

CIE4801 Transportation and spatial modelling

Congested assignment

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Content

- What are we talking about?
- General network assignment problem
- DUE: Deterministic user equilibrium assignment
- DUE: Mathematical formulation
- DUE: Algorithms
- SUE: Stochastic user equilibrium assignment
- Special topics
- Reprise Probit and Logit

1.

What are we talking about?



What do we want to know?

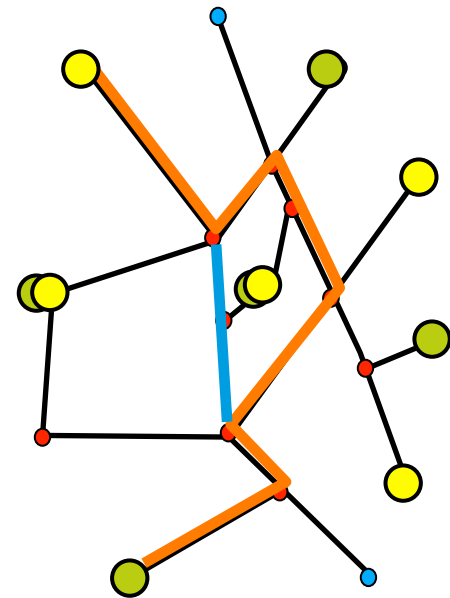
2.

General network assignment problem

General network assignment problem: Main elements

Indices:

origin i
destination j
route r
link a



Network assignment variables

T_{ij} OD travel demand

α_{ijr}^a link-route incidence matrix
(assignment map)

$t_a(q_a)$ link travel time function

Θ flow dispersion parameter
of the stochastic component

INPUT

T_{ijr} route flow

β_{ijr} route choice proportion

t_{ijr} route travel time

q_a link flow

t_a link travel time

OUTPUT

Relationships between variables

1. Link travel times t_a

$$t_a = t_a(q_a)$$

2. Route choice proportions β_{ijr}

$$\beta_{ijr} = \beta_{ijr}(\Theta, t_{ijr})$$

3. Route travel times t_{ijr}

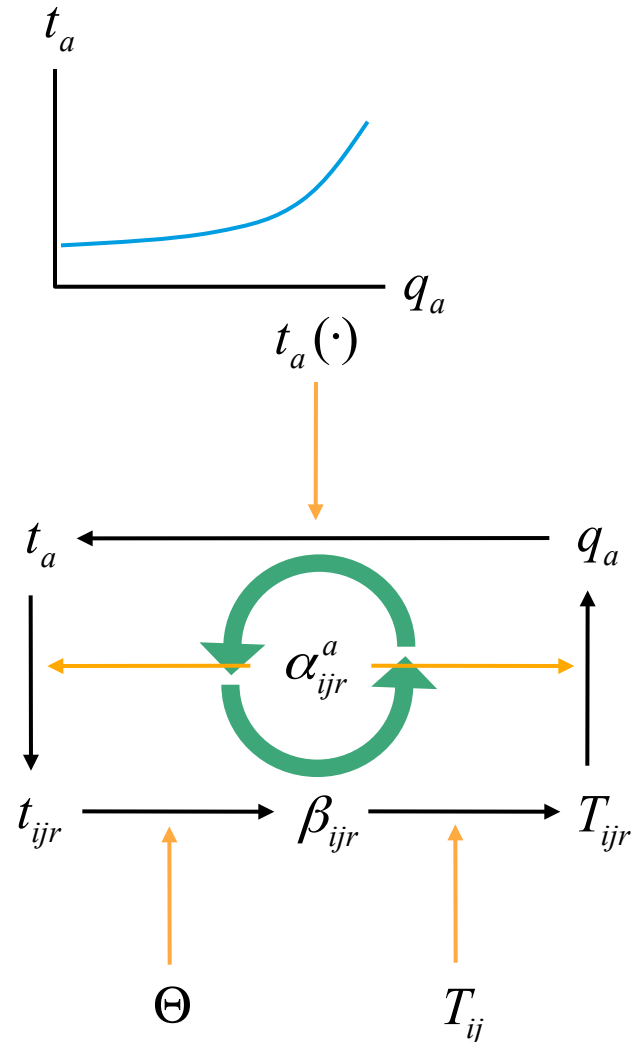
$$t_{ijr} = \sum_a \alpha_{ijr}^a t_a$$

4. Route flows T_{ijr}

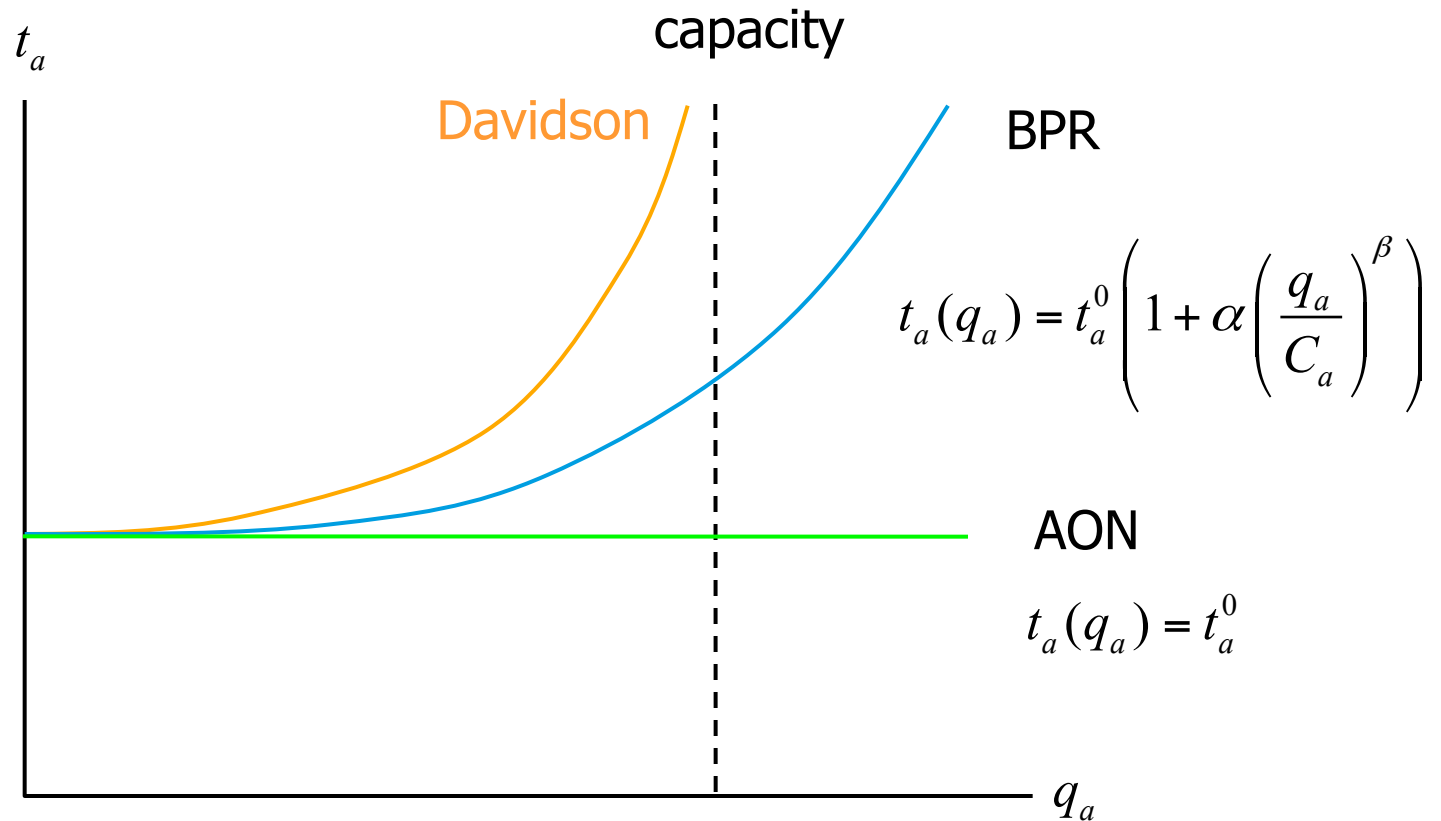
$$T_{ijr} = \beta_{ijr} T_{ij}$$

5. Link flows q_a

$$q_a = \sum_i \sum_j \sum_r \alpha_{ijr}^a T_{ijr}$$

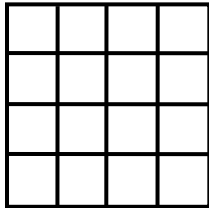


Link performance functions

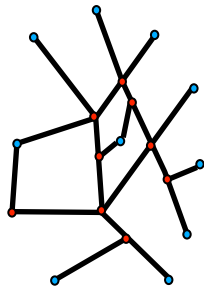


Assignment types

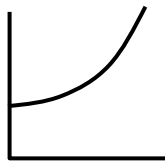
inputs



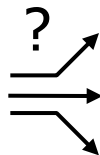
T_{ij}



α_{ijr}^a



$t_a(q_a)$



Θ

$t_a(q_a)$

Θ

$t_a(q_a)$

> 0

SUE

$t_a(q_a)$

$= 0$

DUE

t_a

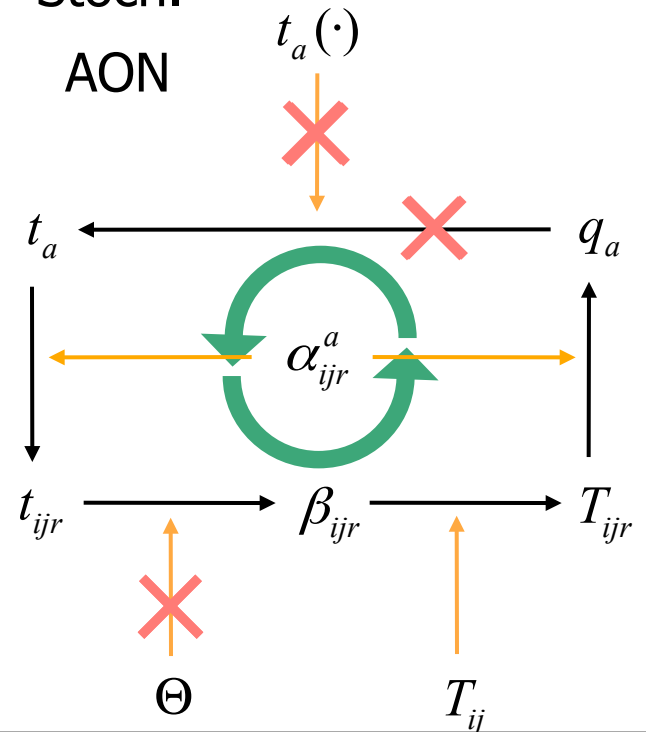
> 0

Stoch.

t_a

$= 0$

AON



Stochastic assignment: reprise

1. Link travel times t_a

$$t_a = t_a^0$$

2. Route choice proportions β_{ijr}

$$\beta_{ijr} = \beta_{ijr}(\Theta, t_{ijr})$$

3. Route travel times t_{ijr}

$$t_{ijr} = \sum_a \alpha_{ijr}^a t_a$$

4. Route flows T_{ijr}

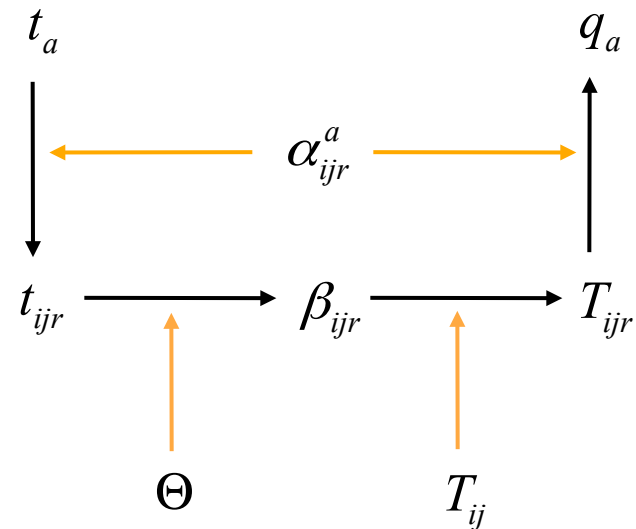
$$T_{ijr} = \beta_{ijr} T_{ij}$$

5. Link flows q_a

$$q_a = \sum_i \sum_j \sum_r \alpha_{ijr}^a T_{ijr}$$

What about probit?

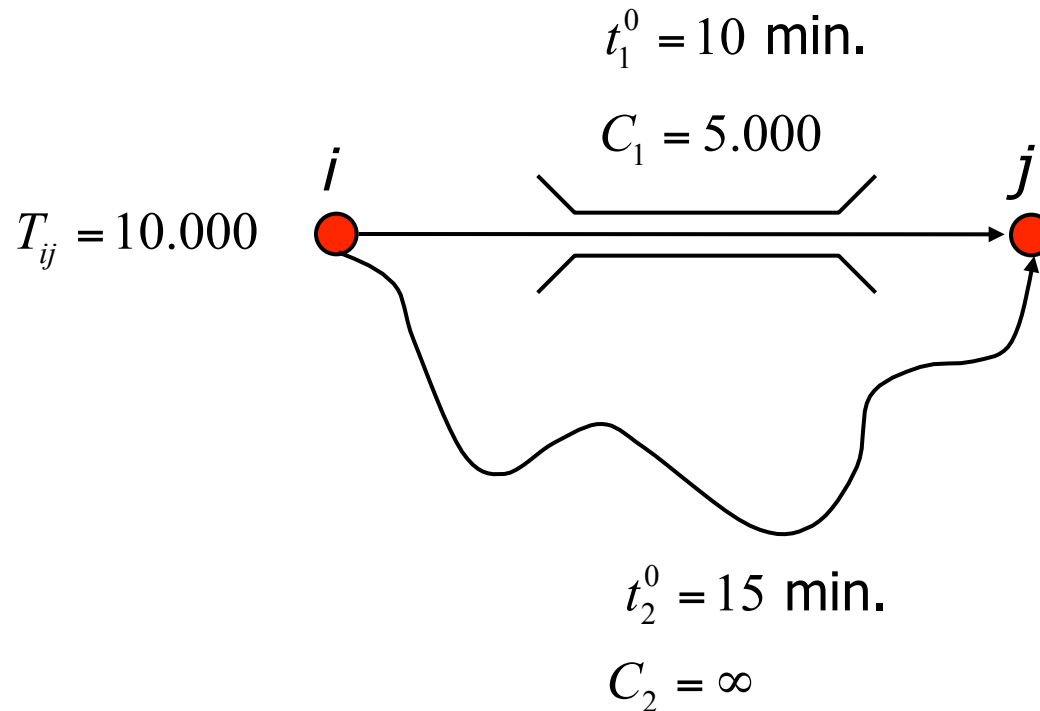
$$t_a = N(\bar{t}_a, \sigma_{t_a})$$



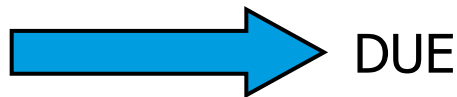
3.1

DUE: Deterministic user equilibrium assignment

Route choice with congestion effects



When congestion is taken into account,
how long will the trip from i to j take along route 1? 15 min.!



DUE assignment

1. Link travel times t_a

$$t_a = t_a(q_a)$$

2. Route choice proportions β_{ijr}

$$\beta_{ijr} = \beta_{ijr}(t_{ijr})$$

3. Route travel times t_{ijr}

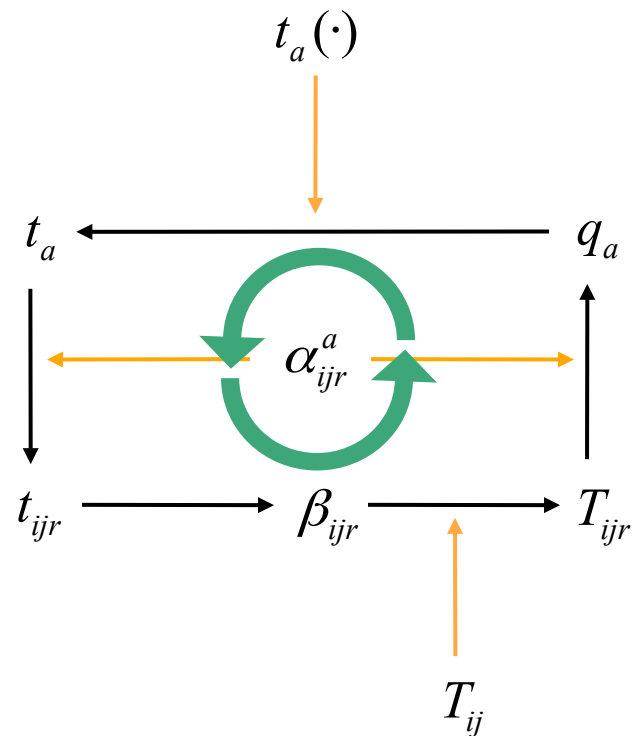
$$t_{ijr} = \sum_a \alpha_{ijr}^a t_a$$

4. Route flows T_{ijr}

$$T_{ijr} = \beta_{ijr} T_{ij}$$

5. Link flows q_a

$$q_a = \sum_i \sum_j \sum_r \alpha_{ijr}^a T_{ijr}$$



Main principle

Deterministic user-equilibrium (DUE) takes **congestion effects** into account and is defined as:

Wardrop's equilibrium law

All travellers choose their optimal route, such that no traveller can improve his/her travel time by unilaterally changing routes.

This equilibrium is reached if the following condition holds:

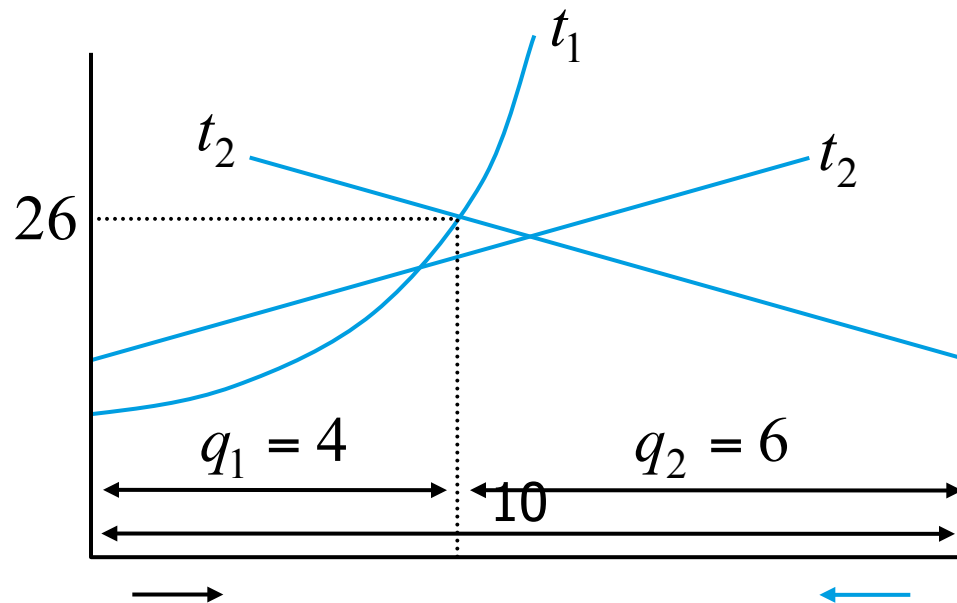
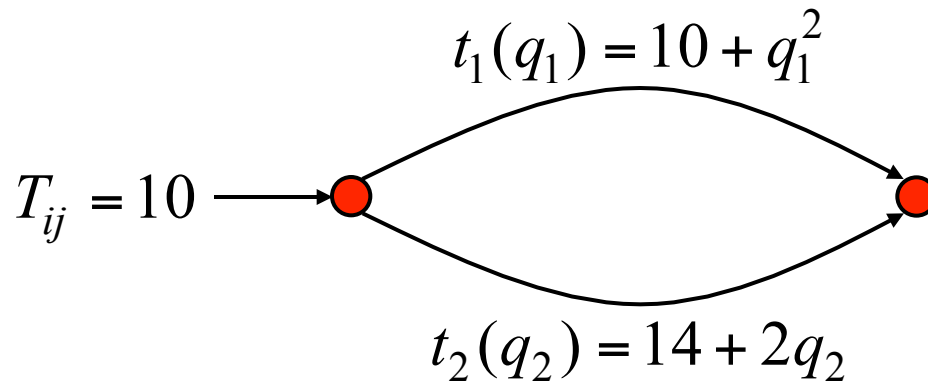
Wardrop's first principle

All used routes have the same travel time which is not greater than the travel time on any unused route.

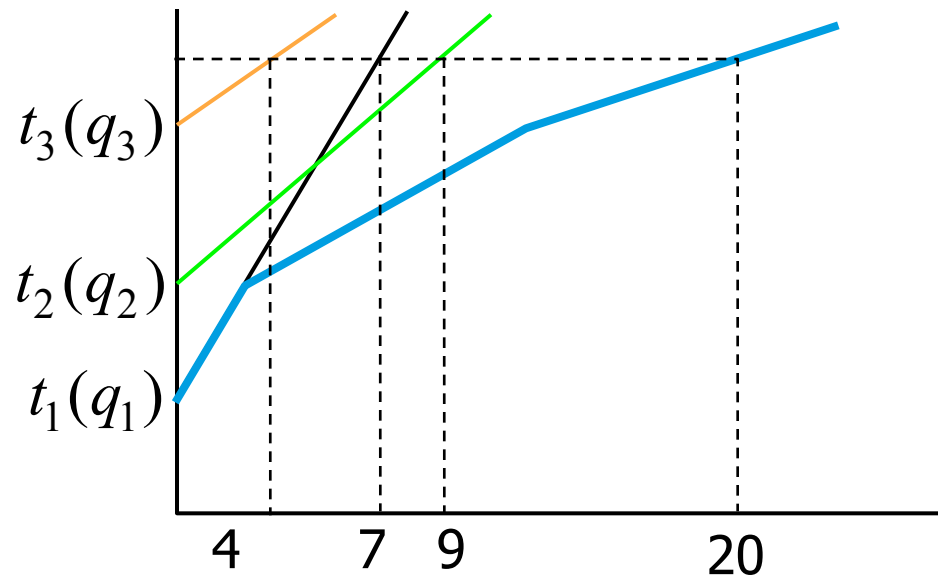
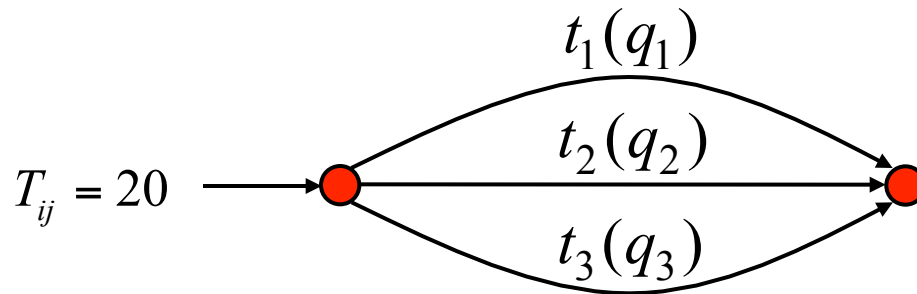
3.2

DUE: Formulations

Example DUE assignment

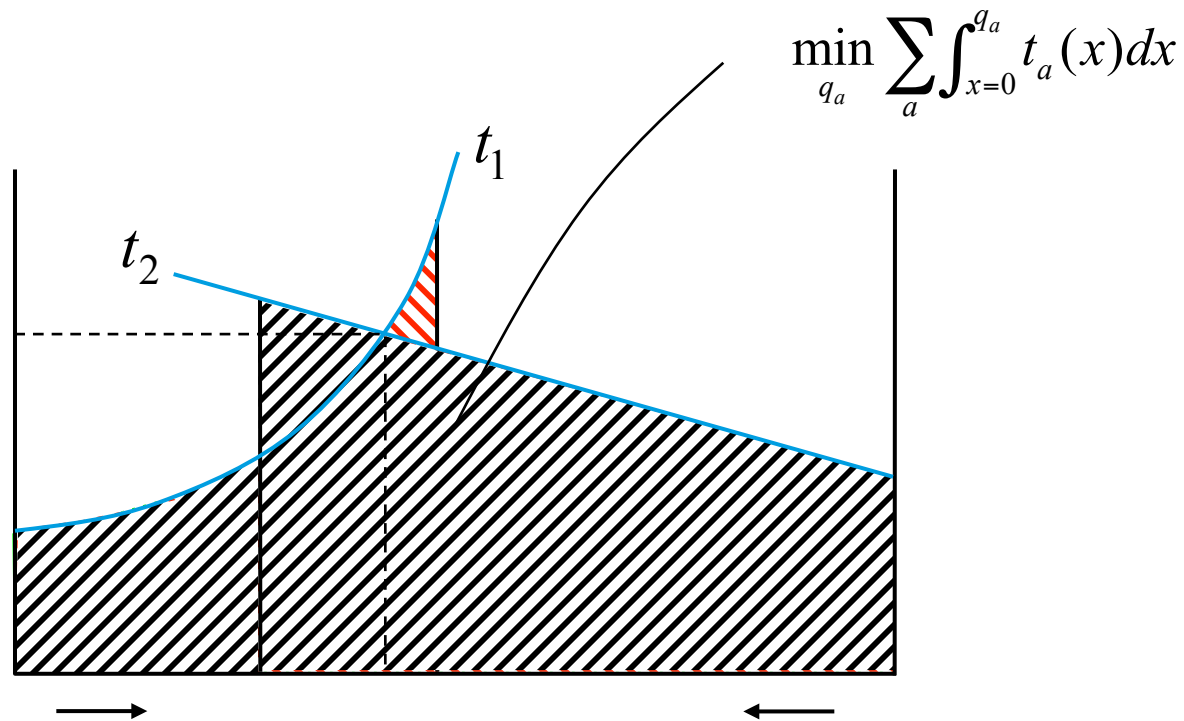


2nd example DUE assignment



DUE assignment: Mathematical formulation

What is the objective function?



Mathematical programming formulation

$$\min_{q_a} \sum_a \int_{x=0}^{q_a} t_a(x) dx$$

subject to: $\sum_r T_{ijr} = T_{ij} \quad \forall i, j$ (flow conservation)

$$q_a = \sum_i \sum_j \sum_r \alpha_{ijr}^a T_{ijr} \quad \forall a$$
 (definition)

$$T_{ijr} \geq 0 \quad \forall i, j, r$$
 (nonnegativity)



non-linear programming problem

Everything in time?

People do not only decide on **travel times**, but may also take other factors into account:

- **fuel costs**
- **toll costs**
- **... etc**

How to take these factors into account?

Replace $t_a(q_a)$ with generalized costs $c_a(q_a)$

For example,

$$c_a(q_a) = \alpha \cdot t_a(q_a) + \theta_a$$

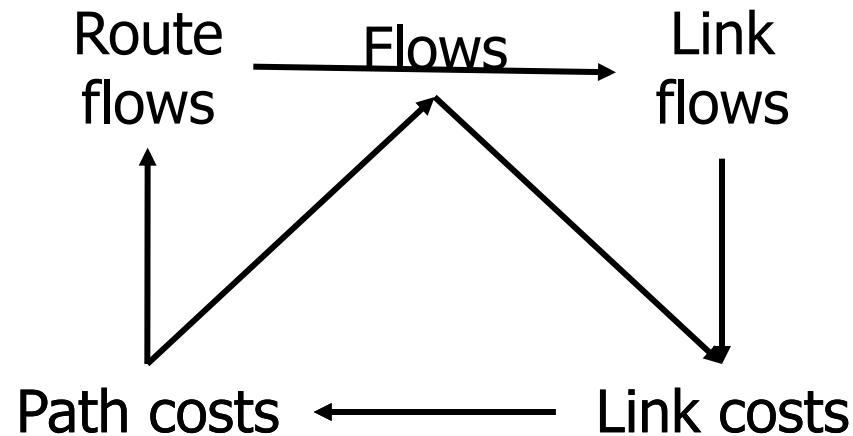
value-of-time (VOT) travel time toll

➔ $\min_{q_a} \sum_a \int_{x=0}^{q_a} c_a(x) dx$

3.3

DUE: Algorithms

Key point assignment problem



General solution scheme:

- Start with an assumption on e.g. link costs
- Follow the arrows until convergence is achieved

Solution principle

1. Set flows of all links equal to 0
2. Determine link costs based on the link flows
 - Thus start with free flow travel times
3. Perform an assignment (AON or MR)
4. Determine link flows
5. Return to step 2

Different approaches for step 4

- Naive: Repeat again and again
- Simple: Average flows with previous results ($1/n$)
- Smart: Average with optimal step sizes

Don't
MSA
Frank-Wolfe

Recall equilibrium example lecture 1

	A	B
time	10.0	15.0
chosen	80	10
time	114.9	15.0
chosen	30	60
time	12.1	23.6
chosen	50	40
time	26.0	16.7
chosen	40	50
time	16.6	19.2
chosen	45	45
time	20.5	17.7
chosen	42	48
time	18.0	18.5
chosen	43.0	47.0
time	18.8	18.3
chosen	42.5	47.5
time	18.4	18.4

Capacity A: 25
Capacity B: 35

Solving the DUE assignment problem

A nonlinear programming problem can be solved using a *steepest descent algorithm*.

The *convex combinations algorithm*, also known as the *Frank-Wolfe algorithm* (1956), is the most common algorithm to solve the DUE assignment problem.

Main principle:

- Find an initial feasible solution
- Linearise the objective function
- Find a new intermediate solution
- Determine a new solution in the direction of the intermediate solution
- Iterate until the new solution does not change anymore

Frank-Wolfe algorithm: step 1

$$\min_{q_a} Z = \sum_a \int_{x=0}^{q_a} t_a(x) dx \quad (1)$$

subject to:

$$\sum_r T_{ijr} = T_{ij} \quad \forall i, j \quad (2)$$

$$q_a = \sum_i \sum_j \sum_r \alpha_{ijr}^a T_{ijr} \quad \forall a \quad (3)$$

$$T_{ijr} \geq 0 \quad \forall i, j, r \quad (4)$$

Step 1: Find an initial feasible solution (iteration 1)

We have to find a solution that satisfies constraints (2), (3), and (4).

An AON assignment yields a feasible solution $q^{(1)}$

Frank-Wolfe algorithm: step 2

$$\min_{q_a} Z = \sum_a \int_{x=0}^{q_a} t_a(x) dx \quad (1)$$

subject to:

$$\sum_r T_{ijr} = T_{ij} \quad \forall i, j \quad (2)$$

$$q_a = \sum_i \sum_j \sum_r \alpha_{ijr}^a T_{ijr} \quad \forall a \quad (3)$$

$$T_{ijr} \geq 0 \quad \forall i, j, r \quad (4)$$

Step 2: Linearise the objective function (iteration i)

$$\begin{aligned} Z(w^{(i)}) &= Z(q^{(i)}) + \sum_a \frac{\partial Z(q^{(i)})}{\partial q_a} (w_a - q_a^{(i)}) \\ &= Z(q^{(i)}) + \sum_a t_a(q_a^{(i)}) (w_a - q_a^{(i)}) \end{aligned}$$

(first-order expansion Taylor polynomial)

Frank-Wolfe algorithm: step 3

$$\min_{q_a} Z = \sum_a \int_{x=0}^{q_a} t_a(x) dx \quad (1)$$

subject to: $\sum_r T_{ijr} = T_{ij} \quad \forall i, j \quad (2)$

$$q_a = \sum_i \sum_j \sum_r \alpha_{ijr}^a T_{ijr} \quad \forall a \quad (3)$$

$$T_{ijr} \geq 0 \quad \forall i, j, r \quad (4)$$

Step 3: Solve the linearised problem (iteration i)

$$\begin{aligned} \min_{w_a^{(i)}} Z(w^{(i)}) &= \min_{w_a^{(i)}} \left\{ Z(q^{(i)}) + \sum_a t_a(q_a^{(i)})(w_a - q_a^{(i)}) \right\} \\ &= \min_{w_a^{(i)}} \sum_a t_a(q_a^{(i)}) w_a^{(i)} \end{aligned}$$

subject to (2), (3), and (4)

Frank-Wolfe algorithm: step 4

$$\min_{q_a} Z = \sum_a \int_{x=0}^{q_a} t_a(x) dx \quad (1)$$

subject to:

$$\sum_r T_{ijr} = T_{ij} \quad \forall i, j \quad (2)$$

$$q_a = \sum_i \sum_j \sum_r \alpha_{ijr}^a T_{ijr} \quad \forall a \quad (3)$$

$$T_{ijr} \geq 0 \quad \forall i, j, r \quad (4)$$

Step 4: Find the new solution (iteration i)

$$w^{(i)} = \arg \min_{0 \leq \alpha \leq 1} Z(q^{(i)} + \alpha(w^{(i)} - q^{(i)}))$$

$$\Rightarrow q^{(i+1)} = q^{(i)} + w^{(i)} (w^{(i)} - q^{(i)})$$

Frank-Wolfe algorithm

Step 1: $i:=1$. Set $q_a^{(i)} = 0$ (assume empty network)

Step 2: Perform a shortest-path AON assignment based on $t_a^{(i)} = t_a(q_a^{(i)})$ yielding link flows $w_a^{(i)}$

Step 3: Compute new solution $q_a^{(i+1)} = q_a^{(i)} + \alpha^{(i)} (w_a^{(i)} - q_a^{(i)})$
with $\alpha^{(i)} = \arg \min_{0 \leq \alpha \leq 1} Z(q^{(i)} + \alpha(w^{(i)} - q^{(i)}))$

Step 4: If no convergence yet, set $i:=i+1$ and return to Step 2.

N.B. Using $\alpha^{(i)} = 1/i$ is called:
Method of Successive Averages (MSA)

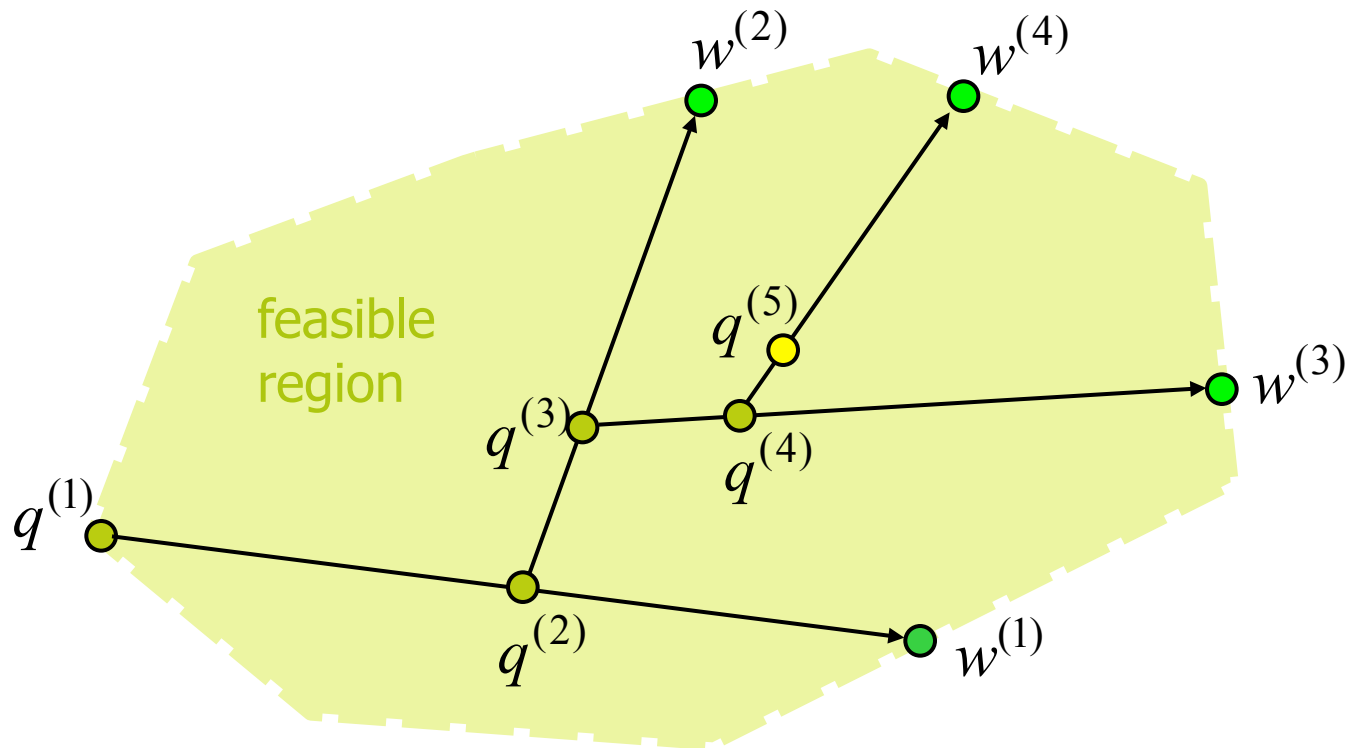
Convergence criteria

- Number of iterations
- Equality of path costs
- Successive link flows (FA): $(q^i - q^{i-1}) / q^{i-1} < d$

- Duality gap
$$\sum_a t_a^i \cdot q_a^i - \sum_o \sum_d T_{od} \cdot \tau_{od}^i$$

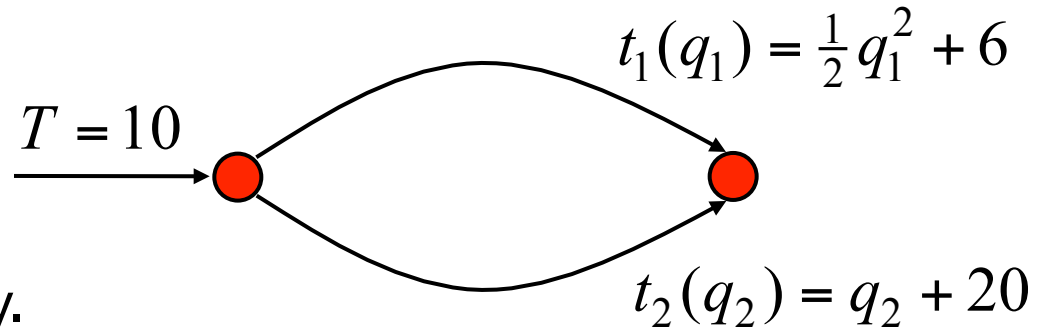
i.e. total travel time based on links minus total travel time based on (latest) shortest paths

Illustration MSA algorithm



MSA: Example

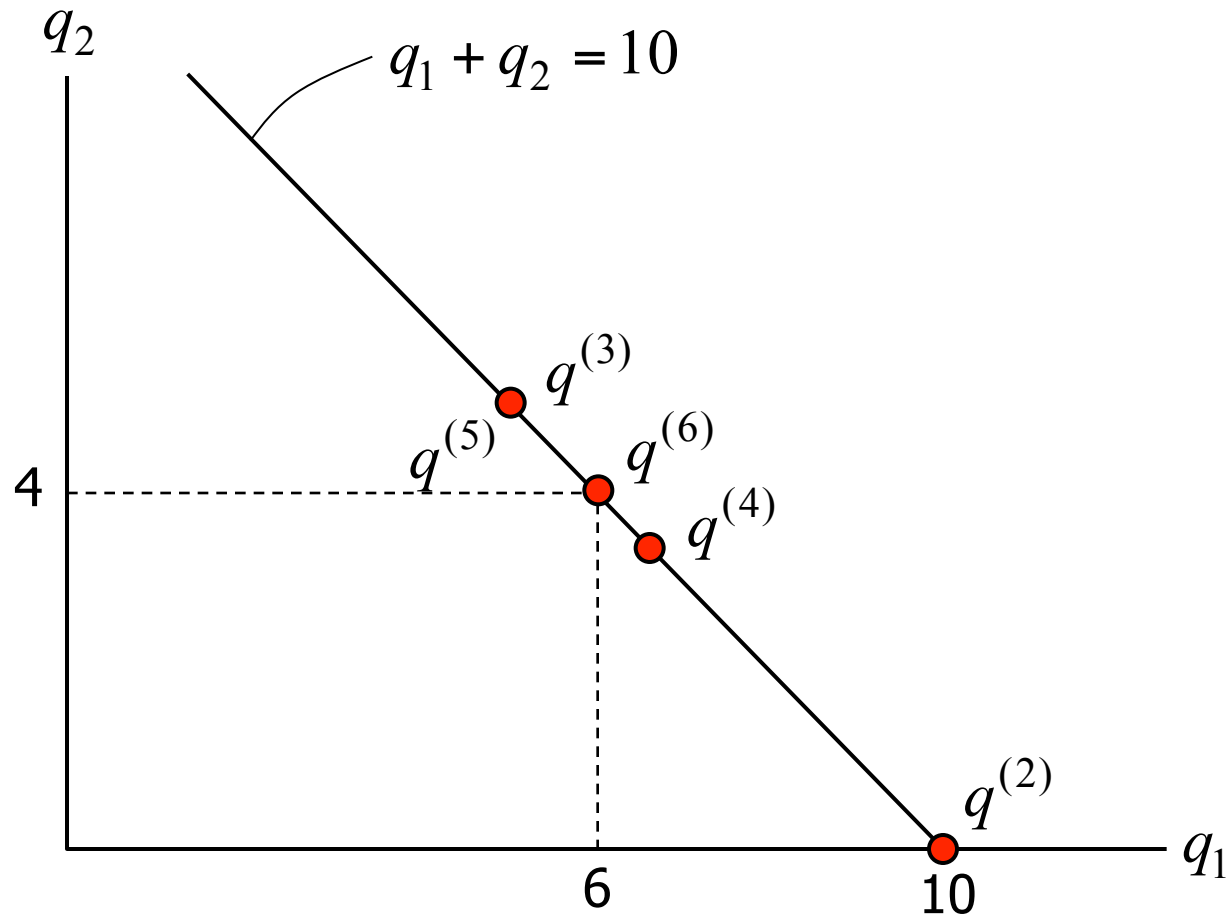
Network



Find DUE iteratively.

q_1	q_2	t_1	t_2	w_1	w_2	α	$q_a^{(i+1)} = q_a^{(i)} + \frac{1}{i}(w_a^{(i)} - q_a^{(i)})$
0	0	6	20	10	0	1	$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \frac{1}{1} \cdot \left[\begin{pmatrix} 10 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$
10	0	56	20	0	10	1/2	$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} + \frac{1}{2} \cdot \left[\begin{pmatrix} 0 \\ 10 \end{pmatrix} - \begin{pmatrix} 10 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$
5	5	18.5	25	10	0	1/3	$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} + \frac{1}{3} \cdot \left[\begin{pmatrix} 10 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 5 \end{pmatrix} \right] = \begin{pmatrix} 6\frac{2}{3} \\ 3\frac{1}{3} \end{pmatrix}$
6.7	3.3	28.2	23.3	0	10	1/4	N N
5	5	18.5	25	10	0	1/5	
6	4	24	24	-	-	-	

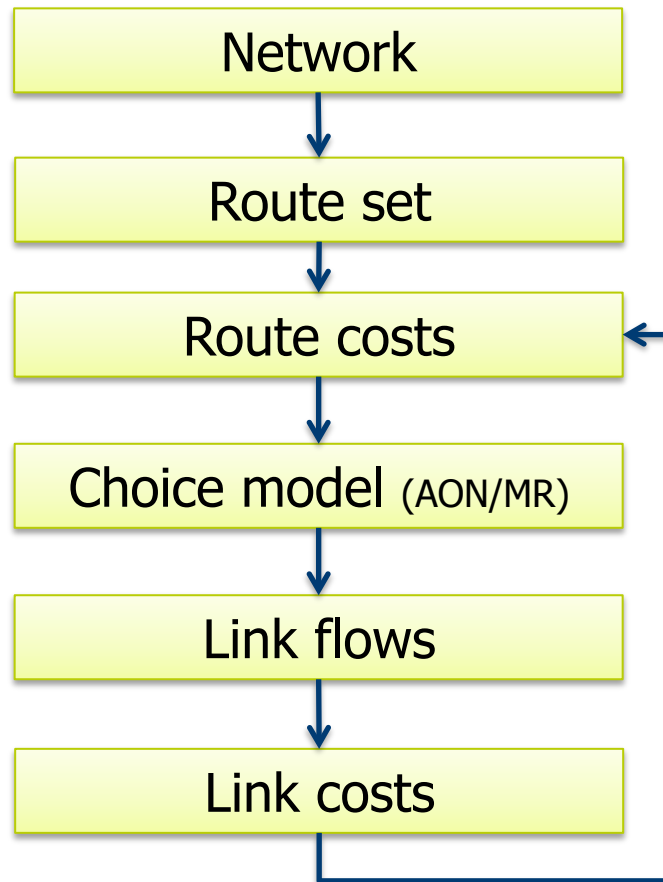
MSA: Graphical representation



Link based versus route based

- Previous slides all referred to a link based formulation
- i.e. repetitive tree searches
- What if you have a set of possible routes?
- Then there's no need to search for routes at each MSA-step

Route based approach: Generic procedure



Generic solution scheme

Route flow averaging

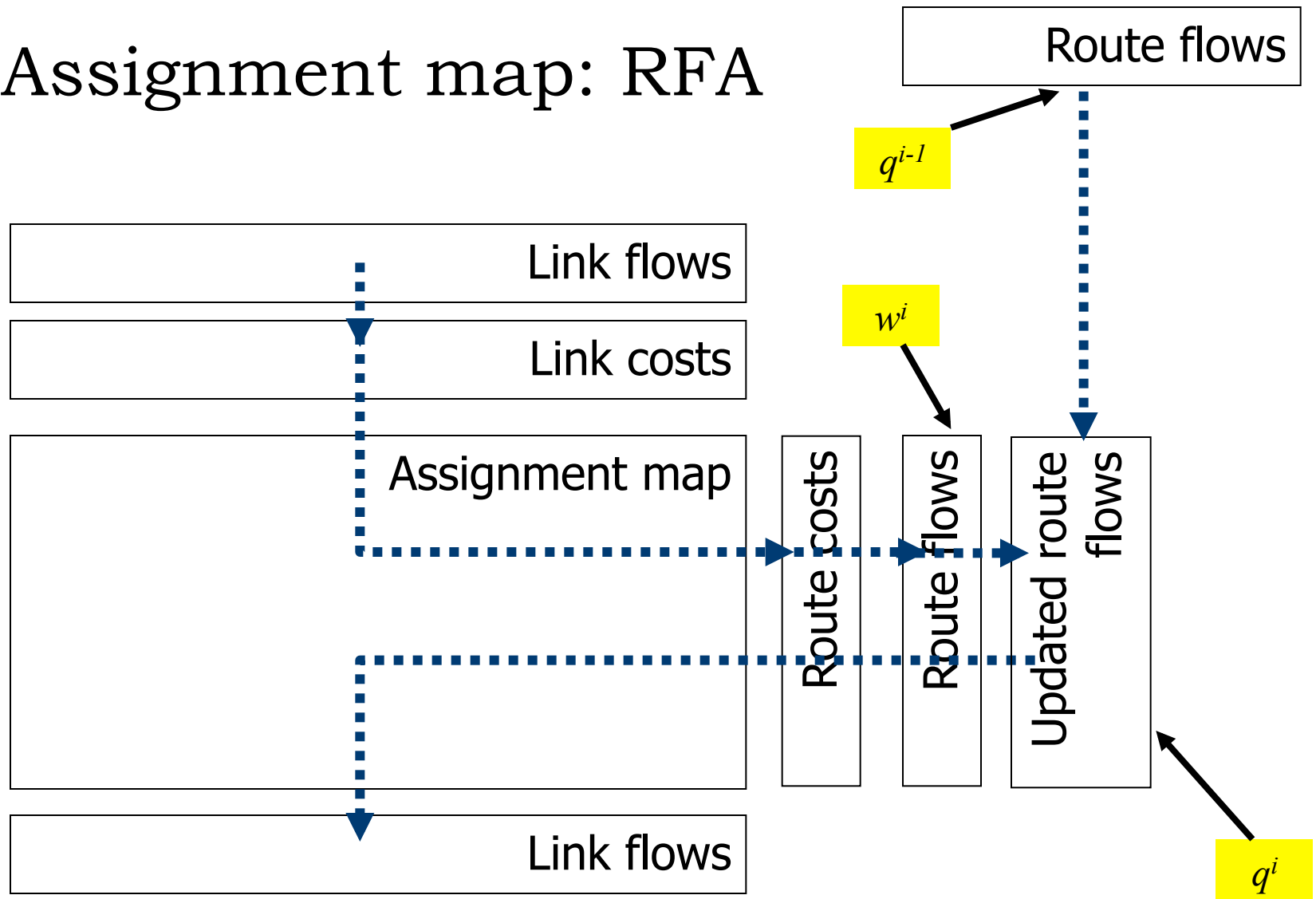
1. Specify routes/choice sets
2. Calculate path costs
3. Assign OD (AON) => route flows
4. Recalculate route flows (MSA)

$$q_r^i = q_r^{i-1} + (w_r^i - q_r^{i-1}) / i$$

where w_r^i = route flow of the intermediate solution

5. Calculate link flows
6. Check convergence
7. Go to step 2 or stop

Assignment map: RFA



Convergence criteria

- Number of iterations
- Successive path flows
- Equality of path costs
- Successive link flows (FA): $(q^i - q^{i-1}) / q^{i-1} < d$
- Successive link costs (CA)
- Duality gap

$$\sum_a t_a^i \cdot q_a^i - \sum_o \sum_d T_{od} \cdot \tau_{od}^i$$

Benefits of a route-based approach

- Full control of routes that are used
 - Freedom in route choice modelling (see SUE)
 - Overlapping routes can explicitly be dealt with
 - Reduced computational efforts in equilibrium assignment
 - Suitable for all kind of network concepts e.g. multimodal networks
-
- Why isn't route-based assignment used in practice?

4.

SUE: Stochastic user equilibrium assignment

Stochastic user equilibrium

- Combination of two concepts
- Congestion: travel time is function of flow
=> equilibrium modelling
- Perception: travellers consider a set of routes
=> route choice
- Adjustment Wardrop
All travellers choose their optimal route, such that no traveller can improve his/her **perceived** travel time by unilaterally changing routes.

Stochastic equilibrium assignment

1. Link travel times t_a

$$t_a = t_a(q_a)$$

2. Route choice proportions β_{ijr}

$$\beta_{ijr} = \beta_{ijr}(\Theta, t_{ijr})$$

3. Route travel times t_{ijr}

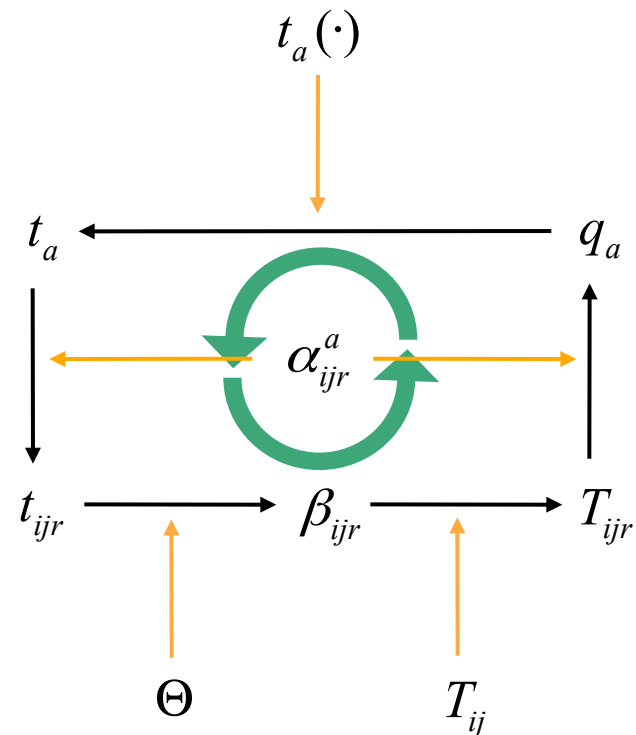
$$t_{ijr} = \sum_a \alpha_{ijr}^a t_a$$

4. Route flows T_{ijr}

$$T_{ijr} = \beta_{ijr} T_{ij}$$

5. Link flows q_a

$$q_a = \sum_i \sum_j \sum_r \alpha_{ijr}^a T_{ijr}$$



Generic solution scheme

Route flow averaging

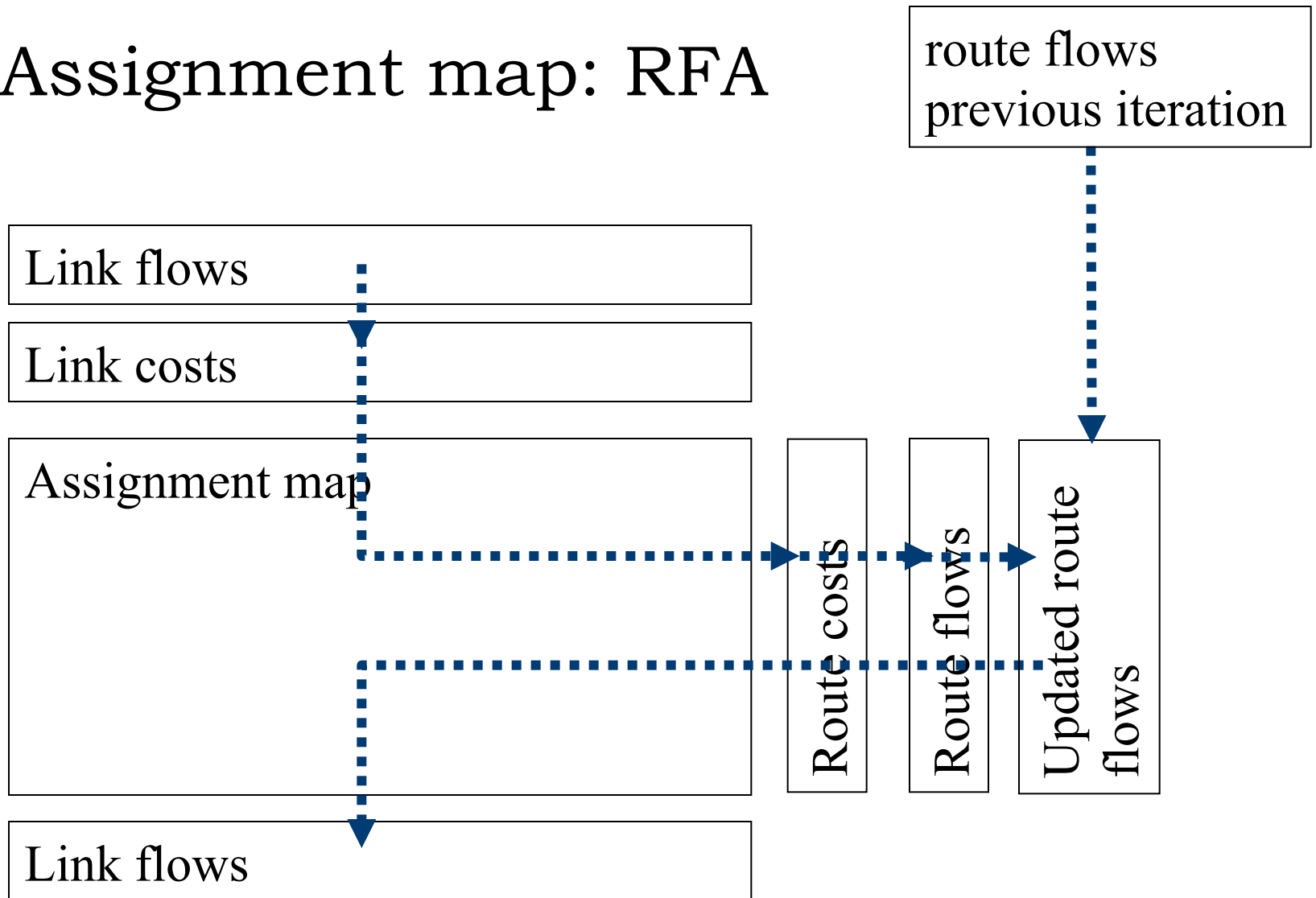
1. Specify paths/choice sets
2. Calculate path costs
3. Assign OD (Logit, Probit)=> route flows
4. Recalculate route flows (MSA)

$$q_r^i = q_r^{i-1} + (w_r^i - q_r^{i-1}) / i$$

where w_r^i = route flow of the intermediate solution

5. Calculate link flows
6. Check convergence
7. Go to step 2 or stop

Assignment map: RFA



Alternative solution scheme

Link flow averaging

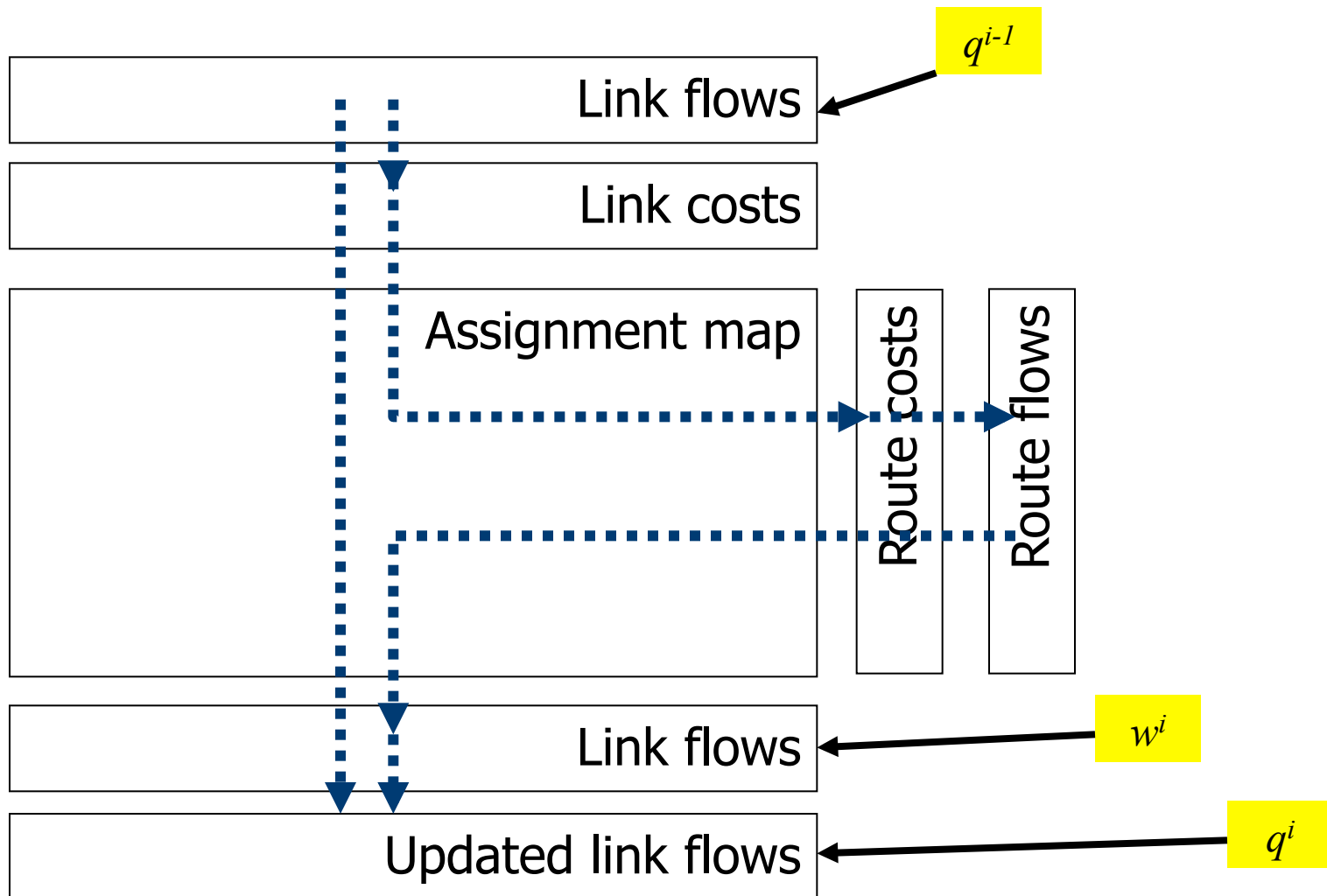
1. Specify paths/choice sets
2. Calculate path costs
3. Assign OD (Probit, Logit)=> route flows
4. Calculate link flows
5. Recalculate link flows (MSA)

$$q_a^i = q_a^{i-1} + (w_a^i - q_a^{i-1}) / i$$

where w_a^i = link flow of the intermediate solution

6. Check convergence
7. Go to step 2 or stop

Assignment map: LFA



When to use what?

- From a behavioural perspective it is better to model route choice
- Note that relatively coarse zones might be an argument as well
- In case of congestion it is obvious to model congestion as well
- So preferred techniques are stochastic assignment for off peak and SUE for peak periods
- Note that for a 24-hour period a stochastic assignment might be suitable as well
- However, in practice AON and DUE are preferred for computational reasons

Route based and link based assignment techniques

- Route based assignment has clear advantages compared to link based techniques
 - More control on the quality of the routes used
 - Possibility to use advanced choice modelling techniques
 - No need for repetitive path searches during assignment
- In practice mostly link based techniques are applied
- This is partly a lock-in phenomenon
 - Nearly all software packages use a link based algorithm
- Conservatism plays a role as well
 - The a-priori choice set generation might not have all relevant routes...

Uniqueness of DUE

- It can be shown that the travel time of a DUE is unique
- However, the flows that lead to these travel times need not to be unique
- For example, in case of multiple OD-pairs using two parallel route parts, the flows can be split up in any way you like as long as the total flows per route part remain constant (i.e. according to the DUE assignment)

5.

Special topics

Special topics

- Trucks
- Uniqueness of DUE
- Convergence speed DUE and SUE
- Duality gap SUE
- Frank-Wolfe or MSA?
- Values for parameters in BPR
- Where's the congestion?

Trucks

Three options

- Sum the OD-matrices of car and truck into OD-matrix vehicles or using a PCU-value
- Assign trucks before performing equilibrium assignment, e.g. using multiple routing, and use the flows as a preload (PCU!)
- Assign trucks and cars simultaneously (again using a PCU-value), i.e. multi-user class assignment

Uniqueness of DUE

- It can be shown that the travel time of a DUE is unique
- However, the flows that lead to these travel times need not to be unique
- For example, in case of multiple OD-pairs using two parallel route parts, the flows can be split up in any way you like as long as the total flows per route part remain constant (i.e. according to the DUE assignment)

Convergence speed DUE and SUE

- In an equilibrium assignment you distribute traffic over a set of routes
- In a DUE you have a single route per MSA-step, in a SUE you have multiple routes per MSA-step
=> SUE needs less MSA-steps
- However, if you use a Probit for SUE, you need iterations for a single MSA-step
=> SUE with Probit takes more computation time
=> SUE with Logit is faster than DUE

Duality gap SUE

- Duality gap in words:
total travel time based on links minus total travel time based on (latest) shortest paths
- For DUE the duality gap should become zero (Wardrop principle)
- In a SUE travellers opt routes that are longer but they perceive to be shortest
=> Total travel time for SUE is higher than for DUE
=> Duality gap > 0

Frank-Wolfe or MSA

- Frank-Wolfe algorithm is a generic mathematical tool
- Theoretically it is only justified if the travel time on a link is a function of the flow on the link
=> so what about intersections?
- MSA is a pragmatic approach, which proves to be rather robust

Values for parameters in BPR

- BPR-function: $t_a(q_a) = t_a^0 \left(1 + \alpha \left(\frac{q_a}{C_a} \right)^\beta \right)$
- Commonly mentioned values: $\alpha = 0.15$ and $\beta = 4$
- However, function differs per road type:
e.g. 0.15 is used for freeways, for regional and urban roads higher values are more suitable

Where's the congestion?

- Net result of assignment: network with flows
- Common unit for analysis: flow-capacity (q/c) ratio
- For which q/c -ratio there is congestion?
- Practice:
 q/c -ratio > 0.85 : congestion
(N.B. q represent average flow, thus $q/c=1$ implies 50% congestion)
- Where's the queue?
- Link having low capacity and not in front of that link

6.

Reprise Probit and logit

Probit and Logit

- Let's first go back to discrete choice theory

Discrete choice theory

Definitions:

Each alternative i has an objective/observable utility V_i

Each individual faces a subjective non-observed utility ε_i for each alternative i .

Utility of alternative i for each individual: $U_i = V_i + \varepsilon_i$

Behavior:

An individual will choose alternative i if this alternative has the highest utility, i.e. if

$$U_i \geq U_j \quad \text{for all } j$$

Discrete choice theory

$$\begin{aligned} p_i &= P(U_i \geq U_j \text{ for all } j) \\ &= P(V_i + \varepsilon_i \geq V_j + \varepsilon_j \text{ for all } j) \\ &= P(\varepsilon_j - \varepsilon_i \leq V_i - V_j \text{ for all } j) \end{aligned}$$

If ε_i 's are all
Gumbel distributed
(independent, with
scale parameter μ),

Logit-model

Easy to solve

$$p_i = \frac{e^{\mu V_i}}{\sum_j e^{\mu V_j}}$$

If ε_i 's are all
normally distributed
(independent),

Probit-model

Can only be solved
by simulation

$$p_i = K .$$

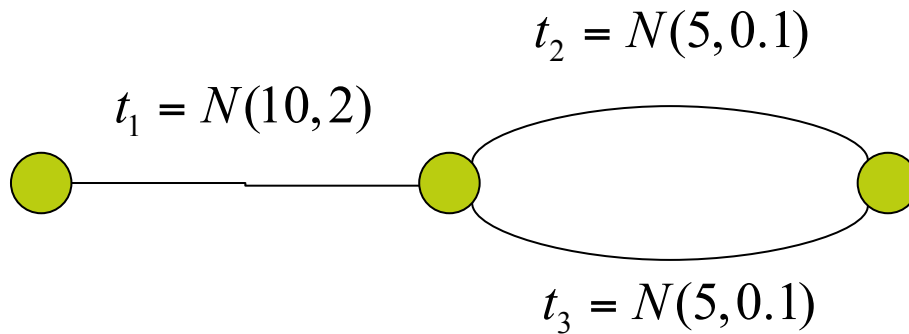
Probit and logit in transport models

- Note that discussion so far is independent on type of choice
- Modelling practice:
 - Demand modelling: dominated by logit
 - Assignment: Probit or logit
- In case of assignment the link is the basic element, thus the error term is defined at link level
- In this case normal distribution is the most likely assumption

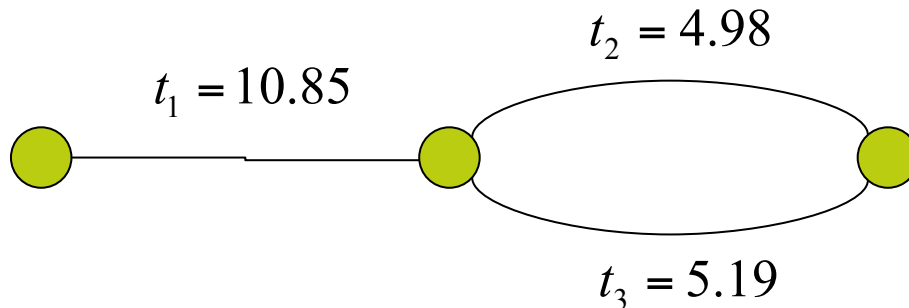
How is the probit assignment solved?

- Iteratively, using MSA approach
- In each iteration a specific network state is considered i.e. a network having specific link times
- These link times are sampled from the distributions that are assumed at link level
- Consequence is that overlapping routes in a specific network state are consistent

Example

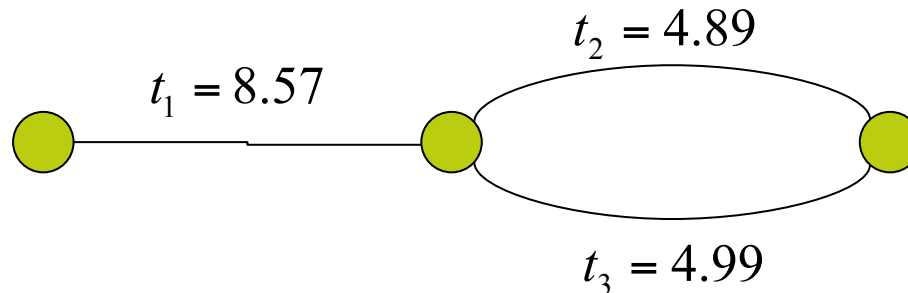


State 1



Route 1=15.83
Route 2=16.04

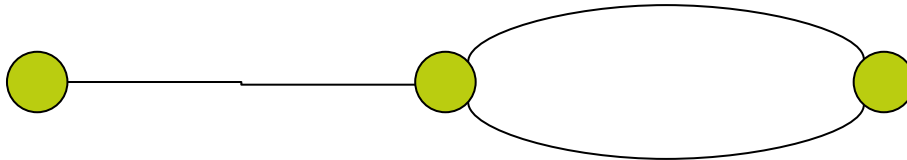
State 2



Route 1=13.46
Route 2=13.56

Switch from link level to route level

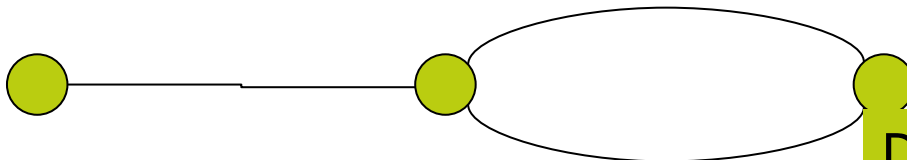
$$t_{r1} = N(15, 2)$$



$$t_{r2} = N(15, 2)$$

$$t_{r1} = 15.85$$

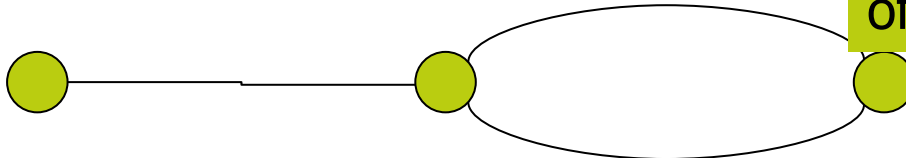
State 1



$$t_{r2} = 14.24$$

$$t_{r1} = 16.78$$

State 2



$$t_{r2} = 13.57$$

Differences between routes is much larger due to (implicit) different assumption for time of link 1 within a given state

Probit and logit at route level

- For the error term at route level both Gumbell distribution ($\Rightarrow$ logit) or normal distribution ($\Rightarrow$ probit) can be assumed
- In both cases the same error is made:
in case of overlapping routes consistency within a network state is not guaranteed
- However, logit is much easier to compute.....
- Today, there are advanced logit models that can correct for overlap