14

Linear equations with multiplicative noise

In this lecture we study stochastic evolution equations with multiplicative noise of the form

$$\begin{cases} dU(t) = AU(t)\,dt + B(U(t))\,dW_H(t), & t \in [0,T], \\ U(0) = u_0. \end{cases} \tag{SCP}$$

Under suitable assumptions on E, the semigroup S generated by A on E, and the function $B : E \to \gamma(H,E)$, we shall prove existence, uniqueness, and Hölder regularity of mild solutions. Such a solution is defined as an adapted process U such that for all $t \in [0,T]$ we have

$$U(t) = S(t)u_0 + \int_0^t S(t-s)B(U(s))\,dW_H(s) \tag{14.1}$$

almost surely. Its existence and uniqueness is proved by a fixed point argument in the completion $V_\theta^p(\Omega;\gamma(L^2(0,T),E))$ of the space of all adapted finite rank step processes $\phi : (0,T) \times \Omega \to E$ such that

$$s \mapsto (t-s)^{-\theta}\phi(s) \text{ belongs to } L^p(\Omega;\gamma(L^2(0,t),E)),$$

uniformly with respect to $0 < t \leqslant T$. The reason for working in this complicated space is the fact that in many applications (e.g. when S is an analytic semigroup) the set $\{t^\theta S(t) :\ t \in (0,T)\}$ is γ-bounded.

The strategy for the fixed point argument is as follows. First, we find conditions on B which guarantee that it acts as a Lipschitz map from $V_\theta^p(\Omega;\gamma(L^2(0,T),E))$ to $V_\theta^p(\Omega;\gamma(L^2(0,T;H),E))$. Note that under these conditions, the stochastic integrals in (14.1) are well defined by the results of the previous lecture. Then, we prove that the process on the right hand side of (14.1) is in $V_\theta^p(\Omega;\gamma(L^2(0,T),E))$ again.

14.1 γ-Lipschitz functions

Let H be a non-zero Hilbert space, let E and F be Banach spaces, and let $(\gamma_{mn})_{m,n=1}^{\infty}$ and $(\gamma_n)_{n=1}^{\infty}$ be Gaussian sequences.

Proposition 14.1. *Let $B : E \to \gamma(H, F)$ be a function such that $Bh : E \to F$ is strongly measurable for all $h \in H$, and let $C \geqslant 0$ be a constant. The following assertions are equivalent:*

(1) *for all orthonormal sequences $(h_m)_{m=1}^{M}$ in H and all sequences $(x_n)_{n=1}^{N}$ and $(y_n)_{n=1}^{N}$ in E,*

$$\mathbb{E}\Big\| \sum_{m=1}^{M} \sum_{n=1}^{N} \gamma_{mn}(B(x_n) - B(y_n))h_m \Big\|^2 \leqslant C^2 \mathbb{E}\Big\| \sum_{n=1}^{N} \gamma_n (x_n - y_n) \Big\|^2;$$

(2) *for all simple functions $\phi_1, \phi_2 : (0,T) \to E$ we have $B(\phi_1)$, $B(\phi_2) \in \gamma(L^2(0,T;H), F)$ and*

$$\|B(\phi_1) - B(\phi_2)\|_{\gamma(L^2(0,T;H),F)} \leqslant C \|\phi_1 - \phi_2\|_{\gamma(L^2(0,T),E)};$$

(3) *for all σ-finite measure spaces $(A, \mathscr{A}, \mu)$ and all μ-simple functions $\phi_1, \phi_2 : A \to E$ we have $B(\phi_1)$, $B(\phi_2) \in \gamma(L^2(A;H), F)$ and*

$$\|B(\phi_1) - B(\phi_2)\|_{\gamma(L^2(A;H),F)} \leqslant C \|\phi_1 - \phi_2\|_{\gamma(L^2(A),E)}.$$

Note that if H is separable, then $Bh : E \to F$ is strongly measurable for all $h \in H$ if and only if $B : E \to \gamma(H, F)$ is strongly measurable; this is proved in Proposition 5.14 (with strong μ-measurability replaced by strong measurability).

Proof. Let us first prove that (1) is equivalent to

(1′) for all orthonormal sequences $(h_m)_{m=1}^{M}$ in H, all sequences $(a_n)_{n=1}^{N}$ of positive real numbers and all sequences $(x_n)_{n=1}^{N}$ and $(y_n)_{n=1}^{N}$ in E,

$$\mathbb{E}\Big\| \sum_{m=1}^{M} \sum_{n=1}^{N} a_n \gamma_{mn}(B(x_n) - B(y_n))h_m \Big\|^2 \leqslant C^2 \mathbb{E}\Big\| \sum_{n=1}^{N} a_n \gamma_n (x_n - y_n) \Big\|^2.$$

For integers a_n, the equivalence follows by applying (1) with the x_n and y_n repeated a_n times and noting that the sum of a_n independent standard Gaussians $\gamma_n^{(1)} + \cdots + \gamma_n^{(a_n)}$ has the same distribution as $a_n \gamma_n$. The case of rational a_n is readily reduced to this, and the general case follows by approximation.

The equivalence of (1′), (2), (3) follows from the following general observation. Let $(A, \mathscr{A}, \mu)$ be any σ-finite measure space. If $(h_m)_{m=1}^{M}$ is orthonormal in H and $\phi_1 = \sum_{n=1}^{N} 1_{A_n} \otimes x_n$ and $\phi_2 = \sum_{n=1}^{N} 1_{A_n} \otimes y_n$ are μ-simple functions with values in E, with the sets A_n disjoint, then by Lemma 5.7 (noting that the sequence $(\frac{1}{\sqrt{\mu(A_n)}} 1_{A_n})_{n=1}^{N}$ is orthonormal in $L^2(A)$),

$$\|B(\phi_1)-B(\phi_2)\|^2_{\gamma(L^2(A;H),F)} = \mathbb{E}\Big\| \sum_{m=1}^M \sum_{n=1}^N \sqrt{\mu(A_n)}\gamma_{nm}(B(x_n)-B(y_n))h_m\Big\|^2$$

and

$$\|\phi_1-\phi_2\|^2_{\gamma(L^2(A),E)} = \mathbb{E}\Big\| \sum_{n=1}^N \sqrt{\mu(A_n)}\gamma_n(x_n-y_n)\Big\|^2. \qquad \square$$

Note that if $(A,\mathscr{A},\mu)$ is a σ-finite measure space with $\mu(A)\neq 0$ and $B: E\to \mathscr{L}(H,F)$ is a function such that $B(\phi)\in\gamma(L^2(A;H),F)$ for every μ-simple function $\phi: A\to E$, then $B(x)\in\gamma(H,F)$ for all $x\in E$. Indeed, consider any set $A_0\in\mathscr{A}$ with $0<\mu(A_0)<\infty$. Then $B(1_{A_0}\otimes x) = 1_{A_0}\otimes B(x)$ belongs to $\gamma(L^2(A;H),F)$, which is only possible if $B(x)\in\gamma(H,F)$. This explains why we restrict ourselves to functions $B: E\to\gamma(H,F)$.

Definition 14.2. *A strongly measurable function $B: E\to\gamma(H,F)$ is called* γ-Lipschitz continuous *if the equivalent conditions of Proposition 14.1 hold. The least possible constant in these conditions is denoted by* $\mathrm{Lip}_\gamma(B)$.

By taking $H=\mathbb{R}$ we obtain the notion of a γ-Lipschitz continuous function from E to F. Clearly, every γ-Lipschitz continuous function $B: E\to F$ is Lipschitz continuous and we have $\mathrm{Lip}(B)\leqslant \mathrm{Lip}_\gamma(B)$.

It is a natural question whether conversely Lipschitz functions are automatically γ-Lipschitz. In this direction we have the following result (cf. Exercise 3), which gives a first example of γ-Lipschitz continuous mappings.

Example 14.3. If F has type 2, then every Lipschitz function $B: E\to\gamma(H,F)$ is γ-Lipschitz continuous and we have $\mathrm{Lip}(B)\leqslant\mathrm{Lip}_\gamma(B)\leqslant T_2^\gamma \mathrm{Lip}(B)$, where T_2^γ is the Gaussian type 2 constant of F.

This result actually characterises the type 2 property; see the Notes at the end of the lecture. Further examples of γ-Lipschitz continuous mappings, relevant for applications to stochastic PDEs, will be given in the next lecture.

14.2 Pisier's property

Our next aim is to prove certain weighted bounds for stochastic convolutions. In order to keep the technicalities at a reasonable level we shall assume an additional geometric property on the underlying Banach space E, first studied by Pisier.

Let $(r_j')_{j=1}^\infty$ and $(r_k'')_{k=1}^\infty$ be Rademacher sequences on probability spaces $(\Omega',\mathscr{F}',\mathbb{P}')$ and $(\Omega'',\mathscr{F}'',\mathbb{P}'')$, and let $(r_{jk})_{j,k=1}^\infty$ be a doubly indexed Rademacher sequence on a probability space $(\Omega,\mathscr{F},\mathbb{P})$. In the next result, recall that $(r_j'r_k'')_{j,k=1}^\infty$ is not a Rademacher sequence (see Exercise 3.2).

Proposition 14.4. *For a Banach space E the following assertions are equivalent:*

(1) *there exists a constant* $0 < C < \infty$ *such that for all finite sequences* $(a_{jk})_{j,k=1}^n$ *in* $\mathbb{R}$ *and* $(x_{jk})_{j,k=1}^n$ *in* E *we have*

$$\mathbb{E}'\mathbb{E}''\Big\| \sum_{j,k=1}^n a_{jk} r_j' r_k'' x_{jk} \Big\|^2 \leqslant C^2 \big(\max_{1\leqslant j,k\leqslant n} |a_{jk}|\big) \mathbb{E}'\mathbb{E}''\Big\| \sum_{j,k=1}^n r_j' r_k'' x_{jk} \Big\|^2;$$

(2) *there exists a constant* $0 < C < \infty$ *such that for all finite sequences* $(x_{jk})_{j,k=1}^n$ *in* E *we have*

$$\frac{1}{C^2}\mathbb{E}\Big\| \sum_{j,k=1}^n r_{jk} x_{jk} \Big\|^2 \leqslant \mathbb{E}'\mathbb{E}''\Big\| \sum_{j,k=1}^n r_j' r_k'' x_{jk} \Big\|^2 \leqslant C^2 \mathbb{E}\Big\| \sum_{j,k=1}^n r_{jk} x_{jk} \Big\|^2.$$

Condition (1) means that the analogue of the Kahane contraction principle holds for double Rademacher sums in E.

Proof. (1)⇒(2): By randomisation and Fubini's theorem, from (1) we obtain

$$\begin{aligned}
&\mathbb{E}\Big\| \sum_{m=1}^M \sum_{n=1}^N r_{mn} x_{mn} \Big\|^2 \\
&\quad = \mathbb{E}'\mathbb{E}''\mathbb{E}\Big\| \sum_{m=1}^M \sum_{n=1}^N r_{mn} r_m' r_n'' x_{mn} \Big\|^2 = \mathbb{E}\mathbb{E}'\mathbb{E}''\Big\| \sum_{m=1}^M \sum_{n=1}^N r_{mn} r_m' r_n'' x_{mn} \Big\|^2 \\
&\quad \leqslant C^2 \mathbb{E}\mathbb{E}'\mathbb{E}''\Big\| \sum_{m=1}^M \sum_{n=1}^N r_m' r_n'' x_{mn} \Big\|^2 = C^2 \mathbb{E}'\mathbb{E}''\Big\| \sum_{m=1}^M \sum_{n=1}^N r_m' r_n'' x_{mn} \Big\|^2.
\end{aligned}$$

This gives the left hand side inequality in (2).

To prove the right hand side inequality in (2) we fix numbers $\varepsilon_{mn} \in \{-1, 1\}$ and use (1) to obtain

$$\begin{aligned}
&\mathbb{E}'\mathbb{E}''\Big\| \sum_{m=1}^M \sum_{n=1}^N r_m' r_n'' x_{mn} \Big\|^2 \\
&\quad = \mathbb{E}'\mathbb{E}''\Big\| \sum_{m=1}^M \sum_{n=1}^N \varepsilon_{mn}^2 r_m' r_n'' x_{mn} \Big\|^2 \leqslant C^2 \mathbb{E}'\mathbb{E}''\Big\| \sum_{m=1}^M \sum_{n=1}^N \varepsilon_{mn} r_m' r_n'' x_{mn} \Big\|^2.
\end{aligned}$$

Taking $\varepsilon_{mn} = r_{mn}(\omega)$ and taking expectations,

$$\begin{aligned}
\mathbb{E}'\mathbb{E}''\Big\| \sum_{m=1}^M \sum_{n=1}^N r_m' r_n'' x_{mn} \Big\|^2 &\leqslant C^2 \mathbb{E}\mathbb{E}'\mathbb{E}''\Big\| \sum_{m=1}^M \sum_{n=1}^N r_{mn} r_m' r_n'' x_{mn} \Big\|^2 \\
&= C^2 \mathbb{E}'\mathbb{E}''\mathbb{E}\Big\| \sum_{m=1}^M \sum_{n=1}^N r_{mn} r_m' r_n'' x_{mn} \Big\|^2 = C^2 \mathbb{E}\Big\| \sum_{m=1}^M \sum_{n=1}^N r_{mn} x_{mn} \Big\|^2.
\end{aligned}$$

(2)⇒(1): This implication follows from the Kahane contraction principle, which may be applied to the outer terms in (2). □

It can be shown that in (1) and (2), the role of Rademacher variables may be replaced by Gaussian variables without changing the class of spaces under consideration; this only affects the numerical value of the constants in the inequalities (a proof of the easy implication is contained in the proof of Proposition 14.7 below). Furthermore, in both formulations the exponent 2 may be replaced by an arbitrary $p \in [1, \infty)$. For Rademacher variables this was shown in the solution to Exercise 3.3; the proof for Gaussian variables is the same.

Definition 14.5. *A Banach space is said to have* Pisier's property *if it satisfies the equivalent conditions of the proposition.*

Example 14.6. If $(A, \mathscr{A}, \mu)$ is a σ-finite measure space, then for all $1 \leqslant p < \infty$ the space $L^p(A)$ has Pisier's property. More generally, if E has Pisier's property, then $L^p(A; E)$ has Pisier's property.

In view of the remarks preceding the definition, the second assertion follows by switching to power p and using Fubini's theorem. For the first assertion it then remains to be verified that $\mathbb{R}$ has Pisier's property. But this is the content of Exercise 3.3; the same argument shows that every Hilbert space has Pisier's property.

The next proposition connects Pisier's property with the theory of γ-radonifying operators.

Proposition 14.7. *Let H be a Hilbert space. If E has Pisier's property, then one has a canonical isomorphism of Banach spaces*

$$\gamma(L^2(0,T), \gamma(L^2(0,T;H), E)) \simeq \gamma(L^2((0,T)^2; H), E).$$

Proof. As in the proof of Theorem 3.12, from the central limit theorem we deduce that condition (2) of Proposition 14.4 implies its Gaussian counterpart

$$\frac{1}{C^2}\mathbb{E}\Big\| \sum_{j,k=1}^{n} \gamma_{jk} x_{jk} \Big\|^2 \leqslant \mathbb{E}'\mathbb{E}''\Big\| \sum_{j,k=1}^{n} \gamma_j' \gamma_k'' x_{jk} \Big\|^2 \leqslant C^2 \mathbb{E}\Big\| \sum_{j,k=1}^{n} \gamma_{jk} x_{jk} \Big\|^2.$$

Let the sets A_j be measurable and disjoint and also let the sets B_j be measurable and disjoint, and let $h_1, \dots, h_n$ be orthonormal in H. Consider the step function

$$f = \sum_{j,k,l=1}^{n} 1_{A_j} \otimes ((1_{B_k} \otimes h_l) \otimes x_{jkl}) = \sum_{j,k,l=1}^{n} ((1_{A_j} \otimes 1_{B_k}) \otimes h_l) \otimes x_{jkl}.$$

The first sum is interpreted as an element of $\gamma(L^2(0,T), \gamma(L^2(0,T;H), E))$ and the second as an element of $\gamma(L^2((0,T)^2; H), E)$. For such f, the estimate

$$\frac{1}{C}\|f\|_{\gamma(L^2((0,T)^2;H),E)} \leqslant \|f\|_{\gamma(L^2(0,T),\gamma(L^2(0,T;H),E))} \leqslant C\|f\|_{\gamma(L^2((0,T)^2;H),E)}$$

is a reformulation of the above Gaussian estimate. The result follows from this by an approximation argument. $\square$

14.3 Stochastic convolutions

We shall now apply Proposition 14.7 to estimate stochastic convolutions.

Let $S : (0,T) \to \mathscr{L}(E,F)$ be strongly measurable in the sense that Sx is strongly measurable for all $x \in E$. Let W_H be an H-cylindrical Brownian motion, adapted to a filtration $(\mathscr{F}_t)_{t\in[0,T]}$. Given an adapted operator-valued process $\Phi : (0,T) \times \Omega \to \mathscr{L}(H,E)$, we introduce the notation

$$(S \diamond \Phi)(t) := \int_0^t S(t-s)\Phi(s)\, dW_H(s), \quad t \in [0,T],$$

provided these stochastic integrals exist.

Lemma 14.8. *Let E and F be Banach spaces, where F is UMD and has Pisier's property, and let $S : (0,T) \to \mathscr{L}(E,F)$ be as above. Let $\Phi : (0,T) \times \Omega \to \mathscr{L}(H,E)$ be H-strongly measurable and adapted. Let $1 < p < \infty$ and fix $0 \leqslant \theta < \frac{1}{2}$. Suppose that:*

(1) *the set $\{t^\theta S(t) : \ t \in [0,T]\}$ is γ-bounded in $\mathscr{L}(E,F)$;*
(2) *the process $t \mapsto (T-t)^{-\theta}\Phi(t)$ belongs to $L^p(\Omega;\gamma(L^2(0,T;H),E))$.*

Then the process $t \mapsto (T-t)^{-\theta}(S\diamond\Phi)(t)$ is well defined, H-strongly measurable and adapted, and defines an element of $L^p(\Omega;\gamma(L^2(0,T),F))$. Moreover,

$$\begin{aligned}\|t \mapsto (T-t)^{-\theta}(S \diamond \Phi)(t)\|_{L^p(\Omega;\gamma(L^2(0,T),F))} \\ \leqslant CT^{\frac{1}{2}-\theta}\|t \mapsto (T-t)^{-\theta}\Phi(t)\|_{L^p(\Omega;\gamma(L^2(0,T;H),E))},\end{aligned}$$

where C is independent of T and Φ.

Proof. Let us first note that $s \mapsto S(t-s)\Phi(s)$ is H-strongly measurable and adapted on $(0,t)$. Moreover, by Theorem 9.14 and the assumptions (1) and (2), this function defines an element of $L^p(\Omega;\gamma(L^2(0,t;H),F))$. Theorem 13.7 therefore shows that it is L^p-stochastically integrable on $(0,t)$ with respect to W_H. This shows that the process $S \diamond \Phi$ is well-defined. The proof that it is H-strongly measurable and adapted is a bit tedious and is left to the reader.

Let $\Delta := \{(t,s) \in (0,T)^2 : \ 0 < s < t < T\}$. We estimate

$$\begin{aligned}
&\|t \mapsto (T-t)^{-\theta}(S \diamond \Phi)(t)\|_{L^p(\Omega;\gamma(L^2(0,T),F))} \\
&\overset{\text{(i)}}{\eqsim} \Big\| t \mapsto (T-t)^{-\theta}\int_0^t S(t-s)\Phi(s)\, dW_H(s)\Big\|_{\gamma(L^2(0,T),L^p(\Omega;F))} \\
&\overset{\text{(ii)}}{\eqsim} \big\| t \mapsto (T-t)^{-\theta}[s \mapsto 1_{(0,t)}(s)S(t-s)\Phi(s)]\big\|_{\gamma(L^2(0,T),L^p(\Omega;\gamma(L^2(0,T;H),F)))} \\
&\overset{\text{(iii)}}{\eqsim} \big\| t \mapsto (T-t)^{-\theta}[s \mapsto 1_{(0,t)}(s)S(t-s)\Phi(s)]\big\|_{\gamma(L^2(0,T),\gamma(L^2(0,T;H),L^p(\Omega;F)))} \\
&\overset{\text{(iv)}}{\eqsim} \big\|(t,s) \mapsto 1_\Delta(t,s)(T-t)^{-\theta}S(t-s)\Phi(s)\big\|_{\gamma(L^2((0,T)^2;H),L^p(\Omega;F)))} \\
&\overset{\text{(v)}}{\lesssim} \big\|(t,s) \mapsto 1_\Delta(t,s)(T-t)^{-\theta}(t-s)^{-\theta}\Phi(s)\big\|_{\gamma(L^2((0,T)^2;H),L^p(\Omega;E)))}.
\end{aligned}$$

The justification of these steps is as follows: (i) follows from the γ-Fubini isomorphism of Theorem 5.22, (ii) combines Theorem 13.7 with the observation that each bounded operator S from E_1 to F_1 canonically induces a bounded operator from $\gamma(L^2(0,T),E_1)$ to $\gamma(L^2(0,T);F_1)$, (iii) follows again from the γ-Fubini isomorphism, (iv) uses Pisier's property of the space $L^p(\Omega;F)$ (cf. Example 14.6) through Proposition 14.7, and (v) follows from the γ-boundedness assumption.

Consider the operator $P : L^2(0,T;H) \to L^2((0,T)^2;H)$ defined by

$$(Pf)(t,s) := 1_\Delta(t,s)(T-t)^{-\theta}(t-s)^{-\theta}(T-s)^{\theta} f(s).$$

This operator is bounded of norm $\|P\| \leqslant CT^{\frac{1}{2}-\theta}$, since

$$\begin{aligned}
&\int_0^T \int_0^t (T-t)^{-2\theta}(t-s)^{-2\theta}(T-s)^{2\theta}\|f(s)\|^2\,ds\,dt \\
&\quad = \int_0^T (T-s)^{2\theta}\|f(s)\|^2\Big(\int_s^T (T-t)^{-2\theta}(t-s)^{-2\theta}\,dt\Big)\,ds \\
&\quad = \int_0^T (T-s)^{1-2\theta}\|f(s)\|^2\Big(\int_0^1 (1-r)^{-2\theta}r^{-2\theta}\,dr\Big)\,ds \\
&\quad \leqslant C^2T^{1-2\theta}\int_0^T \|f(s)\|^2\,ds,
\end{aligned}$$

where $C^2 := \int_0^1 (1-r)^{-2\theta}r^{-2\theta}\,dr$ depends only on θ. Using the right ideal property of Proposition 5.11 it follows that

$$\begin{aligned}
&\big\|(t,s) \mapsto 1_\Delta(t,s)(T-t)^{-\theta}(t-s)^{-\theta}\Phi(s)\big\|_{\gamma(L^2((0,T)^2;H),L^p(\Omega;E))} \\
&\quad \leqslant CT^{\frac{1}{2}-\theta}\big\|s \mapsto (T-s)^{-\theta}\Phi(s)\big\|_{\gamma(L^2(0,T;H),L^p(\Omega;E))} \\
&\quad \eqsim CT^{\frac{1}{2}-\theta}\big\|s \mapsto (T-s)^{-\theta}\Phi(s)\big\|_{L^p(\Omega;\gamma(L^2(0,T;H),E))}. \qquad \square
\end{aligned}$$

For $\theta \geqslant 0$ and $1 \leqslant p < \infty$ we define the Banach space

$$V_\theta^p(\Omega;\gamma(L^2(0,T;H),E))$$

as the completion of the space of all adapted finite rank step processes $\Phi : (0,T)\times\Omega \to \mathscr{L}(H,E)$ with respect to the norm

$$\|\Phi\|_{V_\theta^p(\Omega;\gamma(L^2(0,T;H),E))} := \sup_{t\in(0,T]} \|s \mapsto (t-s)^{-\theta}\Phi(s)\|_{L^p(\Omega;\gamma(L^2(0,t;H),E))}.$$

We write $V_\theta^p(\Omega;\gamma(L^2(0,T),E))$ instead of $V_\theta^p(\Omega;\gamma(L^2(0,T;\mathbb{R}),E))$.

By applying Lemma 14.8 to the subintervals $(0,t)$ we obtain:

Proposition 14.9. *Let E, F, S, θ be as in Lemma 14.8. Then for all $1 < p < \infty$ the stochastic convolution $\Phi \mapsto S \diamond \Phi$ acts as a bounded linear operator from $V_\theta^p(\Omega;\gamma(L^2(0,T),E))$ to $V_\theta^p(\Omega;\gamma(L^2(0,T;H),F))$ of norm $\leqslant CT^{\frac{1}{2}-\theta}$.*

For γ-Lipschitz continuous mappings $B : E \to \mathscr{L}(H, F)$ we have the following mapping property:

Proposition 14.10. *If $B : E \to \mathscr{L}(H, F)$ is γ-Lipschitz continuous, then for all $\theta \geqslant 0$ and $1 \leqslant p < \infty$ the map B acts as a Lipschitz continuous mapping from $V_\theta^p(\Omega; \gamma(L^2(0,T;H),E))$ to $V_\theta^p(\Omega; \gamma(L^2(0,T),F))$ with Lipschitz constant $\leqslant \mathrm{Lip}_\gamma(B)$.*

Proof. For $t \in (0,T)$ let $\mu_{t,\theta}$ be the finite Borel measure on $(0,t)$ defined by

$$\mu_{t,\theta}(A) = \int_A (t-s)^{-2\theta}\, ds, \quad A \in \mathscr{B}(0,t).$$

The result follows from Proposition 14.1 and the observation that for an H-strongly measurable function $\Psi : (0,T) \to \mathscr{L}(H,E)$ the following assertions are equivalent:

(1) $s \mapsto (t-s)^{-\theta}\Psi(s)$ defines an element of $\gamma(L^2(0,t;H),E)$;
(2) $s \mapsto \Psi(s)$ defines an element of $\gamma(L^2((0,t),\mu_{t,\theta};H),E)$.

This equivalence is a consequence of the fact that the functions $h_1, \dots, h_n$ are orthonormal in $L^2((0,t),\mu_{t,\theta};H)$ if and only if the functions $s \mapsto (t-s)^{-\theta}h_1(s),\ \dots,\ s \mapsto (t-s)^{-\theta}h_n(s)$ are orthonormal in $L^2(0,t;H)$. $\square$

14.4 Existence and uniqueness

After these preparations we are ready to prove existence and uniqueness of solutions for the problem (SCP).

We fix an initial value $u_0 \in L^p(\Omega, \mathscr{F}_0; E)$ and consider the fixed point map L_T, initially defined for step functions $\phi : (0,T) \to E$ by

$$L_T(\phi) := Su_0 + S \diamond B(\phi),$$

where for brevity we write $(Su_0)(t) := S(t)u_0$.

The next result formulates a set of conditions ensuring that L_T be well-defined on $V_\theta^p(\Omega; \gamma(L^2(0,T),E))$.

Proposition 14.11. *Let E be a UMD space with Pisier's property and let $1 < p < \infty$. Let A be the generator of a C_0-semigroup S on E such that $\{t^\theta S(t) :\ t \in (0,T)\}$ is γ-bounded for some $0 \leqslant \theta < \frac{1}{2}$, and let $B : E \to \gamma(H,E)$ be γ-Lipschitz continuous. Then the mapping L_T is well-defined and Lipschitz continuous on $V_\theta^p(\Omega; \gamma(L^2(0,T),E))$ and there exists a constant $C \geqslant 0$, independent of T and u_0, such that:*

(1) *for all $\phi \in V_\theta^p(\Omega; \gamma(L^2(0,T),E))$,*

$$\|L_T(\phi)\|_{V_\theta^p(\Omega;\gamma(L^2(0,T),E))} \leqslant CT^{\frac{1}{2}-\theta}\big(T^{1-2\theta} + \|u_0\|_p + \|\phi\|_{V_\theta^p(\Omega;\gamma(L^2(0,T),E))}\big);$$

(2) *for all* $\phi_1, \phi_2 \in V_\theta^p(\Omega;\gamma(L^2(0,T),E))$,

$$\|L_T(\phi_1) - L_T(\phi_2)\|_{V_\theta^p(\Omega;\gamma(L^2(0,T),E))} \leqslant CT^{\frac{1}{2}-\theta}\|\phi_1 - \phi_2\|_{V_\theta^p(\Omega;\gamma(L^2(0,T),E))}.$$

Proof. We begin by estimating the initial value part. By Lemma 10.17 and Theorem 9.14, for all $t \in (0,T]$ the following estimate holds for almost all $\omega \in \Omega$:

$$\begin{aligned}
&\|s \mapsto (t-s)^{-\theta} S(s)u_0(\omega)\|_{\gamma(L^2(0,t),E)} \\
&\quad\lesssim t^\theta \|s \mapsto s^{-\theta}(t-s)^{-\theta} u_0(\omega)\|_{\gamma(L^2(0,t),E)} \\
&\quad= t^{\frac{1}{2}-\theta}\|s \mapsto s^{-\theta}(t-s)^{-\theta}\|_{L^2(0,t)}\|u_0(\omega)\| \\
&\quad\eqsim t^{\frac{1}{2}-\theta}\|u_0(\omega)\|,
\end{aligned}$$

with a constant independent of u_0 and $t \in (0,T)$. In the third line, the equality follows Exercise 5.3. Hence,

$$\begin{aligned}
&\|Su_0\|_{V_\theta^p(\Omega;\gamma(L^2(0,T),E))} \\
&\quad= \sup_{t\in(0,T]} \|s \mapsto (t-s)^{-\theta} S(s)u_0\|_{L^p(\Omega;\gamma(L^2(0,t;H),E))} \lesssim T^{\frac{1}{2}-\theta}\|u_0\|_p.
\end{aligned} \tag{14.2}$$

Fix adapted step processes $\phi_1, \phi_2 : (0,T) \times \Omega \to E$. If B is γ-Lipschitz, Propositions 14.9 and 14.10 show that $B(\phi_k) \in V_\theta^p(\Omega;\gamma(L^2(0,T;H),E))$, $S \diamond B(\phi_k) \in V_\theta^p(\Omega;\gamma(L^2(0,T),E))$, $k = 1,2$, and

$$\begin{aligned}
&\|S \diamond B(\phi_1) - S \diamond B(\phi_2))\|_{V_\theta^p(\Omega;\gamma(L^2(0,T),E))} \\
&\quad\leqslant CT^{\frac{1}{2}-\theta}\|B(\phi_1) - B(\phi_2)\|_{V_\theta^p(\Omega;\gamma(L^2(0,T;H),E))} \\
&\quad\leqslant CT^{\frac{1}{2}-\theta}\mathrm{Lip}_\gamma(B)\|\phi_1 - \phi_2\|_{V_\theta^p(\Omega;\gamma(L^2(0,T),E))}.
\end{aligned}$$

It follows from these estimates that L_T has a unique extension to a Lipschitz continuous mapping on $V_\theta^p(\Omega;\gamma(L^2(0,T),E))$ which satisfies the estimate of (2). The estimate (1) follows from the identity $L_T(\phi) = L_T(0) + (L_T(\phi) - L_T(0))$ and (2), using that from (14.2) and Proposition 14.9 we obtain

$$\begin{aligned}
&\|L_T(0)\|_{V_\theta^p(\Omega;\gamma(L^2(0,T),E))} \\
&\quad\lesssim T^{\frac{1}{2}-\theta}\big(\|u_0\|_p + \|1_{(0,T)} \otimes B(0)\|_{V_\theta^p(\Omega;\gamma(L^2(0,T),E))}\big) \\
&\quad\leqslant T^{\frac{1}{2}-\theta}\big(\|u_0\|_p + T^{1-2\theta}\|B(0)\|_{\gamma(H,E)}\big).
\end{aligned}$$ □

After these preparations we are ready to formulate our main result for existence and uniqueness of mild solutions for the stochastic evolution equation (SCP). We denote by S the C_0-semigroup generated by A.

Definition 14.12. *Let $\theta \geqslant 0$ and $1 \leqslant p < \infty$. A strongly measurable and adapted process $U : [0,T] \times \Omega \to E$ is called a* mild V_θ^p-solution *of the problem* (SCP) *if it belongs to $V_\theta^p(\Omega;\gamma(L^2(0,T),E))$ and for all $t \in [0,T]$ the following identity holds almost surely:*

$$U(t) = S(t)u_0 + (S \diamond B(U))(t).$$

A mild V_0^p-solution is called a mild L^p-solution.

This definition is motivated by the formula $U(t) = S(t)u_0 + (S \diamond B)(t)$ for the unique weak solution of the problem $dU(t) = AU(t)\,dt + B\,dW_H(t)$ which was studied in Lectures 8–10 (and corresponds to the special case $B(x) \equiv B$).

Theorem 14.13 (Existence and uniqueness). *Let E be a UMD space with Pisier's property and let $1 < p < \infty$. Suppose that A is the generator of a C_0-semigroup S on E such that $\{t^\theta S(t) : t \in [0,T]\}$ is γ-bounded for some $0 \leqslant \theta < \frac{1}{2}$, let $B : E \to \gamma(H,E)$ be γ-Lipschitz continuous, and let $u_0 \in L^p(\Omega, \mathscr{F}_0; E)$. Then there exists a unique mild V_θ^p-solution U of* (SCP). *Moreover, there exists a constant $C_T \geqslant 0$, independent of u_0, such that*

$$\|U\|_{V_\theta^p(\Omega;\gamma(L^2(0,T;H),E))} \leqslant C_T(1 + \|u_0\|_p).$$

Here, uniqueness is understood in the sense of $V_\theta^p(\Omega;\gamma(L^2(0,T),E))$. By strong measurability, any two solutions representing the same element in this space are versions of each other.

Proof. By Proposition 14.11 we can find $0 < T_0 \leqslant T$, independent of u_0, such that

$$\|L_{T_0}(\phi_1) - L_{T_0}(\phi_2)\|_{V_\theta^p(\Omega;\gamma(L^2(0,T_0),E))} \leqslant \frac{1}{2}\|\phi_1 - \phi_2\|_{V_\theta^p(\Omega;\gamma(L^2(0,T_0),E))} \quad (14.3)$$

for all $\phi_1, \phi_2 \in V_\theta^p(\Omega;\gamma(L^2(0,T_0),E))$ and

$$\|L_{T_0}(\phi)\|_{V_\theta^p(\Omega;\gamma(L^2(0,T_0),E))} \leqslant \frac{1}{2}\big(1 + \|u_0\|_p + \|\phi\|_{V_\theta^p(\Omega;\gamma(L^2(0,T_0),E))}\big) \quad (14.4)$$

for $\phi \in V_\theta^p(\Omega;\gamma(L^2(0,T_0),E))$. By (14.3) and the Banach fixed point theorem, L_{T_0} has a unique fixed point $\widetilde{U} \in V_\theta^p(\Omega;\gamma(L^2(0,T_0),E))$. Define the strongly measurable adapted process $U : [0,T_0] \times \Omega \to E$ by

$$U(t) := S(t)u_0 + (S \diamond B(\widetilde{U}))(t).$$

Then U is a mild V_θ^p-solution, and clearly we have $U = \widetilde{U}$ as elements of $V_\theta^p(\Omega;\gamma(L^2(0,T_0),E))$. Uniqueness in $V_\theta^p(\Omega;\gamma(L^2(0,T_0),E))$ follows from the uniqueness of the fixed point in that space. Noting that $\widetilde{U} = L_{T_0}(\widetilde{U})$, the estimate (14.4) implies the final estimate on the interval $[0,T_0]$.

Via a standard induction argument we now construct a mild solution on each of the intervals $[T_0, 2T_0], \dots, [(n-1)T_0, nT_0], [nT_0, T]$, where n is an appropriate integer. This results in a mild solution U on $[0,T]$ of (SCP) with the properties as stated in the theorem. Uniqueness on $[0,T]$ follows by induction from the uniqueness on each of the subintervals. We leave the somewhat tedious details as an exercise (see Exercise 5). $\square$

Let us have a closer look at this theorem for the special case where E is a Hilbert space. Then E is a UMD space with Pisier's property, the family $\{S(t) :\ t \in [0,T]\}$ is γ-bounded (since in Hilbert spaces, uniformly bounded families are γ-bounded), and every Lipschitz continuous function $B : E \to \gamma(H,E) = \mathscr{L}_2(H,E)$ is γ-Lipschitz continuous (since Hilbert spaces have type 2, cf. Example 14.3); recall that $\mathscr{L}_2(H,E)$ denotes the space of all Hilbert-Schmidt operators from H to E.

Corollary 14.14 (Hilbert space case). *Let E be a Hilbert space and let $1 < p < \infty$. Suppose that A is the generator of a C_0-semigroup on E, let $B : E \to \mathscr{L}_2(H,E)$ be Lipschitz continuous, and let $u_0 \in L^p(\Omega, \mathscr{F}_0; E)$. Then there exists a unique mild L^p-solution of* (SCP). *Moreover, there exists a constant $C_T \geqslant 0$, independent of u_0, such that*

$$\|U\|_{L^p(\Omega;\gamma(L^2(0,T),E))} \leqslant C_T(1 + \|u_0\|_p).$$

14.5 Space-time regularity

To motivate our approach we return to the proof Theorem 10.19, where space-time Hölder regularity of solutions was proved under the assumption that the semigroup S generated by A is analytic. The crucial ingredient was the γ-boundedness of the family $\{t^\theta S(t) :\ t \in (0,T)\}$ in $\mathscr{L}(E, E_\alpha)$ for $0 \leqslant \alpha < \theta < \frac{1}{2}$. Recall that the spaces E_α have been defined in Lecture 10 as the fractional domain spaces $\mathscr{D}((w - A)^\alpha)$.

As the next proposition shows, for processes in $V_\theta^p(\Omega; \gamma(L^2(0,T),E))$ the proof of Theorem 10.19 can be repeated.

Proposition 14.15. *Let A be the generator of an analytic C_0-semigroup S on a UMD space E. Suppose that $2 < p < \infty$ and $\frac{1}{p} < \theta < \frac{1}{2}$ and let $\alpha \geqslant 0$ and $\beta \geqslant 0$ satisfy $0 \leqslant \alpha + \beta < \theta - \frac{1}{p}$. If $\Phi \in V_\theta^p(\Omega; \gamma(L^2(0,t;H),E))$, then $S \diamond \Phi$ has a version with trajectories in $C^\beta([0,T]; E_\alpha)$ and*

$$\|S \diamond \Phi\|_{L^p(\Omega;C^\beta([0,T];E_\alpha))} \leqslant C_T \|\Phi\|_{V_\theta^p(\Omega;\gamma(L^2(0,t;H),E))},$$

where the constant $C_T \geqslant 0$ is independent of Φ.

Proof. First note that the γ-boundedness of $\{t^\theta S(t) :\ t \in (0,T)\}$ in $\mathscr{L}(E, E_\alpha)$ implies that $(S \diamond \Phi)(t) \in L^p(\Omega; E_\alpha)$ for all $t \in [0,T]$. Fix $0 < s < t \leqslant T$ and write

$$(\mathbb{E}\|S \diamond \Phi(t) - S \diamond \Phi(s)\|_{E_\alpha}^p)^{\frac{1}{p}} \leqslant R_1 + R_2,$$

where

$$R_1 = \Big(\mathbb{E}\Big\| \int_0^s S(t-r) - S(s-r)\Phi(r)\, dW_H(r)\Big\|_{E_\alpha}^p\Big)^{\frac{1}{p}},$$

$$R_2 = \Big(\mathbb{E}\Big\| \int_s^t S(t-r)\Phi(r)\, dW_H(r)\Big\|_{E_\alpha}^p\Big)^{\frac{1}{p}}.$$

Let $\beta' > \beta$ satisfy $\alpha + \beta' < \theta - \frac{1}{p}$ and put $\delta := \beta' + \frac{1}{p}$. Then $\alpha + \delta < \theta$ and $\beta < \delta - \frac{1}{p}$. By Lemma 10.17, and Theorems 9.14 and 13.7, for large w we have

$$\begin{aligned}
R_1^p &\lesssim \mathbb{E}\|r \mapsto S(s-r)(S(t-s)-I)\Phi(r)\|^p_{\gamma(L^2(0,s;H),E_\alpha)} \\
&\eqsim \mathbb{E}\|r \mapsto S(s-r)(S(t-s)-I)(w-A)^{-\delta}\Phi(r)\|^p_{\gamma(L^2(0,s;H),E_{\alpha+\delta})} \\
&\lesssim T^{(\theta-\alpha-\delta)p}\mathbb{E}\|r \mapsto (s-r)^{-\theta}(S(t-s)-I)(w-A)^{-\delta}\Phi(r)\|^p_{\gamma(L^2(0,s;H),E)} \\
&\lesssim T^{(\theta-\alpha-\delta)p}(t-s)^{\delta p}\mathbb{E}\|r \mapsto (s-r)^{-\theta}\Phi(r)\|^p_{\gamma(L^2(0,s;H),E)} \\
&\lesssim T^{(\theta-\alpha-\delta)p}(t-s)^{\delta p}\|\Phi\|^p_{V^p_\theta(\Omega;\gamma(L^2(0,T;H),E))}.
\end{aligned}$$

In the second last estimate we used that $\|(S(t-s)-I)(w-A)^{-\delta}\| \lesssim |t-s|^\delta$ by the analyticity of S (see Lemma 10.15). Similarly,

$$\begin{aligned}
R_2^p &\lesssim \mathbb{E}\|r \mapsto S(t-r)\Phi(r)\|^p_{\gamma(L^2(s,t;H),E_\alpha)} \\
&\lesssim T^{(\theta-\delta-\alpha)p}\mathbb{E}\|r \mapsto (t-r)^{-\theta+\delta}\Phi(r)\|^p_{\gamma(L^2(s,t;H),E)} \\
&\lesssim T^{(\theta-\delta-\alpha)p}(t-s)^{\delta p}\mathbb{E}\|r \mapsto (t-r)^{-\theta}\Phi(r)\|^p_{\gamma(L^2(s,t;H),E)} \\
&\lesssim T^{(\theta-\delta-\alpha)p}(t-s)^{\delta p}\|\Phi\|^p_{V^p_\theta(\Omega;\gamma(L^2(0,T;H),E))},
\end{aligned}$$

where the second last inequality follows by covariance domination. Combining these estimates with Kolmogorov's theorem (Theorem 6.9) and using that $\beta < (\delta p - 1)/p = \delta - \frac{1}{p}$, we obtain a version of $S \diamond \Phi$ which is β-Hölder continuous in E_α. $\square$

We are now ready to formulate our main regularity result for the mild solutions of problem (SCP).

Theorem 14.16 (Hölder Regularity). *Let A be the generator of an analytic C_0-semigroup S on a UMD space E with Pisier's property. Suppose that $B : E \to \mathscr{L}(H,E)$ is γ-Lipschitz continuous and let $u_0 \in L^p(\Omega, \mathscr{F}_0; E)$. For all $\alpha, \beta, \theta \geqslant 0$ satisfying $\alpha + \beta < \theta < \frac{1}{2}$ and all $1 < p < \infty$, the unique mild V^p_θ-solution U of the problem* (SCP) *has a version for which $U - Su_0$ has trajectories in $C^\beta([0,T]; E_\alpha)$.*

Note that if u_0 is sufficiently regular, this result implies that U itself has a version with trajectories in $C^\beta([0,T]; E_\alpha)$.

Proof. The existence of a unique mild V^p_θ-solution follows from Theorem 14.13; the γ-boundedness assumption holds by the analyticity of S.

If U is a mild V^p_θ-solution and $\widetilde{U}$ is mild V^q_θ-solution, where $1 < p \leqslant q < \infty$, then $\widetilde{U}$ is also a mild V^p_θ-solution. Hence by uniqueness, U and $\widetilde{U}$ are equal as elements of $V^p_\theta(\Omega; \gamma(L^2(0,T),E))$, and by strong measurability U and $\widetilde{U}$ are versions of each other. Therefore it suffices to consider the case where $2 < p < \infty$ satisfies $\alpha + \beta < \theta - \frac{1}{p} < \frac{1}{2} - \frac{1}{p}$.

By Proposition 14.9, $S \diamond B(U)$ belongs to $V_\theta^p(\Omega;\gamma(L^2(0,t;H),E_\alpha))$, where U is the mild V_θ^p-solution U of (SCP). Hence, by Proposition 14.15, $U - Su_0 = S \diamond B(U)$ has a version with trajectories in $C^\beta([0,T];E_\alpha)$ and

$$\begin{aligned}\mathbb{E}\|S \diamond B(U)\|^p_{C^\beta([0,T];E_\alpha)} &\leqslant C^p \|B(U)\|^p_{V_\theta^p(\Omega;\gamma(L^2(0,T;H),E))} \\ &\leqslant C^p(1 + \|U\|_{V_\theta^p(\Omega;\gamma(L^2(0,T),E))})^p \leqslant C^p(1+\|u_0\|_p)^p,\end{aligned}$$

where the last of these estimates follows from Theorem 14.13. □

14.6 Exercises

1. Provide the details of the central limit argument in Proposition 14.7.
2. Show that if E is a Banach space with the property that the mapping

$$\sum_{j,k=1}^{n} 1_{A_j}(1_{B_k} \otimes x_{jk}) \mapsto \sum_{j,k=1}^{n} (1_{A_j} 1_{B_k}) \otimes x_{jk}$$

(with notations as in Proposition 14.7) induces an isomorphism

$$\gamma(L^2(0,T),\gamma(L^2(0,T),E)) \simeq \gamma(L^2((0,T)^2),E),$$

then E has Pisier's property (in the formulation using Gaussian random variables; as has been noted without proof, this formulation is equivalent to the one with Rademachers given in the text). This gives a converse to Proposition 14.7.

3. Let H be a Hilbert space, E and F Banach spaces, and assume that E has cotype 2 and F has type 2. Show that every Lipschitz continuous function $B : E \to \gamma(H,F)$ is γ-Lipschitz continuous with

$$\mathrm{Lip}(B) \leqslant \mathrm{Lip}_\gamma(B) \leqslant C_2(E)T_2(F)\mathrm{Lip}(B),$$

where $C_2(E)$ and $T_2(F)$ denote the Gaussian cotype 2 constant of E and the type 2 constant of F, respectively.
Hint: Use the results of Exercise 5.4.

4. Frequently, uniqueness proofs are based on Gronwall's inequality. The purpose of this exercise is to show that the 'γ-Gronwall inequality' fails in spaces without type 2.
   a) Show that if E is a Banach space without type 2, then there exist step functions $\phi_n : (\frac{1}{n+1}, \frac{1}{n}) \to E$ such that

$$\|\phi_n\|_{L^\infty(\frac{1}{n+1},\frac{1}{n};E)} \leqslant 1, \quad \inf_{n\geqslant 1} \|\phi_n\|_{\gamma(L^2(\frac{1}{n+1},\frac{1}{n}),E)} > 0.$$

   b) Prove that the following assertions are equivalent:

(i) the space E has type 2;
(ii) whenever $\phi : (0,1) \to E$ is a strongly measurable function representing an element of $\gamma(L^2(0,1),E)$ and there exists a constant $C = C_\phi \geqslant 0$ such that

$$\|\phi(t)\| \leqslant C\|\phi\|_{\gamma(L^2(0,t),E)} \text{ for almost all } t \in (0,1),$$

we have $\phi = 0$ almost everywhere on $(0,1)$.
Hint: In one direction, consider the function $\phi(t) := \frac{1}{n^2}\phi_n(t)$ for $t \in (t_{n+1}, t_n]$, where ϕ_n is as in a). In the other direction, use Gronwall's inequality.

5. Provide the details of the induction argument that was used at the end of the proof of Theorem 14.13.

Notes. The material of Section 14.1 and 14.3 is based on the paper [83]. Exercise 3 is a variation on a result of that paper. In [80], the following converse is proved: if every Lipschitz function $B : E \to F$ is γ-Lipschitz, then E has cotype 2 and F has type 2.

Pisier's property was introduced, under the name 'property (α)', by Pisier [92] who proved that a Banach lattice has this property if and only if it has finite cotype. Proposition 14.4 and the equivalence with its Gaussian formulation belong to mathematical folklore. It should be noted that the UMD property and Pisier's property are unrelated: the Schatten classes C^p have the UMD property for $1 < p < \infty$ but fail Pisier's property unless $p = 2$, whereas L^1-spaces have Pisier's property but fail the UMD property unless they are finite-dimensional.

Proposition 14.7 is a special case of the more general statement that if H_1 and H_2 are Hilbert spaces and E is a Banach space with Pisier's property, then

$$\gamma(H_1, \gamma(H_2, E)) \simeq \gamma(H_1 \widehat{\otimes} H_2, E)$$

isomorphically, where $H_1 \widehat{\otimes} H_2$ is the Hilbert space tensor product of H_1 and H_2. Exercise 2 can be formulated similarly. Both results are due to Kalton and Weis [58].

The use of Pisier's property can be avoided in Lemma 14.8 and all results depending on it, but it would take a full lecture to explain all the details. The interested reader is referred to [83]. Previous results along these lines for Hilbert spaces can be found in Da Prato and Zabczyk [27]; they were extended to martingale type 2 spaces by Brzeźniak [14]. In this context it should be noted that if S is a C_0-*contraction* semigroup on a Hilbert space E, then by a result of Kotelenez [60] and Tubaro [104] the convolution process

$$t \mapsto \int_0^t S(t-s)\Phi(s)\, dW_H(s)$$

has a continuous version for all adapted and H-strongly measurable $\Phi : (0,T) \times \Omega \to \mathscr{L}_2(H,E)$; see also Da Prato and Zabczyk [27, Theorem

6.10]. As a result, in this situation the solution of Theorem 14.14 has a continuous version.

The results of Sections 14.4 and 14.5 are based on the paper [83]. The main results, Theorem 14.13 and 14.16, are variations of results in that paper and can be extended to semilinear parabolic equations with time-dependent coefficients of the form

$$\begin{cases} dU(t) = (AU(t) + F(t, U(t)))\,dt + B(t, U(t))\,dW_H(t), \\ U(0) = u_0. \end{cases}$$

Sufficient conditions for a mild solution to be a weak solution (which is defined in analogy to Lecture 8) and vice versa are given by Da Prato and Zabczyk [27] and Veraar [106].