. There exists a unique $R \in L^p_{\mathbb{F}^{W_H}}(\Omega; \gamma(L^2(0,T;H),E))$ such that*

$$X = \mathbb{E}X + J_T^{W_H}(R).$$

Proof. Choose a sequence of simple $\mathscr{F}_T^{W_H}$-measurable random variables X_n such that $\lim_{n\to\infty} X_n = X$ in $L^p(\Omega; E)$. Let us write $X_n = \sum_{m=1}^{M_n} 1_{A_{mn}} \otimes x_{mn}$.

By Lemma 13.8, there exist unique processes $\phi_{mn} \in L^p_{\mathbb{F}^{W_H}}(\Omega; L^2(0,T;H))$ such that

$$1_{A_{mn}} = \mathbb{E}1_{A_{mn}} + \int_0^T \phi_{mn}\, dW_H.$$

Put $\Phi_n(t)h := \sum_{m=1}^M [\phi_{mn}, h]x_{mn}$. The process $\Phi_n : (0,T) \times \Omega \to \mathscr{L}(H,E)$ is L^p-stochastically integrable with respect to W_H and

$$X_n = \mathbb{E}X_n + \int_0^T \Phi_n\, dW_H.$$

Let $R_n \in L^p_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;H), E)$ be defined by

$$R_n(\omega)f := \sum_{m=1}^{M_n} \phi_{mn}(\omega) \otimes x_{mn}, \quad f \in L^2(0,T;H).$$

Since $\lim_{n\to\infty} X_n = X$ in $L^p(\Omega;E)$, the isomorphism of Theorem 13.2 implies that the sequence $(R_n)_{n=1}^\infty$ is Cauchy in $L^p(\Omega; \gamma(L^2(0,T;H),E))$. The limit R has the desired properties.

Uniqueness follows from the injectivity of $J_T^{W_H}$. □

As a corollary we observe that the stochastic integral defines an isomorphism of Banach spaces

$$J_T^{W_H} : L^p_{\mathbb{F}^{W_H}}(\Omega; \gamma(L^2(0,T;H),E)) \simeq L^p_0(\Omega, \mathscr{F}_T^{W_H}; E),$$

where $L^p_0(\Omega, \mathscr{F}_T^{W_H}; E)$ is the closed subspace of $L^p(\Omega, \mathscr{F}_T^{W_H}; E)$ consisting of all elements with mean 0.

Proof (Proof of Theorem 13.7 (2)⇒(3) for the filtration $\mathbb{F}^{W_H}$). By the Pettis measurability theorem, the random variable X belongs to $L^p_0(\Omega, \mathscr{F}_T^{W_H}; E)$. The element R provided by Theorem 13.9 has the desired properties. □

13.3 Stochastic integrability of L^p-martingales

We return to the setting where W_H is an H-cylindrical Brownian motion, adapted to a filtration $\mathbb{F}$. The main result of this section states that if E is a UMD space and M is a $\gamma(H,E)$-valued L^p-martingale with respect to $\mathbb{F}$, then M is L^p-stochastically integrable with respect to W^H. The proof has three ingredients: the characterisation of L^p-stochastic integrability (the equivalence (1)⇔(3) of Theorem 13.7)), the γ-multiplier theorem (Theorem 9.14), and the vector-valued Stein inequality (Theorem 12.15).

Theorem 13.10. *Let E be a UMD space and fix $1 < p < \infty$. Let W_H be an H-cylindrical Brownian motion, adapted to a filtration $\mathbb{F}$, and let M :*

$[0,T]\times\Omega\to\gamma(H,E)$ be an L^p-martingale with respect to $\mathbb{F}$. Then M is L^p-stochastically integrable with respect to W_H and we have

$$\Big(\mathbb{E}\Big\|\int_0^T M(t)\,dW_H(t)\Big\|^p\Big)^{\frac1p} \lesssim_{p,E} \sqrt{T}\,\big(\mathbb{E}\|M(T)\|^p_{\gamma(H,E)}\big)^{\frac1p},$$

with a constant depending only on p and E.

Proof. First we prove the result under the additional assumption that $M(T)\in L^\infty(\Omega;\gamma(H,E))$. By the L^∞-contractivity of the conditional expectation we then have $M\in L^\infty((0,T)\times\Omega;\gamma(H,E))$. In particular, for all $x^*\in E^*$ we have $M^*x^*\in L^p(\Omega;L^2(0,T;H))$, and even $M^*x^*\in L^p_{\mathbb{F}}(\Omega;L^2(0,T;H))$ since M is adapted.

Let us write $B:=L^p(\Omega;E)$ for brevity. Define the bounded function $N:[0,T]\to\mathscr{L}(B)$ by

$$N(t)\xi := \mathbb{E}(\xi|\mathscr{F}_t), \qquad \xi\in B,\ t\in[0,T].$$

Since E is a UMD space, by Theorem 12.15 the family $\{N(t): t\in[0,T]\}$ is R-bounded on B, and therefore γ-bounded, with γ-bound depending only on p and E. By Theorem 11.21, for every $\xi\in B$ the function $t\mapsto N(t)\xi$ has left limits at every point $[0,T]$. In particular, these functions are strongly measurable.

By the γ-Fubini isomorphism (Theorem 5.22), for each $t\in[0,T]$ we may identify the random variable $M(t)\in L^p(\Omega;\gamma(H,E))$ with a unique operator $\widetilde{M}(t)\in\gamma(H,B)$ by the formula $(\widetilde{M}(t)h)(\omega)=M(t,\omega)h$. Define a constant function $G:[0,T]\to\gamma(H,B)$ by

$$G(t):=\widetilde{M}(T), \quad t\in[0,T].$$

This function represents the element $R_G\in\gamma(L^2(0,T;H),B)$ satisfying

$$\|R_G\|_{\gamma(L^2(0,T;H),B)} = \sqrt{T}\,\|\widetilde{M}(T)\|_{\gamma(H,B)} \eqsim_p \sqrt{T}\,\big(\mathbb{E}\|M(T)\|^p_{\gamma(H,E)}\big)^{\frac1p},$$

where we used the result of Exercise 5.3.

By the martingale property, for all $t\in[0,T]$ we have $\widetilde{M}(t)=N(t)\widetilde{M}(T)$ in B. Now we apply Theorem 9.14 to conclude that $\widetilde{M}$ represents an element $R\in\gamma(L^2(0,T;H),B)$ satisfying

$$\|R\|_{\gamma(L^2(0,T;H),B)} \lesssim_{p,E} \|R_G\|_{\gamma(L^2(0,T;H),B)}.$$

Using Theorem 5.22 once more, we can identify R with an element $X\in L^p(\Omega;\gamma(L^2(0,T;H),E))$ by the formula $X(\omega)f=(Rf)(\omega)$. Below we check that $X^*x^*=M^*x^*$ in $L^p(\Omega;L^2(0,T;H))$ for all $x^*\in E^*$. Assuming this for the moment, it follows from Theorem 13.7(3)$\Rightarrow$(1) that M is L^p-stochastically integrable and satisfies

$$\Big(\mathbb{E}\Big\|\int_0^T M(t)\,dW_H(t)\Big\|^p\Big)^{\frac{1}{p}} \eqsim_{p,E} \big(\mathbb{E}\|X\|^p_{\gamma(L^2(0,T;H),E)}\big)^{\frac{1}{p}}$$
$$\eqsim_p \|R\|_{\gamma(L^2(0,T;H),B)} \lesssim_{p,E} \sqrt{T}\,\big(\mathbb{E}\|M(T)\|^p_{\gamma(H,E)}\big)^{\frac{1}{p}}.$$

To prove that $X^*x^* = M^*x^*$ for all $x^* \in E^*$, let $f \in L^2(0,T;H)$ and $x^* \in E^*$ be arbitrary and note that for all $A \in \mathscr{F}$,

$$\begin{aligned}
\mathbb{E}(\langle Mf, x^*\rangle 1_A) &= \int_\Omega \int_0^T \langle M(t,\omega)f(t), x^*\rangle 1_A(\omega)\,dt\,dP(\omega) \\
&= \int_0^T \int_\Omega \langle M(t,\omega)f(t), x^*\rangle 1_A(\omega)\,dP(\omega)\,dt \\
&= \mathbb{E}\int_0^T \langle \widetilde{M}(t)f(t), 1_A \otimes x^*\rangle\,dt \\
&= \mathbb{E}\langle Rf, 1_A \otimes x^*\rangle = \mathbb{E}(\langle Xf, x^*\rangle 1_A).
\end{aligned}$$

To conclude the proof we remove the assumption $M(T) \in L^\infty(\Omega;E)$. Choose a sequence of $\mathscr{F}_T$-measurable simple random variables $M_n(T)$ converging to $M(T)$ in $L^p(\Omega;E)$, and define $M_n(t) := \mathbb{E}(M_n(T)|\mathscr{F}_t)$. Since $M_n(T) \in L^\infty(\Omega;E)$, we may apply what we proved above to the martingales M_n. We obtain that each M_n is L^p-stochastically integrable with respect to W_H and

$$\Big(\mathbb{E}\Big\|\int_0^T M_n(t)\,dW_H(t)\Big\|^p\Big)^{\frac{1}{p}} \leqslant C\sqrt{T}\,\big(\mathbb{E}\|M_n(T)\|^p_{\gamma(H,E)}\big)^{\frac{1}{p}},$$

with a constant C independent of n. Similarly, by the above applied to the martingales $M_n - M_m$, we find that the stochastic integrals $\int_0^T M_n(t)\,dW_H(t)$ are Cauchy in $L^p(\Omega;E)$. By the Itô isomorphism, this means that the corresponding elements $R_n \in L^p_{\mathbb{F}}(\Omega;\gamma(L^2(0,T;H),E))$ are Cauchy and therefore converge to a limit $R \in L^p_{\mathbb{F}}(\Omega;\gamma(L^2(0,T;H),E))$. Clearly, $R^*x^* = \lim_{n\to\infty} R_n^*x^* = \lim_{n\to\infty} M_n^*x^* = M^*x^*$ in $L^p(\Omega;L^2(0,T;H))$, and the conclusion of the theorem now follows via Theorem 13.7. □

13.4 Exercises

In the exercises 1-3 we fix $1 < p < \infty$.

1. In this exercise we compare the two notions of adaptedness given in Definitions 13.3 and 13.6.
   a) Show that $R \in L^p(\Omega;\gamma(L^2(0,T;H),E))$ is adapted if and only if the random variables $R(1_{(0,t)}f) : \Omega \to E$ have strongly $\mathscr{F}_t$-measurable versions for all $t \in (0,T)$ and $f \in L^2(0,T;H)$.
      Hint: For the 'if' part, approximate with simple random variables and use that the finite rank step functions are dense in $\gamma(L^2(0,T;H),E)$. To secure adaptedness, build in a small shift before approximating.

Now suppose that $\Phi : (0,T) \to \mathscr{L}(H,E)$ is H-strongly measurable and assume that the conditions of Theorem 13.7 (3) be satisfied; let $R \in L^p(\Omega;\gamma(L^2(0,T;H),E))$ be as in (3).

b) Show that if Φ is adapted, then R is adapted.

2. In the discussion after Definition 13.4 it was observed that a strongly measurable adapted process $\phi : (0,T)\times\Omega \to H$ is L^p-stochastically integrable with respect to W_H if and only if $\phi \in L^p(\Omega; L^2(0,T;H))$. Prove this.
Hint: If $\phi \in L^p(\Omega; L^2(0,T;H))$, then by the previous exercise ϕ is adapted as an element of $L^p(\Omega; L^2(0,T;H))$.

3. Let $\Phi : (0,T)\times\Omega \to \mathscr{L}(H,E)$ be L^p-stochastically integrable with respect to W_H. Show that the stochastic integral process $\bigl(\int_0^t \Phi\,dW_H\bigr)_{t\in[0,T]}$ is a martingale.
Hint: Approximate with finite rank adapted step processes.

If E is a UMD space with type 2 and $\Phi : (0,T)\times\Omega \to \gamma(H,E)$ is an adapted and strongly measurable process such that

$$\mathbb{E}\int_0^T \|\Phi(t)\|^2_{\gamma(H,E)}\,dt < \infty,$$

then Φ is stochastically integrable with respect to H-cylindrical Brownian motions W_H; this follows from Theorem 13.7 and Exercise 5.4. In the next two exercises we show that the UMD assumption can essentially be dropped from this statement.

4. A Banach space E has *martingale type* $p \in [1,2]$ if there exists a constant $M_p(E)$ such that for any L^p-martingale sequence $(d_n)_{n=1}^N$ with values in E we have
$$\mathbb{E}\Bigl\|\sum_{n=1}^N d_n\Bigr\|^p \leqslant (M_p(E))^p \sum_{n=1}^N \mathbb{E}\|d_n\|^p.$$
a) Show that every martingale type p space has type p.
b) Show that every UMD space with type p has martingale type p.

In both cases, give relations between the constants.

c) Deduce that L^p-spaces, $1 < p < \infty$, have martingale type $\min\{p,p'\}$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

5. Let E be a martingale type 2 space.
a) Show that if W_H is an H-cylindrical Brownian motion and $\Phi : (0,T)\times\Omega \to \mathscr{L}(H,E)$ is an adapted finite rank step process, then
$$\mathbb{E}\Bigl\|\int_0^T \Phi\,dW_H\Bigr\|^2 \leqslant (M_2(E))^2\;\mathbb{E}\int_0^T \|\Phi(t)\|^2_{\gamma(H,E)}\,dt.$$

b) Conclude that if $\Phi : (0,T) \times \Omega \to \gamma(H,E)$ is an adapted strongly measurable process satisfying

$$\mathbb{E}\int_0^T \|\Phi(t)\|^2 \, dt < \infty,$$

then Φ is stochastically integrable with respect to W_H, with the same estimate as before.

Notes. A systematic treatment of decoupling inequalities is presented in the monograph of Gine and de la Peña [31]. The proof of the decoupling inequality (Theorem 13.1) is based on an idea of Montgomery-Smith [78].

The idea to use decoupling inequalities for obtaining bounds on stochastic integrals in UMD spaces was first used by Garling [40], who only considered step processes and used the resulting estimates to investigate certain geometric properties of UMD spaces. Using a more delicate decoupling result together with Burkholder's characterisation of UMD spaces through ζ-convexity (Theorem 12.17), McConnell [75] proved that a UMD-valued process is stochastically integrable if almost surely its trajectories are stochastically integrable with respect to an independent copy of the Brownian motion. In view of Theorem 6.17 this result can be viewed as an 'almost sure' version of the implication (3)⇒(1) of Theorem 13.7.

Our approach to vector-valued stochastic integration in UMD spaces via γ-radonifying norms is taken from [82], where Theorem 13.7 was proved. In that paper, McConnell's result is recovered using a stopping time argument.

The equivalence of norms in (13.4) is a special case of an inequality of Burkholder, Davis, Gundy which, in the more general situation where the integrator is a continuous-time martingale M, relates the norms of stochastic integrals to the norms of the quadratic variation process of M. For more details we refer to Karatzas and Shreve [59], Revuz and Yor [94], or Kallenberg [55].

An alternative proof of the implication (2)⇒(3) of Theorem 13.7, which is based on finite-dimensional approximations, covariance domination, and the theorem of Hoffmann-Jorgensen and Kwapień (Theorem 5.9) is given in [82]. A detailed proof of the implication (3)⇒(1) is contained in [81].

A systematic theory of stochastic integration in martingale type 2 space has been developed by Neidhardt [85], Dettweiler [33], and Brzeźniak [13]. The first two authors assumed that E be 2-uniformly smooth, a property which was subsequently shown to be equivalent to the martingale type 2 property by Pisier [91]. For an overview, see Brzeźniak [15].