# 15

# Applications to stochastic PDE

In this final lecture we present some applications of the theory developed in this course to stochastic partial differential equations. We concentrate on two specific examples: the wave equation and the heat equation.

## 15.1 Space-time white noise

It has been mentioned already in Lecture 6 that for $H = L^2(D)$, where $D$ is a domain in $\mathbb{R}^d$, $H$-cylindrical Brownian motions can be used to model space-time white noise on $D$. We begin by making this idea more precise.

**Definition 15.1.** *Let $(A, \mathscr{A}, \mu)$ be a $\sigma$-finite measure space and denote by $\mathscr{A}_0$ the collection of all $B \in \mathscr{A}$ such that $\mu(B) < \infty$. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. A* white noise *on $(A, \mathscr{A}, \mu)$ is a mapping $w : \mathscr{A}_0 \to L^2(\Omega)$ such that:*

(i) *each $w(B)$ is centred Gaussian with*

$$\mathbb{E}(w(B))^2 = \mu(B);$$

(ii) *if $B_1 \cap \cdot \cap B_N = \varnothing$, then $w(B_1), \dots, w(B_N)$ are independent and*

$$w\Big(\bigcup_{n=1}^N B_n\Big) = \sum_{n=1}^N w(B_n).$$

It follows from the general theory of Gaussian processes that such mappings always exist. We shall not go into the details of this, since in all applications the white noise is assumed to be given.

**Definition 15.2.** *A white noise $w$ on $[0, T] \times D$, where $D$ is a domain in $\mathbb{R}^d$, will be called a* space-time white noise *on $D$.*

Canonically associated with such $w$ is an $L^2(D)$-cylindrical Brownian motion $W$, defined by

$$W(t)1_B := w([0,t] \times B), \quad B \in \mathscr{B}_0(D);$$

this definition is extended to simple functions by linearity. To see that $W$ is indeed an $L^2(D)$-cylindrical Brownian motion note that for disjoint $B_1, \dots, B_N \in \mathscr{B}_0(D)$ and real numbers $c_1, \dots, c_N$ we have, by (i) and (ii),

$$\begin{aligned} \mathbb{E}\Big(W(t)\sum_{n=1}^N c_n 1_{B_n}\Big)^2 &= \sum_{n=1}^N c_n^2 \mathbb{E}(w([0,t] \times B_n))^2 \\ &= \sum_{n=1}^N c_n^2 t|B_n| = t\Big\|\sum_{n=1}^N c_n 1_{B_n}\Big\|^2_{L^2(D)}. \end{aligned}$$

## 15.2 The stochastic wave equation

In this section we study the stochastic wave equation with Dirichlet boundary conditions, driven by multiplicative space-time white noise:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,\xi) = \Delta u(t,\xi) + B\Big(u(t,\xi), \dfrac{\partial u}{\partial t}(t,\xi)\Big)\dfrac{\partial w}{\partial t}(t,\xi), & \xi \in D,\ t \in [0,T], \\ u(t,\xi) = 0, & \xi \in \partial D,\ t \in [0,T], \\ u(0,\xi) = u_0(\xi), & \xi \in D, \\ \dfrac{\partial u}{\partial t}(0,\xi) = v_0(\xi), & \xi \in D. \end{cases} \tag{WE}$$

Here $w$ is a space-time white noise on a bounded domain $D$ in $\mathbb{R}^d$ with smooth boundary $\partial D$.

In order to keep the technicalities at a minimum we discuss two special cases in detail: the case where the operator-valued function $B$ is of rank one, which is equivalent to the formulation (WE1) below, and the case where $D$ is the unit interval in $\mathbb{R}$ and $B = I$.

### 15.2.1 Rank one multiplicative noise

We begin with the following special case of (WE):

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,\xi) = \Delta u(t,\xi) + b\Big(u(t,\xi), \dfrac{\partial u}{\partial t}(t,\xi)\Big)\dfrac{\partial W}{\partial t}(t,\xi), & \xi \in D,\ t \in [0,T], \\ u(t,\xi) = 0, & \xi \in \partial D,\ t \in [0,T], \\ u(0,\xi) = u_0(\xi), & \xi \in D, \\ \dfrac{\partial u}{\partial t}(0,\xi) = v_0(\xi), & \xi \in D, \end{cases} \tag{WE1}$$

where $W$ is a standard Brownian motion. We assume that the diffusion term $b : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the growth condition

$$|b(\xi_1, \xi_2)|^2 \leqslant C_1^2(|\xi_1|^2 + |\xi_2|^2)$$

and the Lipschitz condition

$$|b(\xi_1, \xi_2) - b(\eta_1, \eta_2)|^2 \leqslant C_2^2(|\xi_1 - \eta_1|^2 + |\xi_2 - \eta_2|^2).$$

The initial values $u_0$ and $v_0$ are taken in $W^{1,2}(D)$ and $L^2(D)$, respectively.

Writing the first equation as a system of two first order equations,

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,\xi) = v(t,\xi), \\ \dfrac{\partial v}{\partial t}(t,\xi) = \Delta u(t,\xi) + b(u(t,\xi), v(t,\xi))\dfrac{\partial W}{\partial t}(t,\xi), \end{cases} \quad \xi \in D,\ t \in [0,T], \tag{15.1}$$

we reformulate the problem (WE) as a first order stochastic evolution equation as follows. Let $\Delta$ denote the Dirichlet Laplacian on $L^2(D)$ with domain $\mathscr{D}(\Delta) = W^{2,2}(D) \cap W_0^{1,2}(D)$; see Examples 7.21. On the Hilbert space

$$\mathscr{H} := \mathscr{D}((-\Delta)^{1/2}) \times L^2(D) = W^{1,2}(D) \times L^2(D)$$

we define the operator

$$A := \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix}$$

with domain $\mathscr{D}(A) := \mathscr{D}(\Delta) \times \mathscr{D}((-\Delta)^{1/2}) = (W^{2,2}(D) \cap W_0^{1,2}(D)) \times W^{1,2}(D)$. As in Example 7.22, this operator is the generator of a bounded $C_0$-group on $\mathscr{H}$, and we may reformulate the problem (15.1) as an abstract stochastic evolution equation of the form

$$\begin{cases} dU(t) = AU(t)\,dt + B(U(t))\,dW(t), \\ U(0) = U_0, \end{cases} \tag{15.2}$$

where $W$ is a Brownian motion and the function $B : \mathscr{H} \to \mathscr{H}$ is the *Nemytskii map* associated with $b$,

$$B\begin{bmatrix} f \\ g \end{bmatrix} := \begin{bmatrix} 0 \\ b(f,g) \end{bmatrix}, \qquad \begin{bmatrix} f \\ g \end{bmatrix} \in \mathscr{H},$$

and $U_0 := \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in \mathscr{H}$.

**Proposition 15.3.** *Under the above assumptions on $b$, the Nemytskii map $B : \mathscr{H} \to \mathscr{H}$ is well defined, Lipschitz continuous with* $\mathrm{Lip}(B) \leqslant \mathrm{Lip}(b)$*, and of linear growth.*

*Proof.* For all $(f,g) \in \mathscr{H}$ we have

$$\begin{aligned}\|B(f,g)\|_{\mathscr{H}}^2 &= \int_D |b(f(\xi),g(\xi))|^2 \, d\xi \\ &\lesssim \int_D |f(\xi)|^2 + |g(\xi)|^2 \, d\xi \lesssim \|f\|_2^2 + \|g\|_2^2 \leqslant \|(f,g)\|_{\mathscr{H}}^2.\end{aligned}$$

A similar estimate gives that $B$ is Lipschitz continuous from $\mathscr{H}$ to $\mathscr{H}$ with $\mathrm{Lip}(B) \leqslant \mathrm{Lip}(b)$. □

We say that a measurable adapted process $u : [0,T] \times \Omega \times D \to \mathbb{R}$ is a *mild $L^p$-solution* of (WE) if $U(t,\omega) := \begin{bmatrix} u(t,\omega,\cdot) \\ \frac{\partial u}{\partial t} u(t,\omega,\cdot) \end{bmatrix}$ belongs to $\mathscr{H}$ for all $(t,\omega) \in [0,T] \times \Omega$ and the resulting process $U : [0,T] \times \Omega \to \mathscr{H}$ is a mild $L^p$-solution of the problem (15.2).

**Theorem 15.4.** *Under the above assumptions, for all $1 < p < \infty$ the problem* (WE) *admits a unique mild $L^p$-solution.*

Here, uniqueness is understood in the sense of $L^p(\Omega; \gamma(L^2(0,T), \mathscr{H}))$.

*Proof.* By Proposition 15.3, the Nemytskii operator $B$ associated with $b$ is Lipschitz continuous. Moreover, as we have seen in Example 7.22, the operator $A$ is the generator of a $C_0$-group on $\mathscr{H}$. We have thus checked all assumptions of Corollary 14.14 (with $H = \mathbb{R}$ and $E = \mathscr{H}$) and conclude that for all $1 < p < \infty$ the problem (WE) admits a unique mild $L^p$-solution. □

### 15.2.2 Additive space-time white noise

Our next example concerns the stochastic wave equation with additive space-time white noise on the unit interval $(0,1)$ in $\mathbb{R}$:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,\xi) = \Delta u(t,\xi) + \dfrac{\partial w}{\partial t}(t,\xi), & \xi \in (0,1), \quad t \in [0,T], \\ u(t,0) = u(t,1) = 0, & t \in [0,T], \\ u(0,\xi) = u_0(\xi), & \xi \in (0,1), \\ \dfrac{\partial u}{\partial t}(0,\xi) = v_0(\xi), & \xi \in (0,1). \end{cases} \tag{WE2}$$

Here $w$ is a space-time white noise on $(0,1)$. We model this problem as an abstract stochastic evolution equation on the Hilbert space $\mathscr{H} = W^{1,2}(0,1) \times L^2(0,1)$ as

$$\begin{cases} dU(t) = AU(t)\,dt + d\begin{bmatrix} 0 \\ W_{L^2}(t) \end{bmatrix}, \\ U(0) = U_0, \end{cases} \tag{15.3}$$

where as before $A = \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix}$, with $\Delta$ the Dirichlet Laplacian on $L^2(0,1)$, and $W_{L^2}$ is the $L^2(0,1)$-cylindrical Brownian motion canonically associated with $w$; see Section 15.1.

To analyse the problem (15.3) we use the functional calculus for self-adjoint operators. Using this calculus it can be checked that the $C_0$-group $S$ generated by $A$ is of the form

$$S(t) = \begin{bmatrix} \cos(t(-\Delta)^{1/2}) & (-\Delta)^{-1/2}\sin(t(-\Delta)^{1/2}) \\ -(-\Delta)^{1/2}\sin(t(-\Delta)^{1/2}) & \cos(t(-\Delta)^{1/2}) \end{bmatrix}.$$

By Theorem (8.6), the unique weak solution $U$ of (15.3) is given by

$$U(t) = \int_0^t S(t-s)\,d\begin{bmatrix} 0 \\ W(s) \end{bmatrix} = \int_0^t \begin{bmatrix} (-\Delta)^{-1/2}\sin((t-s)(-\Delta)^{1/2}) \\ \cos((t-s)(-\Delta)^{1/2}) \end{bmatrix} dW_{L^2}(s), \tag{15.4}$$

provided both integrands are stochastically integrable with respect to $W_{L^2}$. Noting that the trigonometric functions $h_n(\xi) := \sqrt{2}\sin(n\pi\xi)$, $n \geqslant 1$, form an orthonormal basis of eigenfunctions for $\Delta$, by using Theorems 5.19 and 6.17 this is the case if and only if the following two conditions are satisfied:

$$\begin{aligned} &\int_0^T \sum_{n=1}^\infty [\sin^2(t(-\Delta)^{1/2})h_n, h_n]\,dt < \infty, \\ &\int_0^T \sum_{n=1}^\infty [\cos^2(t(-\Delta)^{1/2})h_n, h_n]\,dt < \infty. \end{aligned} \tag{15.5}$$

But if these conditions hold, then by adding we obtain $\int_0^T \sum_{n=1}^\infty [h_n, h_n]\,dt < \infty$, which is obviously false. We conclude that the problem (15.3) fails to have a weak solution in $\mathscr{H}$.

Instead of looking for a solution in $\mathscr{H}$, we could try to look for a solution in the larger space

$$\mathscr{G} := L^2(0,1) \times W^{-1,2}(0,1),$$

where $W^{-1,2}(0,1)$ denotes the completion of $L^2(0,1)$ with respect to the norm $\|f\|_{W^{-1,2}(0,1)} := \|(-\Delta)^{-1/2}f\|$. This definition of $W^{-1,2}(0,1)$ is motivated by the fact that $W^{1,2}(0,1)$ can be characterised as the domain of $(-\Delta)^{1/2}$. The space $\mathscr{G}$ is the so-called *extrapolation space* of $\mathscr{H}$ with respect to $(-\Delta)^{1/2}$; we refer to Exercise 5 for a more systematic discussion.

As is easy to check, the semigroup $S$ extends to a $C_0$-semigroup on $\mathscr{G}$, and the stochastic convolution (15.4) is well defined in $\mathscr{G}$ if and only if

$$\begin{aligned} &\int_0^T \sum_{n=1}^\infty [(-\Delta)^{-1}\sin^2(t(-\Delta)^{1/2})h_n, h_n]\,dt < \infty, \\ &\int_0^T \sum_{n=1}^\infty [(-\Delta)^{-1}\cos^2(t(-\Delta)^{1/2})h_n, h_n]\,dt < \infty. \end{aligned} \tag{15.6}$$

These conditions are indeed satified, as is clear from the identity $(-\Delta)^{-1}h_n = (n\pi)^{-2}h_n$.

Let us call a measurable adapted process $u : [0,T] \times \Omega \times (0,1) \to \mathbb{R}$ an *extrapolated weak solution* of (WE2) if $U(t,\omega) := \begin{bmatrix} u(t,\omega,\cdot) \\ \frac{\partial u}{\partial t} u(t,\omega,\cdot) \end{bmatrix}$ belongs to $\mathscr{G}$ for all $(t,\omega) \in [0,T] \times \Omega$ and the resulting process $U : [0,T] \times \Omega \to \mathscr{G}$ is a weak solution of the problem (WE2). Summarising the above discussion, we have proved:

**Theorem 15.5.** *The stochastic wave equation* (WE2) *admits a unique extrapolated weak solution.*

Here, uniqueness is understood in the sense of $\gamma(L^2(0,T;L^2(0,1)),\mathscr{G})$.

## 15.3 The stochastic heat equation

Next we consider two stochastic heat equations with Dirichlet boundary values, driven by multiplicative space-time white noise on a domain $D$ in $\mathbb{R}^d$:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,\xi) = \Delta u(t,\xi) + B(u(t,\xi))\dfrac{\partial w}{\partial t}(t,\xi), & \xi \in D, \quad t \in [0,T], \\ u(t,\xi) = 0, & \xi \in \partial D, \quad t \in [0,T], \\ u(0,\xi) = u_0(\xi), & \xi \in D, \end{cases}$$

Again we discuss two particular cases of this problem: multiplicative rank one noise and additive space-time white noise. In both cases, the proofs of the main results can only be sketched, as they depend on a fair amount of interpolation theory and results from the theory of PDE. We refer to the Notes for references on this material.

### 15.3.1 Rank one multiplicative noise

Let $D$ be a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial D$. Our first example concerns the following stochastic heat equation driven by a rank one multiplicative noise:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,\xi) = \Delta u(t,\xi) + b(u(t,\xi))\dfrac{\partial W}{\partial t}(t), & \xi \in D, \quad t \in [0,T], \\ u(t,\xi) = 0, & \xi \in \partial D, \quad t \in [0,T], \\ u(0,\xi) = u_0(\xi), & \xi \in D. \end{cases} \tag{HE1}$$

Here $W$ is standard real-valued Brownian motion. We assume that the function $b : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous.

We fix $1 < p < \infty$ and assume that the initial value $u_0$ belongs to $L^p(D)$. We say that a measurable adapted process $u : [0,T] \times \Omega \times D \to \mathbb{R}$ is a *mild $V_\theta^p$-solution* of (HE1) if $\xi \mapsto u(t,\omega,\xi)$ belongs to $L^p(D)$ for all $(t,\omega) \in [0,T] \times \Omega$

and the resulting process $U : [0,T] \times \Omega \to L^p(D)$ is a mild $V_\theta^p$-solution of the stochastic evolution equation

$$\begin{cases} dU(t) = AU(t)\,dt + B(U(t))\,dW(t), \\ U(0) = u_0. \end{cases} \tag{15.7}$$

Here $A$ is the Dirichlet Laplacian on $L^p(D)$ and $B : L^p(D) \to L^p(D)$ is the Nemytskii map associated with $b$,

$$(B(u))(\xi) := b(u(\xi)).$$

**Proposition 15.6.** *Under the above assumptions on $b$, the Nemytskii map $B : L^p(D) \to L^p(D)$ is well defined and $\gamma$-Lipschitz continuous with $\mathrm{Lip}_\gamma(B) \leqslant C_p\mathrm{Lip}(b)$, where $C_p$ is a constant depending only on $p$.*

*Proof.* Let us first note that $B(f) \in L^p(D)$ for all $f \in L^p(D)$, so $B$ is well defined.

It follows from the Kahane-Khintchine inequality that for all $f_1, \dots, f_N$ and $g_1, \dots, g_N$ in $L^p(D)$,

$$\begin{aligned}
&\Big(\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n(B(f_n) - B(g_n))\Big\|_{L^p(D)}^2\Big)^{\frac12} \\
&\qquad \eqsim_p \Big(\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n(B(f_n) - B(g_n))\Big\|_{L^p(D)}^p\Big)^{\frac1p} \\
&\qquad = \Big(\int_D \mathbb{E}\Big|\sum_{n=1}^N \gamma_n(b(f_n(\xi)) - b(g_n(\xi)))\Big|^p\,d\xi\Big)^{\frac1p} \\
&\qquad \eqsim_p \Big(\int_D \Big(\mathbb{E}\Big|\sum_{n=1}^N \gamma_n(b(f_n(\xi)) - b(g_n(\xi)))\Big|^2\Big)^{\frac{p}{2}}\,d\xi\Big)^{\frac1p} \\
&\qquad = \Big(\int_D \Big(\sum_{n=1}^N |b(f_n(\xi)) - b(g_n(\xi))|^2\Big)^{\frac{p}{2}}\,d\xi\Big)^{\frac1p} \\
&\qquad \leqslant \mathrm{Lip}(b)\Big(\int_D \Big(\sum_{n=1}^N |f_n(\xi) - g_n(\xi)|^2\Big)^{\frac{p}{2}}\,d\xi\Big)^{\frac1p} \\
&\qquad \eqsim_p \mathrm{Lip}(b)\Big(\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n(f_n - g_n)\Big\|_{L^p(D)}^2\Big)^{\frac12},
\end{aligned}$$

where the last equivalence is obtained by doing the same computation backwards. Now we apply Proposition 14.1 (with $H = \mathbb{R}$). $\square$

Note that this result can be extended to Nemytskii maps on spaces $L^p(A)$, where $(A, \mathscr{A}, \mu)$ is any $\sigma$-finite measure space.

Let us say that an adapted process $u : [0,T] \times \Omega \times D \to \mathbb{R}$ is a *mild* $V_\theta^p$*-solution* of the problem (HE1) if if $\xi \mapsto u(t,\omega,\xi)$ belongs to $V_\theta^p$ for all $(t,\omega) \in [0,T] \times \Omega$ and the resulting process $U : [0,T] \times \Omega \to V_\theta^p$ is a mild $V_\theta^p$-solution of the problem (15.7).

**Theorem 15.7.** *Let $1 < p < \infty$, $\alpha \geqslant 0$, $\beta \geqslant 0$, $\theta \geqslant 0$ be such that $\alpha + 2\beta + d/p < 2\theta < 1$. Then the problem* (HE1) *has a unique mild $V_\theta^p$-solution $u$. This solution has a version with the property that $u - Su_0$ has trajectories in $C^\beta([0,T]; C^\alpha(\overline{D}))$, where $S$ denotes the semigroup generated by the Dirichlet Laplacian on $L^p(D)$.*

*Proof (Sketch).* We check the conditions of Theorem 14.16.

The space $E = L^p(D)$ is UMD and has Pisier's property and by Proposition 15.6, $B$ is $\gamma$-Lipschitz continuous from $E$ to $E$.

The Dirichlet Laplacian $A$ generates an analytic $C_0$-semigroup $S$ on $E$ (see Exercise 3). Choose numbers $0 \leqslant \eta < \eta' < \frac{1}{2}$ such that $\alpha + d/p < 2\eta$ and $\eta' + \beta < \theta$. The fractional domain space $E_{\eta'}$ associated with $A$ equals, up to an equivalent norm, the complex interpolation space $[E, \mathscr{D}(A)]_{\eta'}$.

Let $W^{2\eta,p}(D)$ be the Sobolev-Slobodetskii space of all functions $f : D \to \mathbb{R}$ such that

$$\|f\|_{W^{2\eta,p}(D)}^d := \|f\|_{L^p(D)}^p + \int_D \int_D \frac{|f(\xi) - f(\eta)|^p}{|\xi - \eta|^{d+2\eta p}} \, d\xi \, d\eta < \infty.$$

This space equals, up to an equivalent norm, the real interpolation space $(E, \mathscr{D}(A))_{\eta,p}$.

By general results in interpolation theory, we have a continuous embedding $[E, \mathscr{D}(A)]_{\eta'} \hookrightarrow (E, \mathscr{D}(A))_{\eta',p}$. By the above identifications, this results in a continuous embedding $E_{\eta'} \hookrightarrow W^{2\eta,p}(D)$.

Now we apply Theorem 14.16, which tells us that $U - Su_0$ has a version in with trajectories in $C^\beta([0,T]; E_{\eta'})$. By the above, this space embeds into $C^\beta([0,T]; W^{2\eta,p}(D))$. The proof is finished by an appeal to the Sobolev embedding theorem, which asserts that for $0 \leqslant \alpha < 2\eta - d/p$ we have a continuous embedding $W^{2\eta,p}(D) \hookrightarrow C^\alpha(\overline{D})$. □

### 15.3.2 Additive space-time white noise

Our final example is the stochastic heat equation driven by an additive space-time white noise:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,\xi) = \Delta u(t,\xi) + \dfrac{\partial w}{\partial t}(t,\xi), & \xi \in (0,1), \quad t \in [0,T], \\ u(t,0) = u(t,1) = 0, & \qquad\qquad\quad t \in [0,T], \\ u(0,\xi) = u_0(\xi), & \xi \in (0,1). \end{cases} \tag{HE2}$$

Here $w$ is a space-time white noise on the unit interval $(0,1)$.

We formulate the problem (HE2) as an abstract stochastic evolution equation in $L^2(0,1)$ of the form

$$\begin{cases} dU(t) = AU(t)\,dt + dW_{L^2}(t), \quad t \geqslant 0, \\ U(0) = u_0, \end{cases} \tag{15.8}$$

where $A$ is the Dirichlet Laplacian on $L^2 := L^2(0,1)$ and $W_{L^2}$ is the $L^2$-cylindrical Brownian motion canonically associated with $W$. By a computation similar to (15.9) below (see Exercise 3) it is easy to check that the assumptions of Theorem 8.6 are satisfied, and therefore for initial values $u_0 \in L^2$ we obtain the existence of a unique weak solution $U$ of (15.8) in $L^2$. Note that in contrast to the situation for the wave equation, here it is not necessary to pass to an extrapolation space. The reason behind this is that the regularising effect of the heat semigroup takes us back into $L^2$; the wave semigroup does not have any such effect. It is nevertheless useful to consider the equation in a suitable extrapolation scale, as this enables us to obtain precise Hölder regularity results.

To this end we shall apply Theorem 10.19 in a suitable extrapolation space of $L^p := L^p(0,1)$. Fix $\delta > \frac{1}{4}$ and let $L^p_{-\delta}$ denote the *extrapolation space* of order $\delta$ associated with the Dirichlet Laplacian $A_p$ on $L^p$, that is, $L^p_{-\delta}$ is the completion of $L^p$ with respect to the norm $\|x\|_{-\delta} := \|(-A_p)^{-\delta}x\|$. Since $A_p$ is invertible on $L^p$ (see Exercise 4), $(-A_p)^\delta$ acts as an isomorphism from $L^p$ onto $L^p_{-\delta}$. We will show next that the identity operator $I$ on $L^2$ extends to a bounded embedding from $L^2$ into $L^p_{-\delta}$ which is $\gamma$-radonifying. Then, we will exploit the regularising effect of the semigroup $S$ to get back into a suitable Sobolev space contained in $L^p$ and use this to deduce regularity properties of the solution.

As is well known,

$$H_1 := \mathscr{D}(A) = W^{2,2}(0,1) \cap W^{1,2}_0(0,1)$$

and

$$E_1 := \mathscr{D}(A_p) = W^{2,p}(0,1) \cap W^{1,p}_0(0,1)$$

with equivalent norms.

The functions $h_n(\xi) := \sqrt{2}\sin(n\pi\xi)$, $n \geqslant 1$, form an orthonormal basis in $L^2$ of eigenfunctions for $A$ with eigenvalues $-\lambda_n$, where $\lambda_n = (n\pi)^2$. If we endow $H_1$ with the equivalent Hilbert norm $\|f\|_{H_1} := \|Af\|_2$, the functions $\lambda_n^{-1}h_n$ form an orthonormal basis for $H_1$ and we have

$$\begin{aligned} \mathbb{E}\Big\| \sum_{n=M}^{N} \gamma_n \lambda_n^{-1} h_n \Big\|^2_{L^p_{1-\delta}} &= \mathbb{E}\Big\| \sum_{n=M}^{N} \gamma_n \lambda_n^{-1} (-A_p)^{1-\delta} h_n \Big\|^2_p \\ &= \mathbb{E}\Big\| \sum_{n=M}^{N} \gamma_n (n\pi)^{-2\delta} h_n \Big\|^2_p \overset{(*)}{\lesssim} \sum_{n=M}^{N} (n\pi)^{-4\delta}, \end{aligned} \tag{15.9}$$

where $(*)$ follows from a square function estimate as in the proof of Proposition 15.6 together with the fact that $\|h_n\|_p \leqslant \sqrt{2}$. The right hand side of (15.9) tends to 0 as $M, N \to \infty$ since we took $\delta > \frac{1}{4}$. It follows that the identity operator on $H_1$ extends to a continuous embedding from $H_1$ into $L^p_{1-\delta}$ which is $\gamma$-radonifying. Denoting this embedding by $i_{-\delta}$, we obtain a commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{I_{-\delta}} & L^p_{-\delta} \\ {\scriptstyle A^{-1}}\big\downarrow & & \big\uparrow{\scriptstyle A_p} \\ H_1 & \xrightarrow{i_{-\delta}} & L^p_{1-\delta} \end{array}$$

where the top mapping $I_{-\delta} : H \to L^p_{-\delta}$ is injective and $\gamma$-radonifying by the ideal property.

We are now in a position to apply Theorem 10.19. As before we assume that $u_0 \in L^2$. We say that a measurable adapted process $u : [0,T]\times\Omega\times(0,1) \to \mathbb{R}$ is a *weak solution* of (HE2) if $\xi \mapsto u(t,\omega,\xi)$ belongs to $L^2$ for all $(t,\omega) \in [0,T]\times\Omega$ and the resulting process $U : [0,T]\times\Omega \to L^2$ is a weak solution of the problem (15.8).

**Theorem 15.8.** *The problem* (HE2) *admits a unique weak solution* $u$. *For all* $\alpha \geqslant 0$ *and* $\beta \geqslant 0$ *satisfying* $\alpha+2\beta < \frac{1}{2}$, *the process* $u - Su_0$ *has a version with trajectories in* $C^\beta([0,T]; C^\alpha_0[0,1])$, *where* $S$ *denotes the semigroup generated by the Dirichlet Laplacian on* $L^2(0,1)$.

*Proof (Sketch).* Fix arbitrary real numbers $\alpha \geqslant 0$ and $\beta \geqslant 0$ satisfying $\alpha + 2\beta < \frac{1}{2}$. Replacing $\delta$ by a smaller number if necessary, we can find $\theta \geqslant 0$ such that $\frac{1}{4} < \delta < \theta$, $\beta+\theta < \frac{1}{2}$, and $\alpha+2\delta < 2\theta$. Put $\eta := \theta-\delta$. As is easy to check, (the extrapolation of) $A_p$ generates an analytic $C_0$-semigroup in $L^p_{-\delta}$. Hence we may apply Theorem 10.19 in the space $L^p_{-\delta}$ to obtain a weak solution $U$ of the problem

$$\begin{cases} dU(t) = AU(t)\,dt + I_{-\delta}\,dW_H(t), \quad t \in [0,T], \\ U(0) = 0, \end{cases}$$

with paths in the space $C^\beta\big([0,T]; (L^p_{-\delta})_\theta\big) = C^\beta\big([0,T]; L^p_\eta\big)$; the identity $(L^p_{-\delta})_\theta = L^p_\eta$ is a generalisation of Lemma 10.8. Along the embedding $L^2 \hookrightarrow L^p_{-\delta}$, this solution is consistent with the weak solution $U$ of this problem in $L^2$.

Noting that $\alpha < 2\eta$ we choose $p$ so large that $\alpha + \frac{1}{p} < 2\eta$. We have

$$L^p_\eta = W^{2\eta,p}_0(0,1)$$

with equivalent norms, and by the Sobolev embedding theorem,

$$W^{2\eta,p}(0,1) \hookrightarrow C^\alpha[0,1]$$

with continuous inclusion. We denote $C_0^\alpha[0,1] = \{f \in C^\alpha[0,1] : f(0) = f(1) = 0\}$. Putting things together we obtain a continuous inclusion

$$L^p_\eta \hookrightarrow C_0^\alpha[0,1].$$

In particular it follows that $U$ takes values in $L^p$. Almost surely, the trajectories of $U$ belong to $C^\beta([0,T]; C_0^\alpha[0,1])$.

If we compare Theorems 15.7 and 15.8 (for $d = 1$ and $D = (0,1)$), we notice that we get better Hölder regularity for the former ($\alpha + 2\beta < 1$ in the limit $p \to \infty$) than for the latter ($\alpha + 2\beta < \frac{1}{2}$). The explanation for this is the additional $\delta > \frac{1}{4}$ needed in Theorem 15.8 to get the $\gamma$-radonification of $B := I_{-\delta}$. In Theorem 15.7, $\gamma$-radonification came for free.

## 15.4 Exercises

1. Let $(A, \mathscr{A}, \mu)$ be a $\sigma$-finite measure space and put $H := L^2(A)$. Let $(W_H(t))_{t\in[0,T]}$ be an $H$-cylindrical Brownian motion. Show that

$$w([0,t] \times B) := W_H(t)1_B, \quad t \in [0,T],\ B \in \mathscr{A},$$

uniquely defines a space-time white noise $w$ on $A$.

2. Check the computations leading to the conditions (15.5) and (15.6).
*Hint:* A bounded operator $R : H_1 \to H_2$, where $H_1$ and $H_2$ are separable Hilbert spaces, is Hilbert-Schmidt if and only if $RR^* : H_2 \to H_2$ has finite trace.

3. In this exercise we take a look at the following stochastic heat equation with additive space-time white noise on the domain $D = (0,1)^d$ in $\mathbb{R}^d$.

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,\xi) = \Delta u(t,\xi) + \dfrac{\partial W}{\partial t}(t,\xi), & \xi \in D, \quad t \in [0,T], \\ u(t,\xi) = 0, & \xi \in \partial D, \quad t \in [0,T], \\ u(0,\xi) = u_0(\xi), & \xi \in D. \end{cases}$$

We model this problem as a stochastic evolution equation of the form (15.8).
   a) Prove that the Dirichlet Laplacian generates an analytic $C_0$-semigroup on $L^2(D)$.
   b) Show that the problem (15.8) has a weak solution in $L^2(D)$ if and only if $d = 1$.

*Hint for a) and b):* Find an orthonormal basis of eigenvectors.

4. Show that the heat semigroup generated by the Dirichlet Laplacian on $L^2(0,1)$ extends to an analytic $C_0$-semigroup on $L^p(0,1)$, $1 < p < \infty$, and show that its generator is invertible.

5. In this exercise we take a closer look at extrapolation spaces. Let $A$ be a densely defined closed operator on a Banach space $E$ and denote by $\mathscr{G}(A)$ its graph,
$$\mathscr{G}(A) = \{(x, Ax) \in E \times E : \ x \in \mathscr{D}(A)\}.$$
Define the *extrapolation space* of $E$ with respect to $A$ as the quotient space
$$E_{-1} := (E \times E)/\mathscr{G}(A).$$
 a) Show that the mapping $x \mapsto (0, x)$ defines a bounded dense embedding $E \hookrightarrow E_{-1}$.
 b) Show that $A_{-1} : x \mapsto (-x, 0)$ defined a bounded operator from $E$ to $E_{-1}$ which extends $A$.
 c) Show that if $\lambda \in \varrho(A)$, then the identity map on $E$ extends to an isomorphism of Banach spaces $E_{-1} \simeq E_{-1}^{\lambda}$, where the latter is defined as the completion of $E$ with respect to the norm $\|x\|_{E_{-1}^{\lambda}} := \|R(\lambda, A)x\|$.

**Notes.** The literature on stochastic partial differential equations is enormous and various approaches are possible. The functional analytic approach taken here, where the equation is reformulated as a stochastic evolution equation on some infinite-dimensional state space, goes back to Hille and Phillips in the deterministic case and give rise to the theory of $C_0$-semigroups. In the setting of Hilbert spaces, the theory of stochastic evolution equations was pioneered by Da Prato and Zabczyk and their schools. We refer to their monograph [27] for further references. See also Curtain and Pritchard [25] for some earlier references.

Our definition of a space-time white noise in Section 15.2 follows the lecture notes of Walsh [107], where also Theorem 15.5 can be found.

The presentation of Section 15.2.2 follows Da Prato and Zabczyk [27, Example 5.8].

Concerning problem (HE2), the existence of a solution in $C^{\alpha}([0,T]\times[0,1])$ for $0 \leqslant \alpha < \frac{1}{4}$ was proved by Da Prato and Zabczyk by very different methods; see [27, Theorem 5.20]. Theorem 15.8 was obtained by Brzeźniak [14] under more general assumptions. The approach taken here is from [34].

The results on interpolation theory needed in the proofs of Theorems 15.8 and 15.7 can be found in the book of Triebel [103].