Nonlinear Theory of Elasticity

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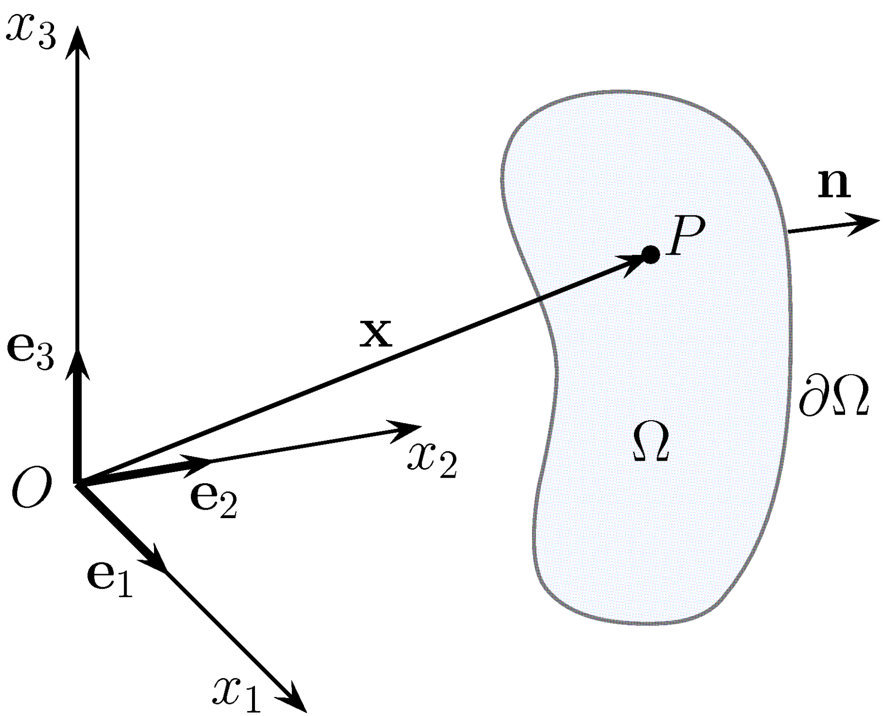
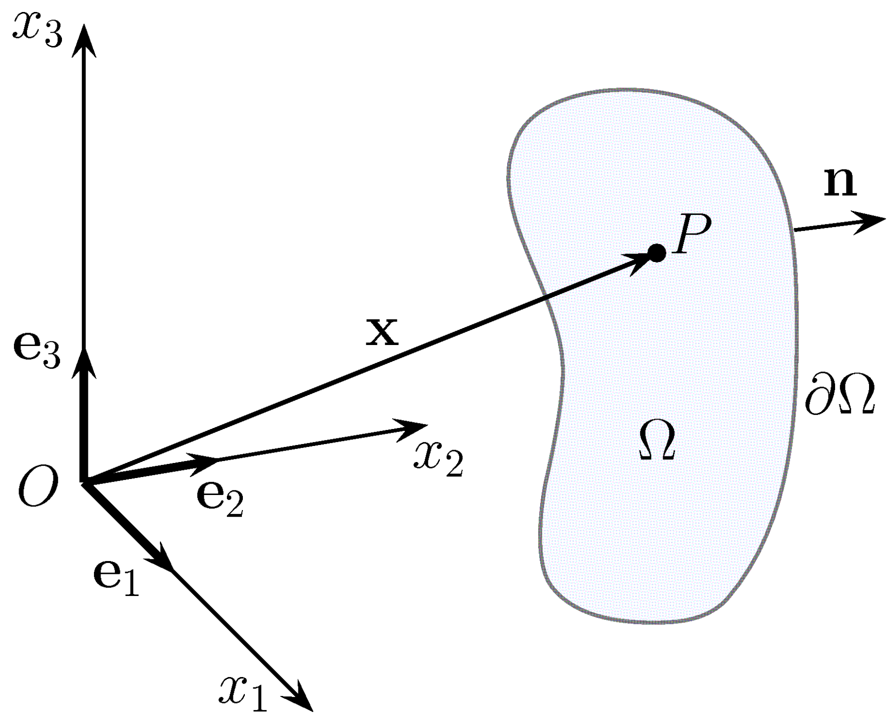
Cartesian global coordinate system with base vectors of the Euclidian space



- orthonormal basis
- origin *O*
- point *P*
- domain Ω of a deformable body
- closed domain surface $\partial\Omega$

$$\mathbf{e}_i \; := \; \frac{\partial \mathbf{x}}{\partial x_i} \; \in \mathbb{R}^3, \qquad \mathbf{x}_i^T \mathbf{x}_j \; = \; \delta_{ij}$$

geometry description

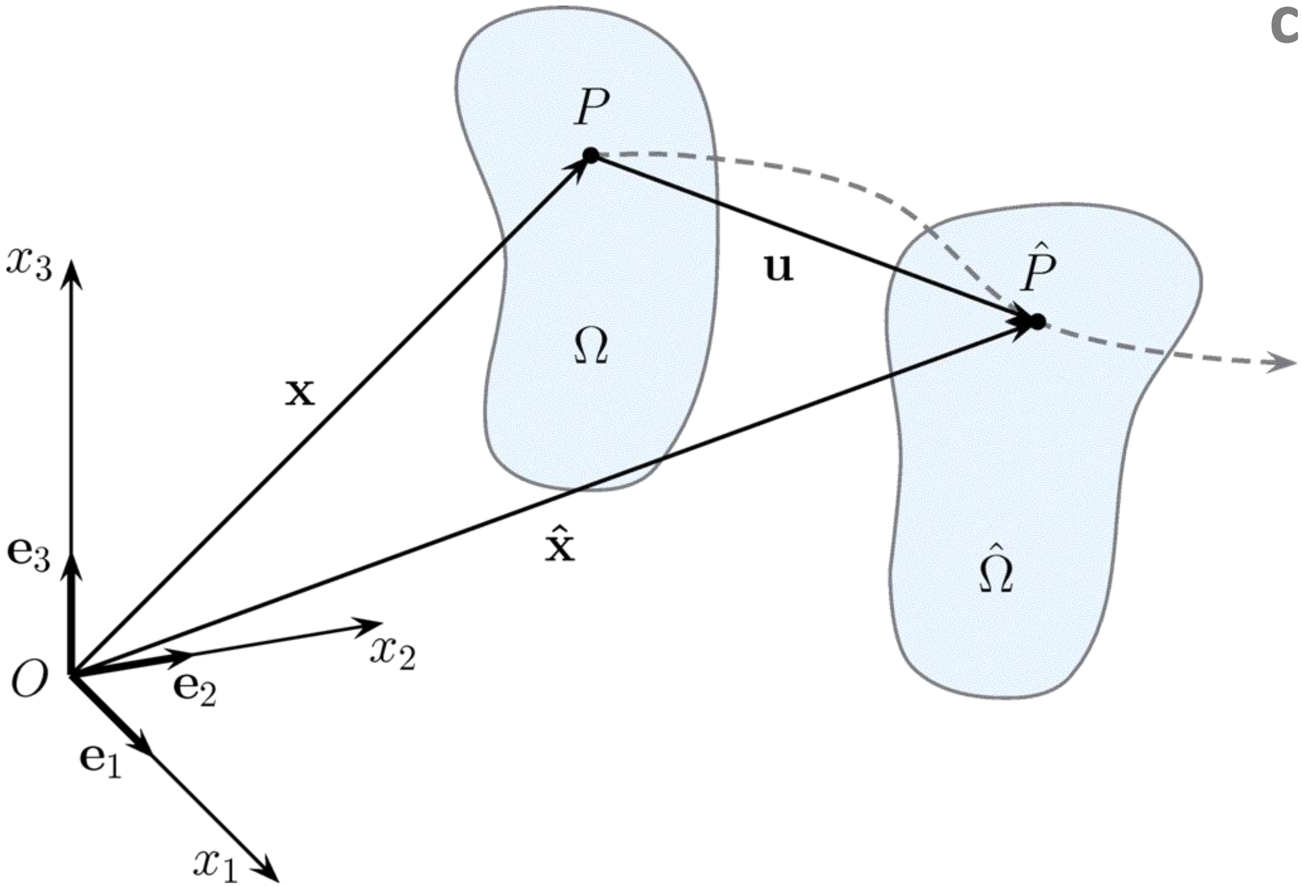


$$\mathbf{x} = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 + x_3\,\mathbf{e}_3$$

$$\mathbf{n} = n_1\,\mathbf{e}_1 + n_2\,\mathbf{e}_2 + n_3\,\mathbf{e}_3$$

solid body	neighboring points remain neighboring points independent of time
rigid body	distant between points remains constant during displacement
deformable body	distance between neighboring points may change with time

configurations



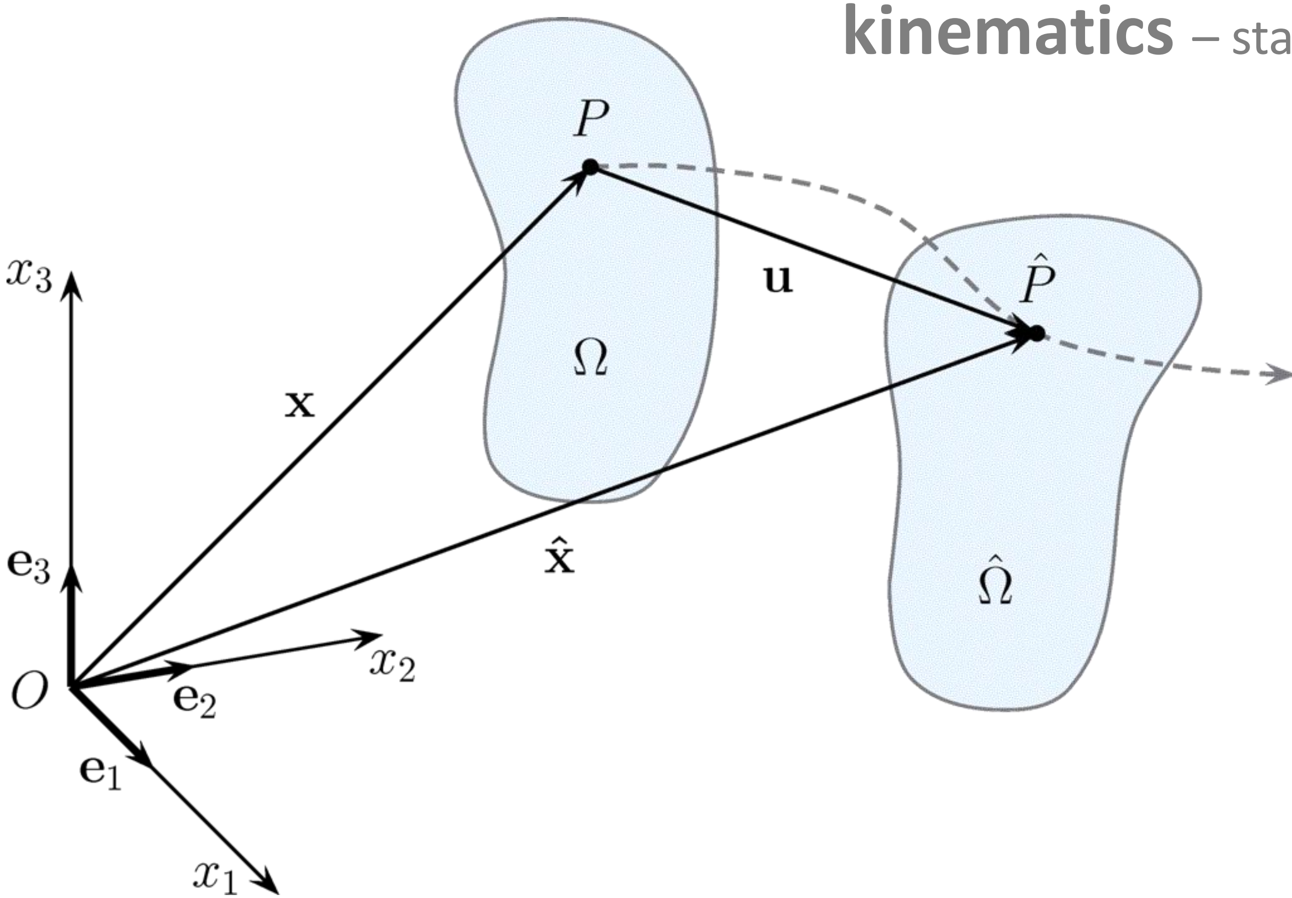
reference configuration Ω

- often: state at time $t = 0$
- material points $P(\mathbf{x}, t)$

instant configuration $\hat{\Omega}$

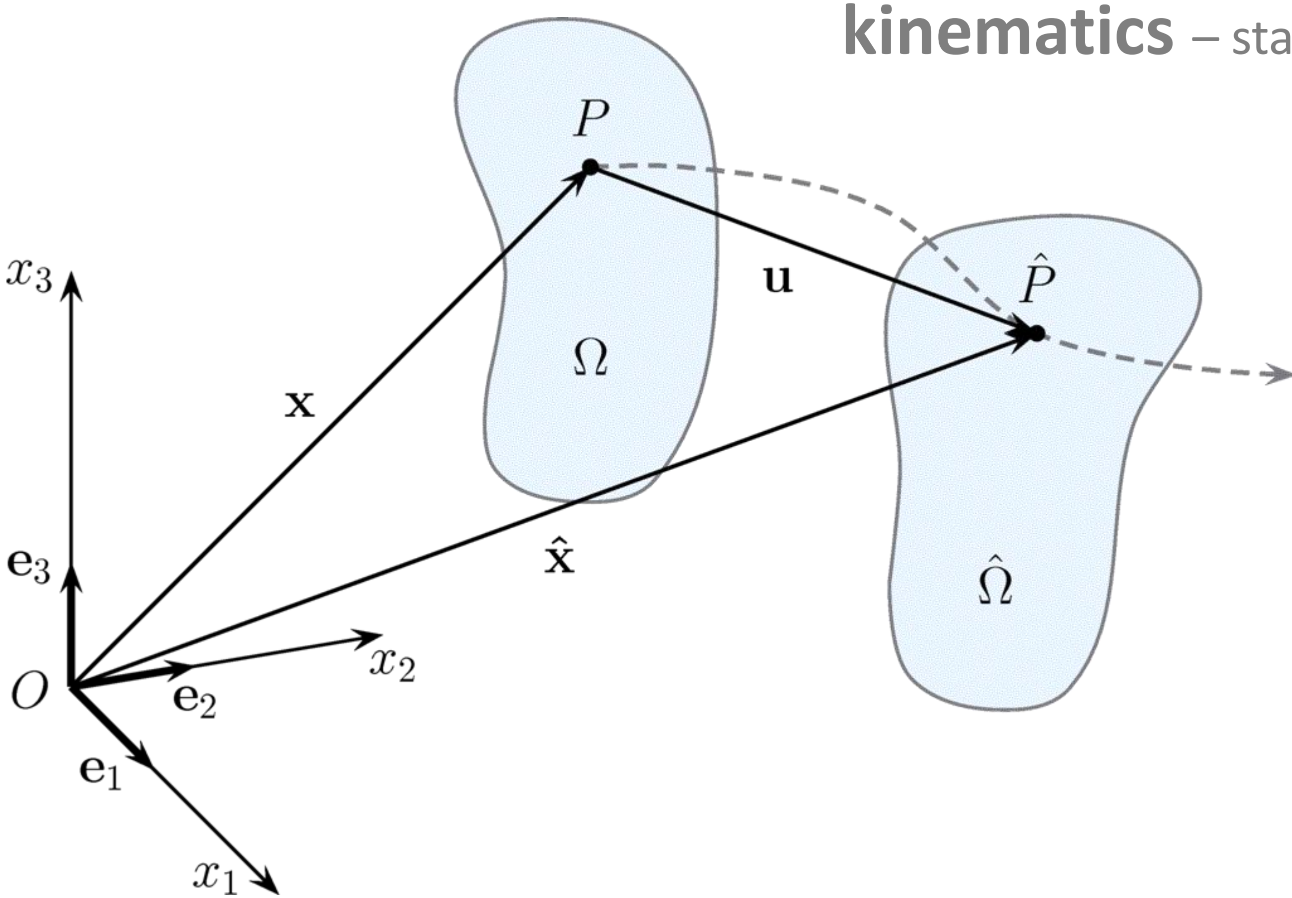
- often: state at time $\hat{t} \neq t$
- material points $P(\hat{\mathbf{x}}, t)$

kinematics – state of displacements



$$\mathbf{u} := \hat{\mathbf{x}} - \mathbf{x} \qquad \mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$

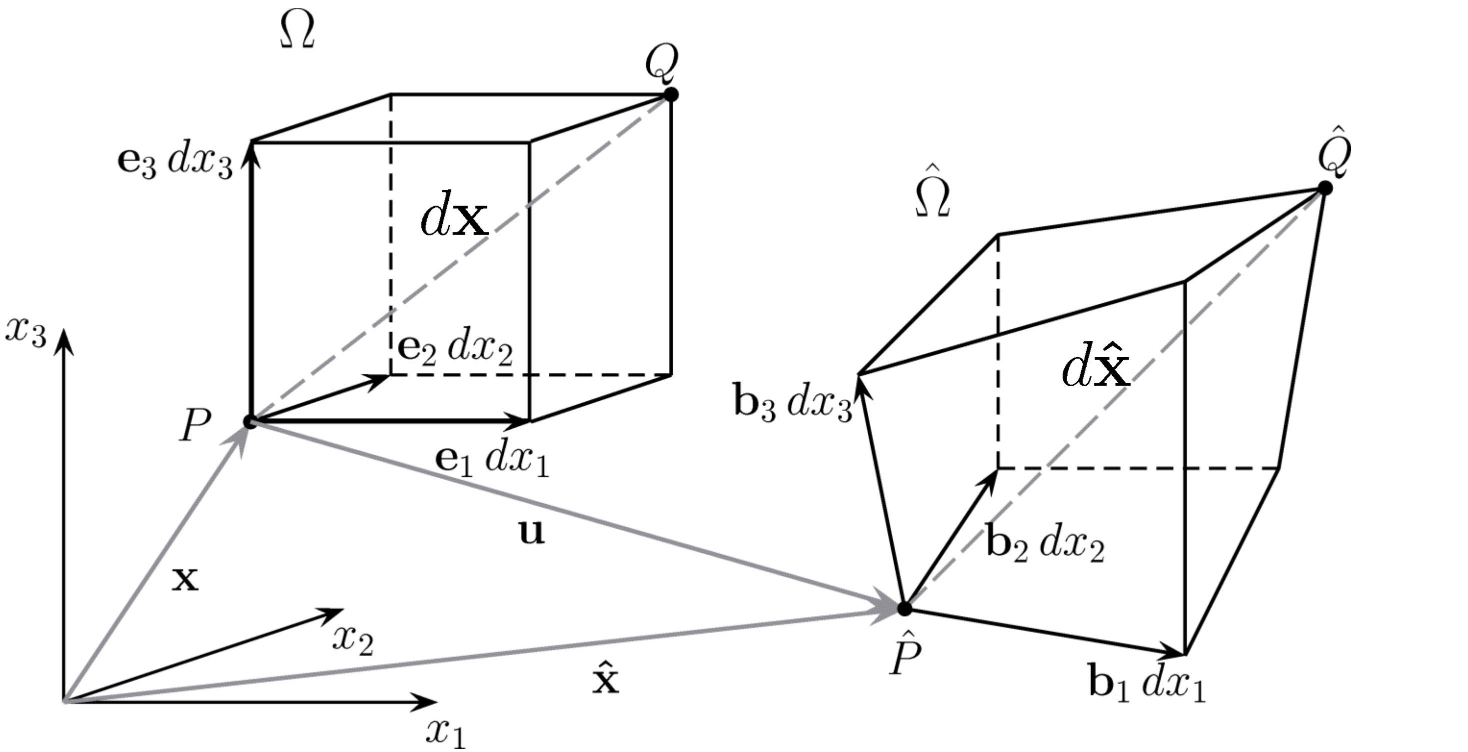
$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \hat{\mathbf{x}} = \hat{x}_1\mathbf{e}_1 + \hat{x}_2\mathbf{e}_2 + \hat{x}_3\mathbf{e}_3$$



displacement $\mathbf{u}$ is a combination of

- rigid body movement/rotation (AB = const for any two points)
- deformation (AB ≠ const)

A, B neighboring points of the deformable body

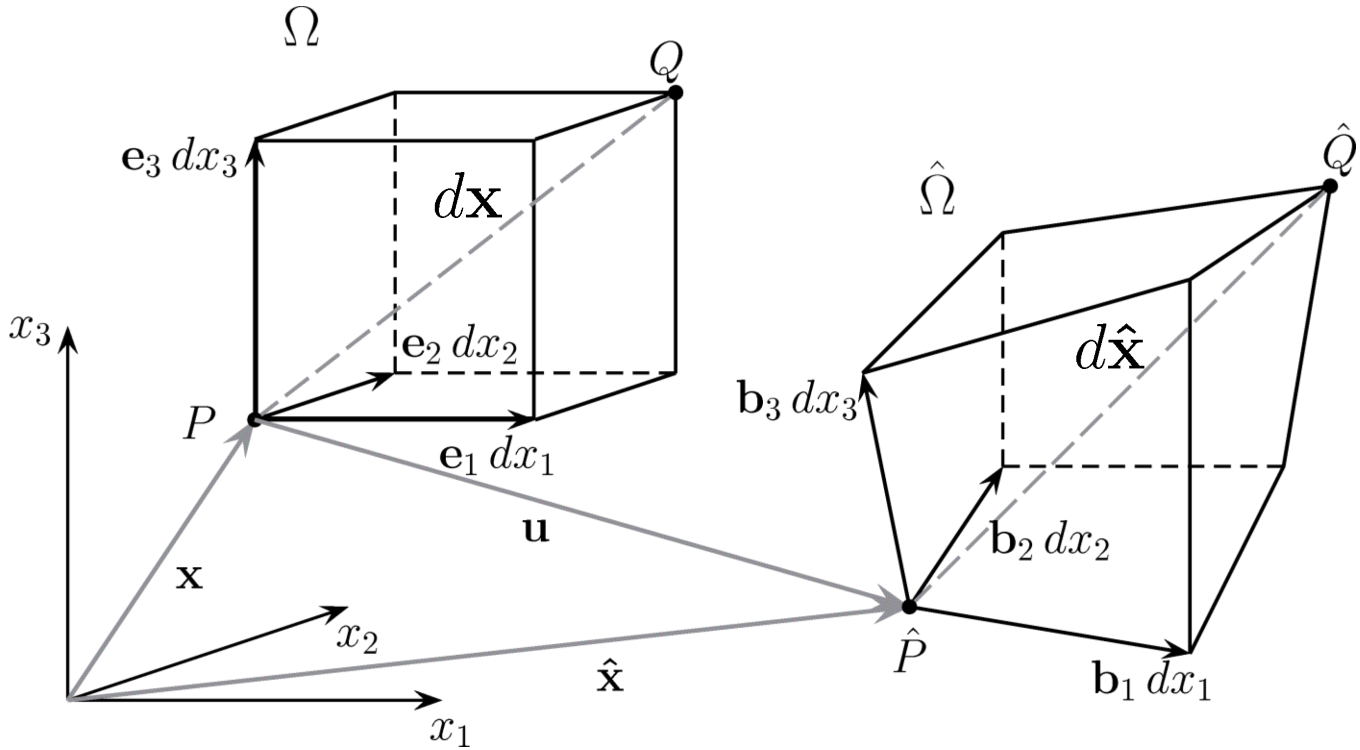


- infinitesimal volume considered
- base vectors of reference and instant configuration

$$\begin{aligned} d\mathbf{x} &= \mathbf{e}_1\,dx_1 + \mathbf{e}_2\,dx_2 + \mathbf{e}_3\,dx_3 \\ d\hat{\mathbf{x}} &= \mathbf{b}_1\,dx_1 + \mathbf{b}_2\,dx_2 + \mathbf{b}_3\,dx_3 \end{aligned}$$

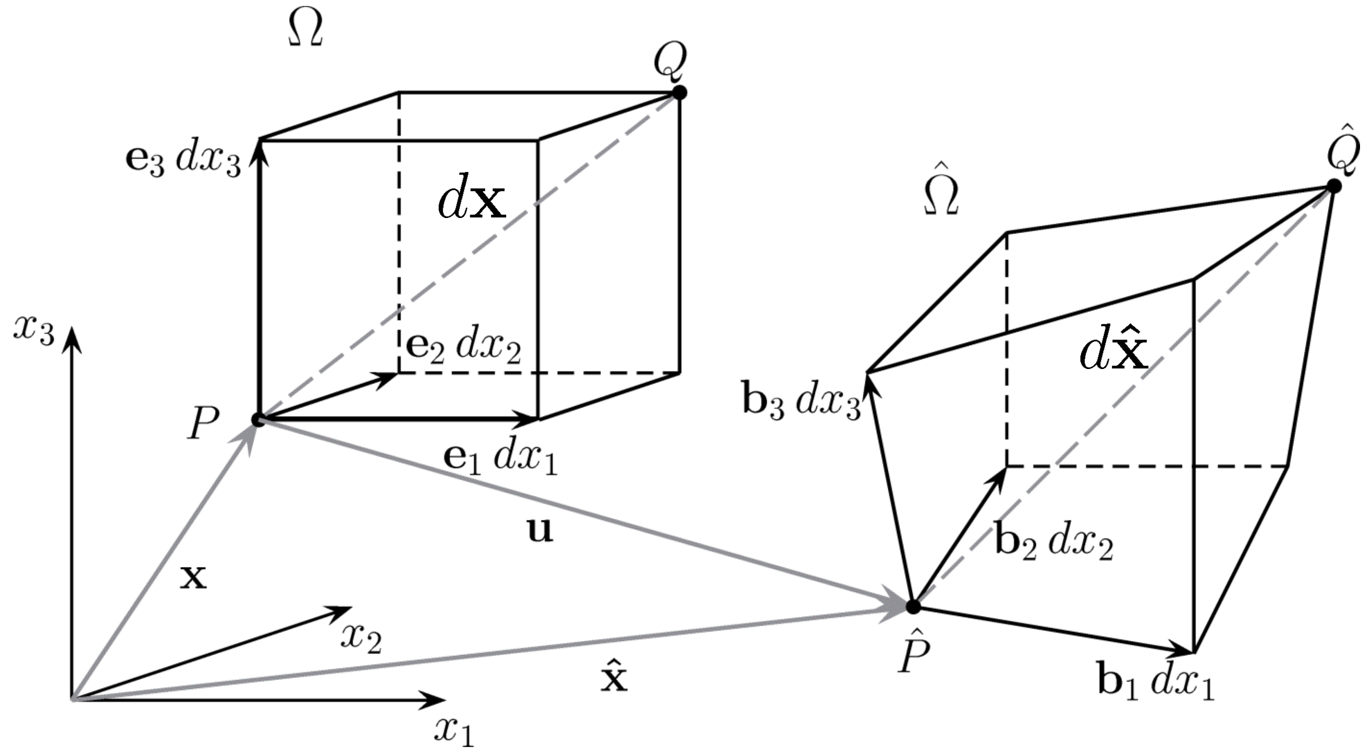
$$dx_i \quad : \quad \text{infinitesimal edge length}$$

$$\mathbf{e}_i \;:=\; \frac{\partial \mathbf{x}}{\partial x_i} \qquad i \in \{1,2,3\} \qquad \mathbf{b}_i \;:=\; \frac{\partial \hat{\mathbf{x}}}{\partial x_i} \;=\; \frac{\partial(\mathbf{x}+\mathbf{u})}{\partial x_i} \;=\; \mathbf{e}_i + \frac{\partial \mathbf{u}}{\partial x_i}$$



method of **Lagrange**

- material particle identification in the reference configuration Ω
- particle location $\mathbf{X}$ at time $\hat{t}$ is a function of $(\mathbf{x}, t)$ and $\hat{t}$
- analog for state variables



method of **Euler**

- material particle identification in the instant configuration $\hat{\Omega}$
- particle location $\mathbf{x}$ at time $\hat{t}$ is a function of $(\hat{\mathbf{x}}, t)$ and $\hat{t}$
- analog for state variables

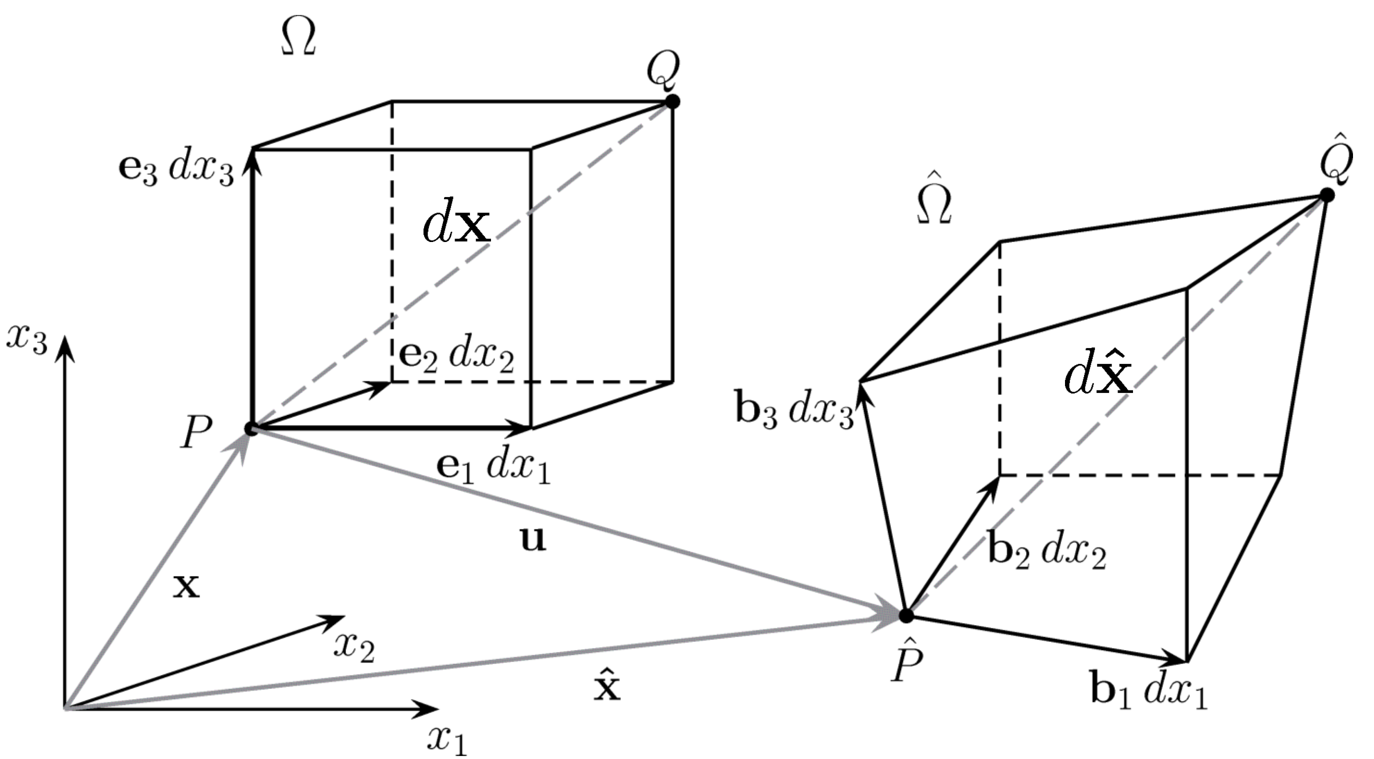
- material deformation gradient **F**
- representation of diagonal $d\hat{\mathbf{x}}$ as function of diagonal $d\mathbf{x}$
- columns of **F** are the instant base vectors $\mathbf{b}_k$ (k=1,2,3)

$$d\hat{\mathbf{x}} = \mathbf{F}\,d\mathbf{x}$$

$$\begin{bmatrix} d\hat{x}_1 \\ d\hat{x}_2 \\ d\hat{x}_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial \hat{x}_1}{\partial x_1} & \frac{\partial \hat{x}_1}{\partial x_2} & \frac{\partial \hat{x}_1}{\partial x_3} \\ \frac{\partial \hat{x}_2}{\partial x_1} & \frac{\partial \hat{x}_2}{\partial x_2} & \frac{\partial \hat{x}_2}{\partial x_3} \\ \frac{\partial \hat{x}_3}{\partial x_1} & \frac{\partial \hat{x}_3}{\partial x_2} & \frac{\partial \hat{x}_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

- use the deformation gradient **F** to show the change in volume for the infinitesimal volume in instant and reference configuration



- split of the deformation gradient **F** into a unit matrix **I** and a matrix **H**
- **H** contains the partial derivatives of **u** w.r.t. coordinates of ref. config.

$$d\hat{\mathbf{x}} = (\mathbf{I} + \mathbf{H})\, d\mathbf{x}$$

$$\begin{bmatrix} d\hat{x}_1 \\ d\hat{x}_2 \\ d\hat{x}_3 \end{bmatrix} = \begin{bmatrix} 1 + \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & 1 + \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & 1 + \frac{\partial u_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

$$\text{with} \quad \mathbf{H} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

- consider the change the length of $d\mathbf{x}$
- deformation measure referred to the reference configuration
- results in the strain tensor of **Green**

$$
\begin{aligned}
d\hat{\mathbf{x}}^T d\hat{\mathbf{x}} - d\mathbf{x}^T d\mathbf{x} &= d\mathbf{x}^T (\mathbf{I}+\mathbf{H})^T (\mathbf{I}+\mathbf{H}) d\mathbf{x} - d\mathbf{x}^T d\mathbf{x} \\
&= d\mathbf{x}^T (\mathbf{H}+\mathbf{H}^T+\mathbf{H}^T\mathbf{H}) d\mathbf{x}
\end{aligned}
$$

$$
(\mathbf{H}+\mathbf{H}^T+\mathbf{H}^T\mathbf{H}) := 2\,\mathbf{E}
$$

$$
\begin{aligned}
\text{strain tensor of } \hat{\Omega}: \quad \mathbf{E} &= \frac{1}{2}(\mathbf{H}+\mathbf{H}^T+\mathbf{H}^T\mathbf{H}) \\
\text{tensor coordinates:} \quad e_{im} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_m} + \frac{\partial u_m}{\partial x_i} + \sum_k \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_m}\right) \\
& \quad i, m, k \in \{1, 2, 3\}
\end{aligned}
$$

- consider the change the length of $d\mathbf{x}$
- deformation measure referred to the reference configuration
- results in the strain tensor of **Green-Lagrange**

$$
\begin{aligned}
d\hat{\mathbf{x}}^T d\hat{\mathbf{x}} - d\mathbf{x}^T d\mathbf{x} &= d\mathbf{x}^T (\mathbf{I} + \mathbf{H})^T (\mathbf{I} + \mathbf{H}) d\mathbf{x} - d\mathbf{x}^T d\mathbf{x} \\
&= d\mathbf{x}^T (\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T \mathbf{H}) d\mathbf{x}
\end{aligned}
$$

$$
(\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T \mathbf{H}) := 2\,\mathbf{E}
$$

$$
\text{strain tensor of } \hat{\Omega}: \quad \mathbf{E} = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T \mathbf{H})
$$

$$
\text{tensor coordinates:} \quad e_{im} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_m} + \frac{\partial u_m}{\partial x_i} + \sum_k \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_m}\right)
$$

LINEAR THEORY $\quad i, m, k \in \{1, 2, 3\}$

$$\text{strain tensor of } \hat{\Omega}: \quad \mathbf{E} \quad = \quad \frac{1}{2}(\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T\mathbf{H})$$

$$\text{tensor coordinates:} \quad e_{im} \quad = \quad \frac{1}{2}\left(\frac{\partial u_i}{\partial x_m} + \frac{\partial u_m}{\partial x_i} + \sum_k \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_m}\right)$$

$$i, m, k \in \{1, 2, 3\}$$

$$\mathbf{E} \quad = \quad \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}$$

- diagonal coefficients e_{ii} — *stretch: measure of fibre elongation*
- off-diagonal coefficients e_{im} — *shear: measure of the angle between fibre angles*

... applying Voigt notation

$$\boldsymbol{\epsilon} = \mathbf{D}\,\mathbf{u}$$

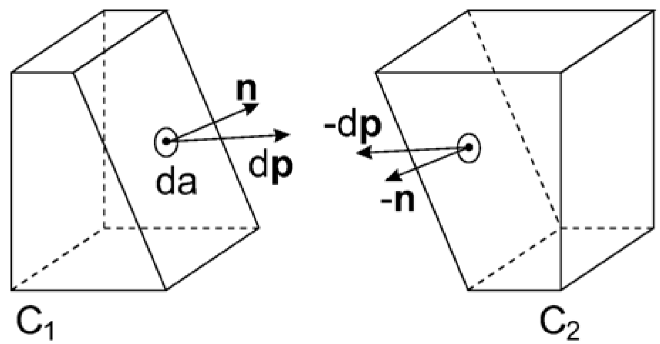
$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{31} \\ \epsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\epsilon_{ii} = e_{ii} = \left(\frac{\partial u_i}{\partial x_i} + \frac{1}{2} \sum_k \frac{\partial u_k}{\partial u_i} \frac{\partial u_k}{\partial u_i} \right)$$

$$\epsilon_{ij} = 2\,e_{ij} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_k \frac{\partial u_k}{\partial u_i} \frac{\partial u_k}{\partial u_j} \right)$$

stress vector – stress tensor

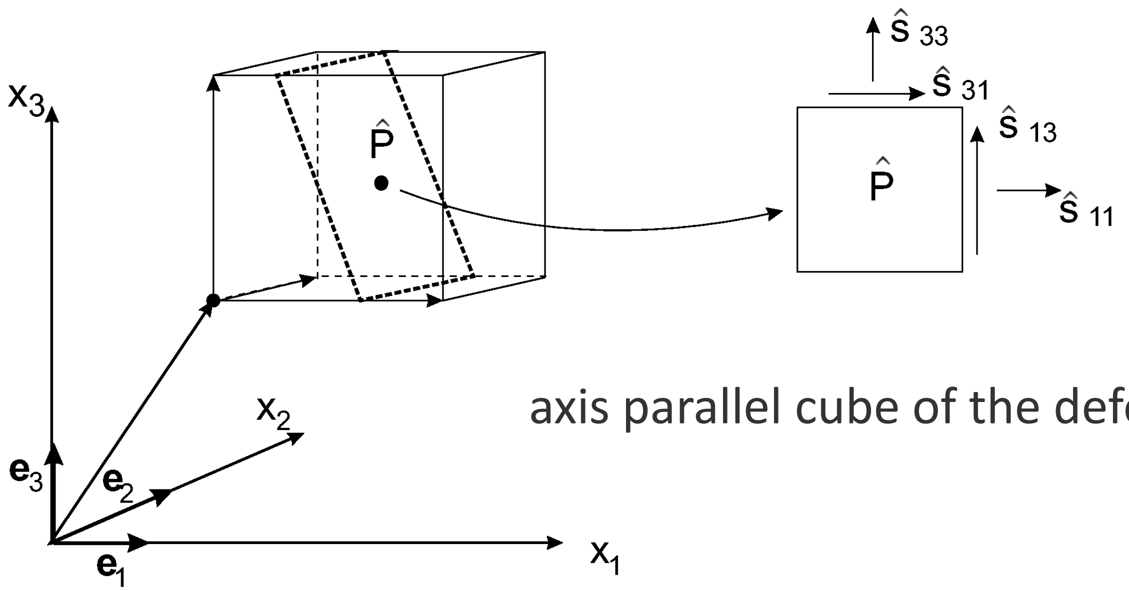
- stress vector in direction of the surface normal **n**
- action of subfield C_1 on C_2 is replaced by a fictitious force d**p**



$$\hat{\mathbf{t}} = \lim_{da \to 0} \frac{d\mathbf{p}}{da}$$

stress vector at point $\hat{P}$

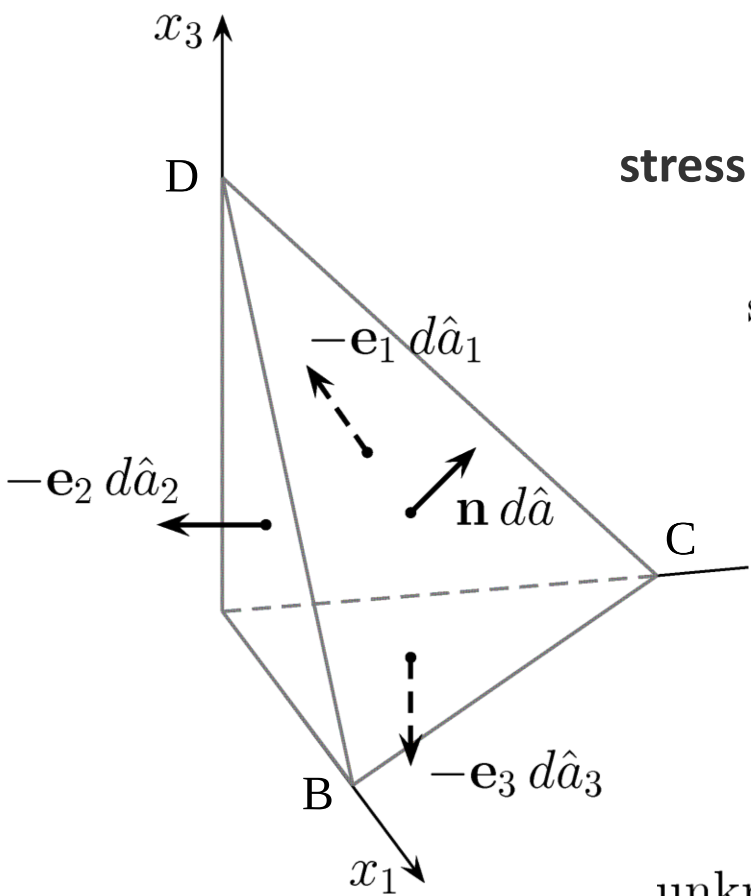
$$\begin{bmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \end{bmatrix} = \hat{t}_1 \, \mathbf{e}_1 + \hat{t}_2 \, \mathbf{e}_2 + \hat{t}_3 \, \mathbf{e}_3$$



axis parallel cube of the deformed configuration

$$\hat{\mathbf{S}} = \begin{bmatrix} \hat{\mathbf{s}}_1 & \hat{\mathbf{s}}_2 & \hat{\mathbf{s}}_3 \end{bmatrix} = \begin{bmatrix} \hat{s}_{11} & \hat{s}_{12} & \hat{s}_{13} \\ \hat{s}_{21} & \hat{s}_{22} & \hat{s}_{23} \\ \hat{s}_{31} & \hat{s}_{32} & \hat{s}_{33} \end{bmatrix}$$

- stress tensor of the **deformed configuration**
- columns of the Cauchy stress tensor are the stress vectors on the positive faces of the element



stress vector on a section

sum of surface vectors $\mathbf{e}_i d\hat{a}_i$ equals zero, since the tetrahedron is closed

$$\hat{\mathbf{n}} d\hat{a} - \mathbf{e}_1 d\hat{a}_1 - \mathbf{e}_2 d\hat{a}_2 - \mathbf{e}_3 d\hat{a}_3 \;=\; \mathbf{0}$$

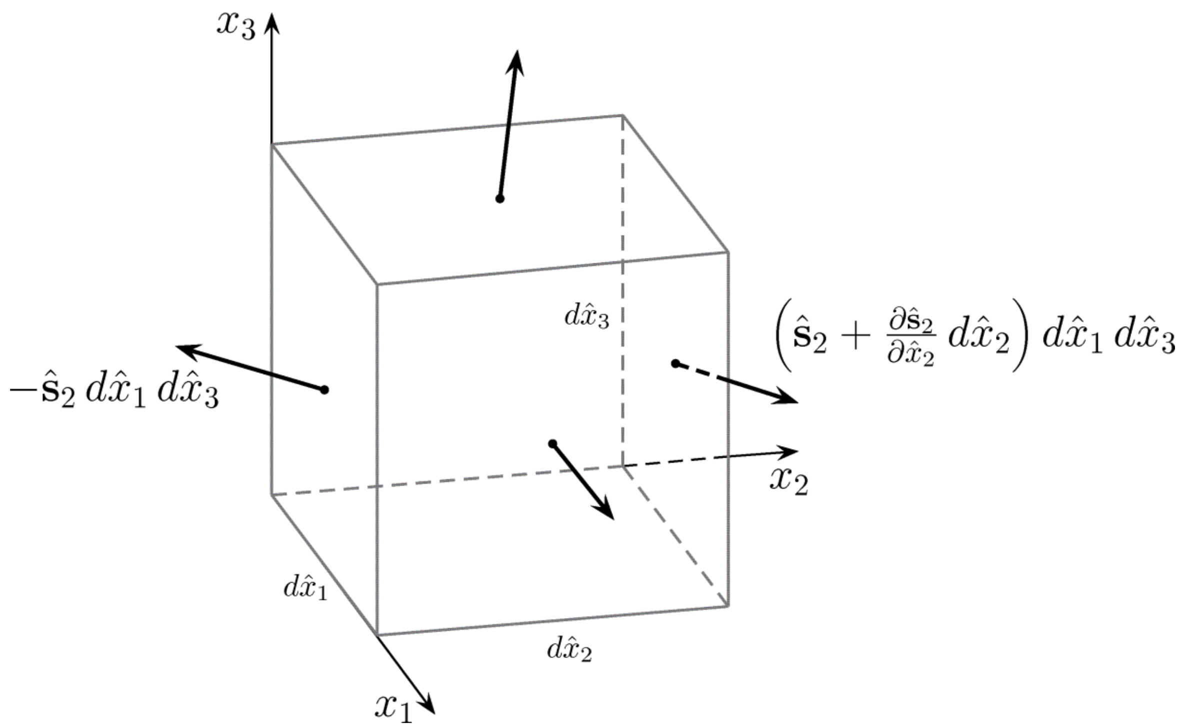
$$\text{with} \quad d\hat{a}_m \;=\; \mathbf{e}_m^T \hat{\mathbf{n}}\, da$$

unknown stress vector $\hat{\mathbf{t}}$ on face BCD follows from equillibrium state of the tetrahedron

$$0 \;=\; \hat{\mathbf{t}}\, da - \hat{\mathbf{s}}_1 da_1 - \hat{\mathbf{s}}_2 da_2 - \hat{\mathbf{s}}_3 da_3$$

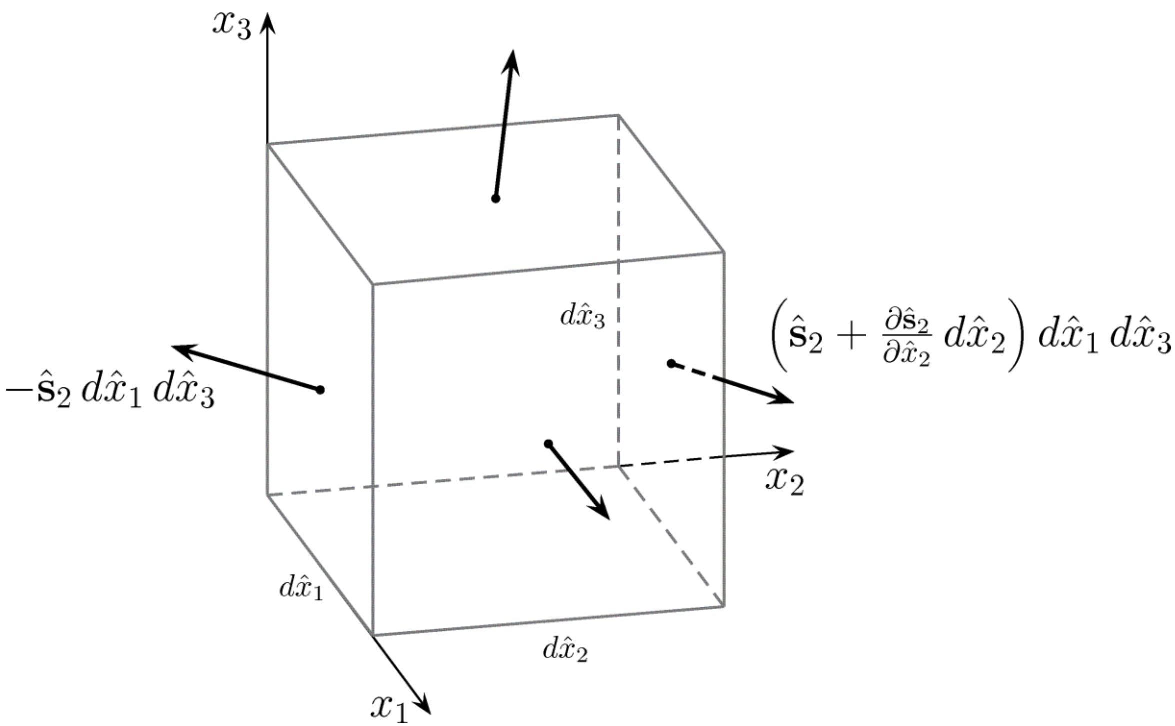
$$0 \;=\; \hat{\mathbf{t}}\, da - \hat{\mathbf{s}}_1 n_1 da - \hat{\mathbf{s}}_2 n_2 da - \hat{\mathbf{s}}_3 n_3 da$$

$$\hat{\mathbf{t}} \;=\; \hat{\mathbf{S}}\, \mathbf{n}$$



forces in x_2- and x_3-direction analogous
load acting on the unit volume $\rightarrow \hat{\rho}\mathbf{p}$

$$\sum F = \mathbf{0} = -\hat{\mathbf{s}}_1\, d\hat{x}_2\, d\hat{x}_3 + (\hat{\mathbf{s}}_1 + \frac{\partial \hat{\mathbf{s}}_1}{\partial \hat{x}_1} d\hat{x}_1)\, d\hat{x}_2\, d\hat{x}_3 + \hat{\rho}\mathbf{p}\, d\hat{x}_1\, d\hat{x}_2\, d\hat{x}_3$$



$$\frac{\partial \hat{\mathbf{s}}_1}{\partial \hat{x}_1} + \frac{\partial \hat{\mathbf{s}}_2}{\partial \hat{x}_2} + \frac{\partial \hat{\mathbf{s}}_3}{\partial \hat{x}_3} + \hat{\rho}\mathbf{q} \;=\; \mathbf{0}$$

sum of the moments acting on the element is null in the state of equilibrium

→ from this follows the symmetry of the Cauchy stress tensor

$$\hat{s}_{ik} \;=\; \hat{s}_{ki}$$

- consider a surface element of the reference configuration Ω which is replaced to the instant configuration $\hat{\Omega}$

$$d\mathbf{a}(=\mathbf{n}\,da) \quad \rightarrow \quad d\hat{\mathbf{a}}(=\mathbf{n}\,d\hat{a})$$

- 1st Piola-Kirchhoff tensor causes the same force $d\mathbf{f}$ (definition!) on $d\mathbf{a}(=\mathbf{n}\,da)$ & $d\hat{\mathbf{a}}(=\mathbf{n}\,d\hat{a})$

$$d\mathbf{f} \;=\; \mathbf{P}\,d\mathbf{a} \;=\; \hat{\mathbf{S}}\,d\hat{\mathbf{a}} \;=\; (det\mathbf{F})\,\hat{\mathbf{S}}\,\hat{\mathbf{F}}^T\,d\mathbf{a}$$

$$\mathbf{P} \;=\; (det\mathbf{F})\,\hat{\mathbf{S}}\,\hat{\mathbf{F}}^T$$

1st Piola Kirchhoff tensor

- stress coordinates are referred to the global base vectors
- 1st PK stress tensor is **unsymmetric**, in general, not in use!

- PK1 force vector referred to the basis of the instant configuration $\hat{\Omega}$

$$\mathbf{p}_k = p_{1k}\mathbf{e}_1 + p_{2k}\mathbf{e}_2 + p_{3k}\mathbf{e}_3$$

$$\mathbf{p}_k = s_{1k}\mathbf{b}_1 + s_{2k}\mathbf{b}_2 + s_{3k}\mathbf{b}_3$$

- bases vectors in $\hat{\Omega}$ are the columns of the deformation gradient

$$\mathbf{P} = \mathbf{F}\,\mathbf{S}$$

- relation between Cauchy stress tensor and 2nd PK tensor

$$\mathbf{S} = (det\mathbf{F})\hat{\mathbf{F}}\,\hat{\mathbf{S}}\,\hat{\mathbf{F}}^t$$

$$\hat{\mathbf{S}} = (det\hat{\mathbf{F}})\mathbf{F}\,\mathbf{S}\,\mathbf{F}^t$$

2nd Piola Kirchhoff tensor is symmetric!

energetically conjugate stress tensor to the Green strain tensor

... linear elasticity – Hooke's law

$$\boldsymbol{\sigma} = \mathbf{C}\,\boldsymbol{\epsilon}$$

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{31}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} a & b & b & 0 & 0 & 0\\ b & a & b & 0 & 0 & 0\\ b & b & a & 0 & 0 & 0\\ 0 & 0 & 0 & c & 0 & 0\\ 0 & 0 & 0 & 0 & c & 0\\ 0 & 0 & 0 & 0 & 0 & c\end{bmatrix} \begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\\ \epsilon_{23}\\ \epsilon_{31}\\ \epsilon_{12}\end{bmatrix}$$

$$a = \frac{E\,(1-\nu)}{(1+\nu)(1-2\nu)}$$

$$b = a\,\frac{\nu}{(1-\nu)} \qquad c = a\,\frac{(1-2\nu)}{2(1-\nu)}$$

governing equations

$$
\begin{array}{llcll}
\text{strain} - \text{displm.} & e_{im} & = & \frac{1}{2}(u_{i,m} + u_{m,i} + \sum_k u_{k,m}u_{k,i}) & \hat{\mathbf{x}} \in \hat{\Omega} \\
\text{stress} - \text{strain} & \hat{s}_{im} & = & C_{imkl}\, e_{kl} & \hat{\mathbf{x}} \in \hat{\Omega} \\
\text{equilibrium} & 0 & = & \hat{s}_{i1,1} + \hat{s}_{i2,2} + \hat{s}_{i3,3} + \hat{\rho} q_i & \hat{\mathbf{x}} \in \hat{\Omega} \\
\text{stress vector} & \hat{t}_i & = & \hat{\mathbf{s}}_i^T\, \mathbf{n} & \hat{\mathbf{x}} \in \partial\hat{\Omega}
\end{array}
$$

e_{im} / $\hat{s}_{im}$ Green-Lagrange/Cauchy strain/stress tensor coordinates

Dirichlet (prescribed displacements)

$$\mathbf{x} \in \partial\hat{\Omega} \;\wedge\; \mathbf{u} \in \hat{\Gamma}_u \;:\; u_i = u_{i0}$$

Neumann (prescribed stresses)

$$\mathbf{x} \in \partial\hat{\Omega} \;\wedge\; \mathbf{t} \in \hat{\Gamma}_t \;:\; t_i = t_{i0}$$

$\hat{\Gamma}_u$: boundary of prescribed displacements components

$\hat{\Gamma}_t$: boundary of prescribed stress vector components

weighted residual approach, cf linear theory of elasticity

- choice of a suited approximation rule for the displacement state
- definition of residuals which are not a priori satisfied
- choose of admissible/suited weight functions
- here: Bubnov-Galerkin approach: variation of displacements
- multiply residuals with weight functions
- integration over volume of the **instant configuration**

$$\int_{\hat{\Omega}} g(\hat{\mathbf{x}})\, r(\hat{\mathbf{x}})\, d\hat{\mathbf{x}} \;=\; 0$$

1st integral form

$$\int_{\hat{\Omega}} \sum_i \sum_m (\delta u_i \frac{\partial \hat{s}_{im}}{\partial \hat{x}_m})\, d\hat{v} \;+\; \int_{\hat{\Omega}} \sum_i \delta u_i\, \hat{\rho} q_i\, d\hat{v} +$$
$$\int_{\delta\hat{\Omega}} \sum_i \delta u_i\, (\hat{t}_i - \sum_m \hat{s}_{im}\, n_m)\, d\hat{a} +$$
$$\int_{\hat{\Gamma}_t} \sum_i \delta u_i\, (\hat{t}_i - \hat{t}_{i_0})\, d\hat{a} + \int_{\hat{\Gamma}_u} \sum_i \delta \hat{t}_i\, (\hat{u}_i - \hat{u}_{i_0})\, d\hat{a} \;=\; 0$$

2nd integral form (Principle of virtual work)

$$\int_{\hat{\Omega}} \sum_i \sum_m \hat{s}_{im}\, \delta \left(\frac{\partial u_i}{\partial \hat{x}_m} \right) d\hat{v} \;=\; \int_{\hat{\Omega}} \sum_i \delta u_i\, \hat{\rho} q_i\, d\hat{v} + \int_{\hat{\Gamma}_t} \sum_i \delta u_i\, \hat{t}_{i_0}\, d\hat{a}$$

$$u_i \;=\; u_{i_0} \qquad \hat{x}_i \in \hat{\Gamma}_u$$

- spatial integral form derived for volume elements of the instant config.
- volume of the body in $\hat{\Omega}$ is unknown!

integral equation is referred to the known reference configuration

replace ...

- unknow volume $d\hat{v}$ *with* known volume dv
- Cauchy coordinates $\hat{s}_{im}$ *with* 2nd Piola-Kirchhoff coordinates s_{im}
- instant coordinate $\hat{x}_i$ *with* $x_i + u_i$

on the left hand follows

$$\sum_i \sum_m \hat{s}_{im}\, \delta\left(\frac{\partial u_i}{\partial \hat{x}_m}\right) d\hat{v} \;=\; \sum_i \sum_m \delta e_{im}\, s_{im}\, dv$$

on the right hand follows in analogy

$$\sum_i \delta u_i\, \hat{t}_i\, d\hat{a} \;=\; \sum_i \delta u_i\, p_i\, da$$

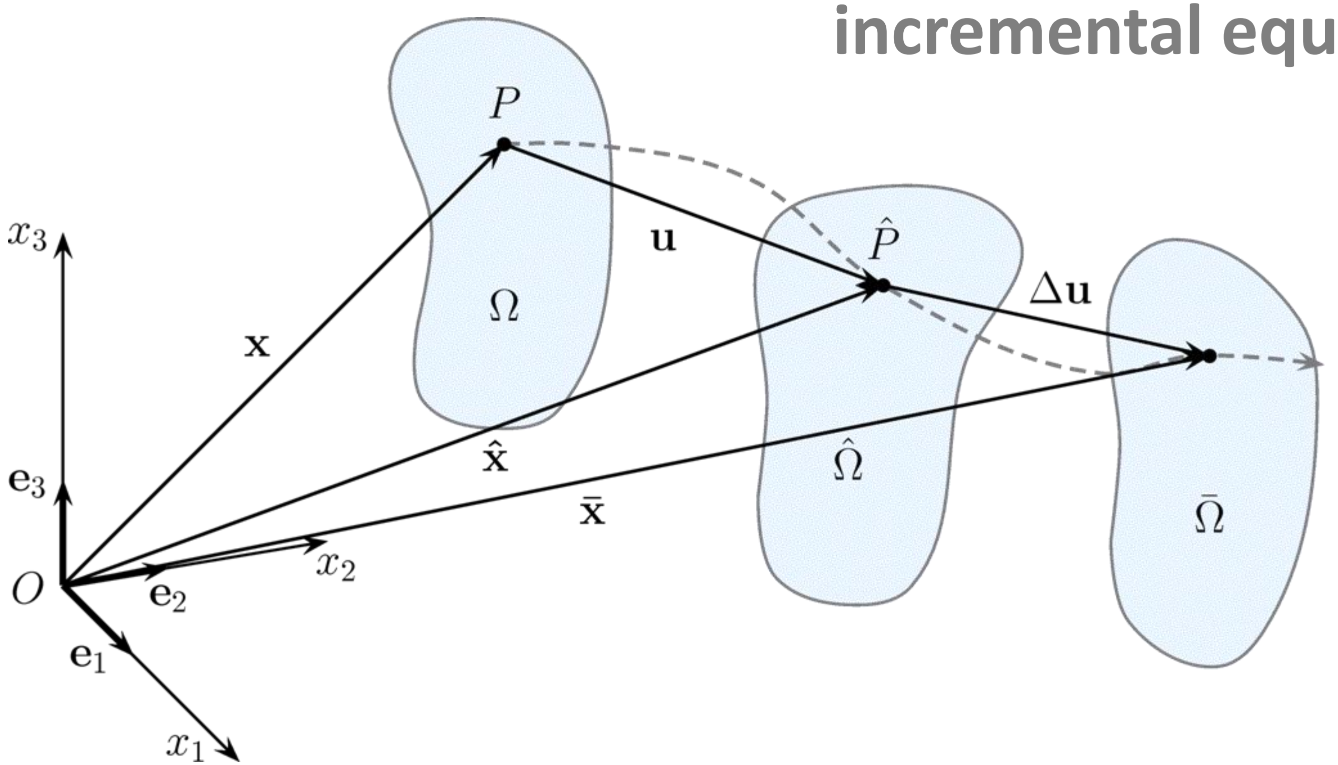
Principle of virtual work

$$\int_\Omega \sum_i \sum_m \delta e_{im}\, s_{im}\, dv \;=\; \int_\Omega \sum_i \delta u_i\, q_i\, dv + \int_{\Gamma_t} \sum_i \delta u_i\, p_{i0}\, da$$

$$\wedge \quad u_i \;=\; u_{i0} \quad \mathbf{x} \in \Gamma_u$$

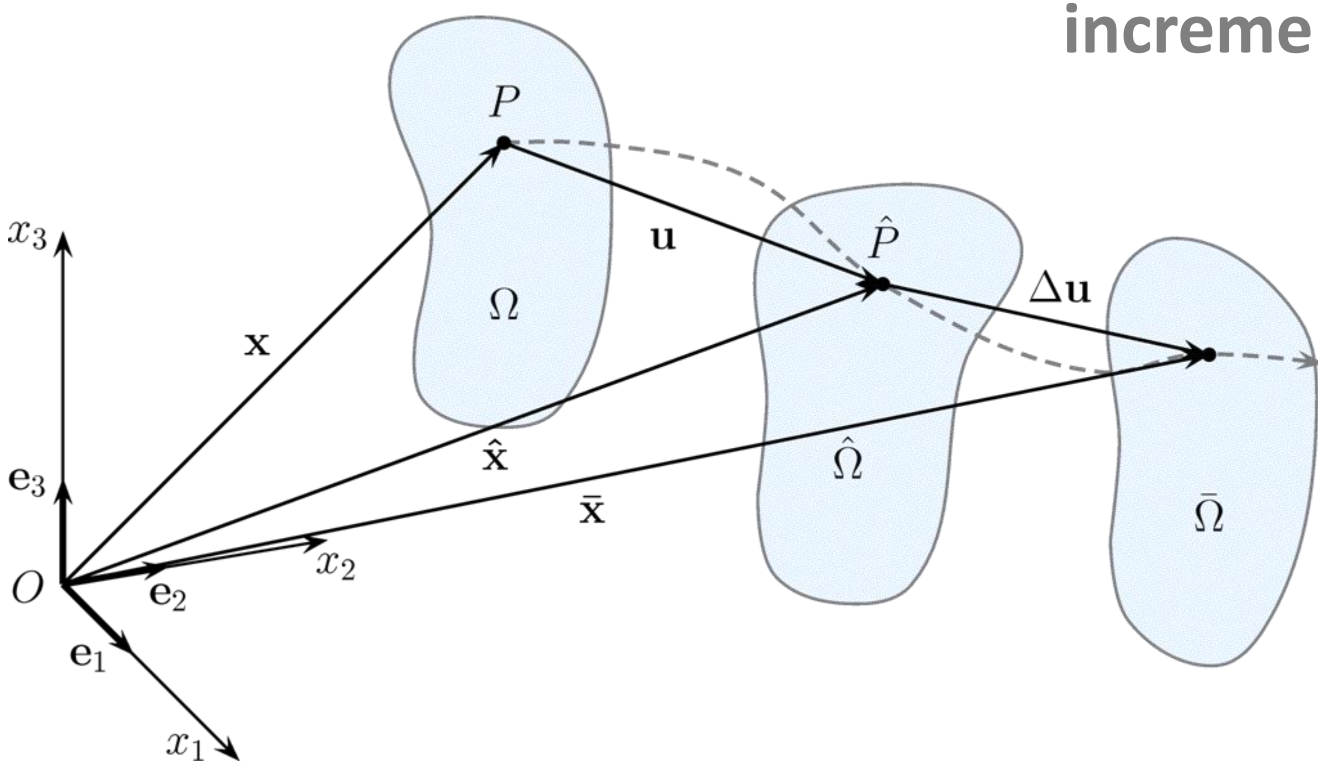
- strains are nonlinear functions of the derivatives (Green-Lagrange)
- stresses (2nd PK) are referred to base vectors of reference & instant config.
- conservative loads are assumed → independent of the displacements

incremental equations – strategy



- stepwise solution for the nonlinear equations $0, \Delta t, 2\Delta t, \ldots, t$
- initial configuration is assumed to be known
- solution at the end of each step
- governing equations are incremental equations
- consistent linearization leads to incremental equations

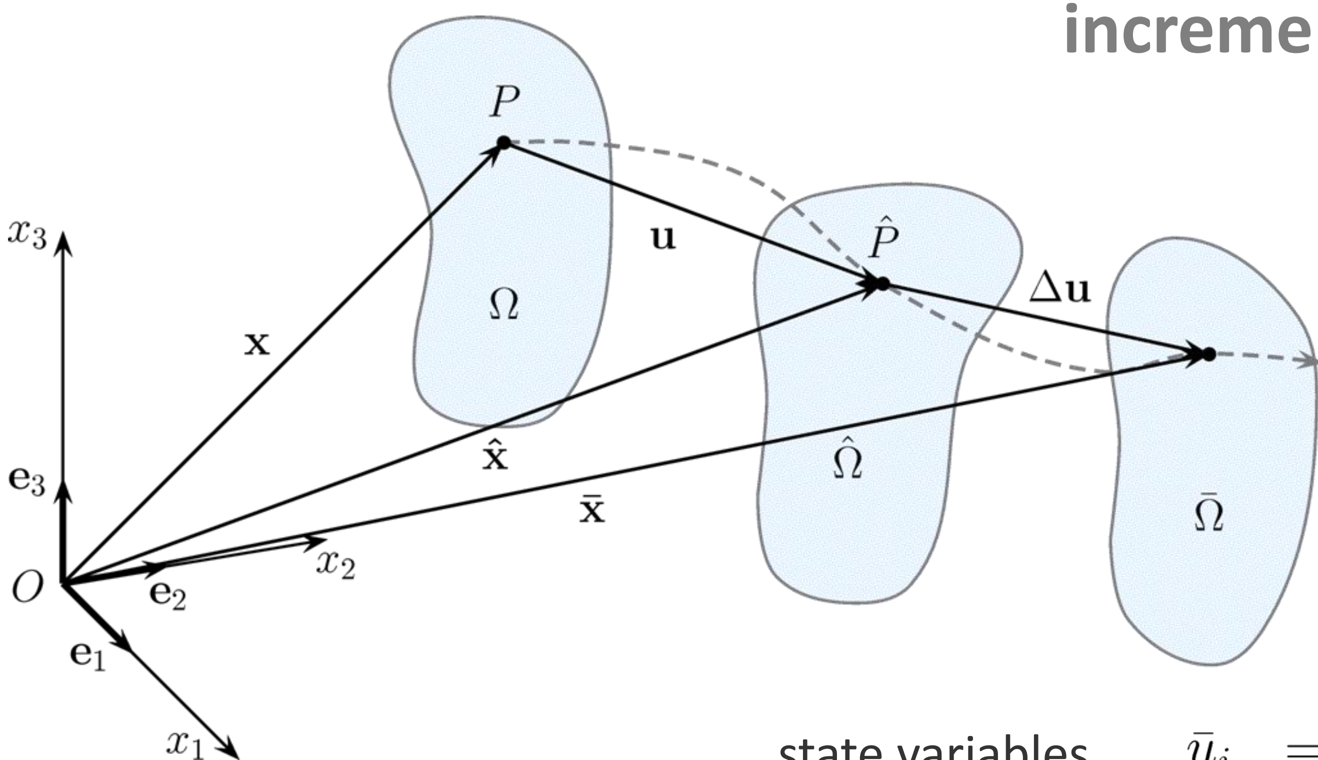
incremental equations



Ω	reference configuration
$\hat{\Omega}$	instant configuration I, known from previous step
$\bar{\Omega}$	instant configuration II, unknown

$\bar{\mathbf{u}}$	unknown displacement state at the end of step *i*
$\mathbf{u}$	known displacement state at beginning of step *i*
$\Delta\mathbf{u}$	displacement increment from $\hat{\Omega}$ to $\bar{\Omega}$

incremental equations



state variables

$$
\begin{aligned}
\bar{u}_i &= u_i + \Delta u_i \\
\bar{q}_{i0} &= q_{i0} + \Delta q_{i0} \\
\bar{e}_{ij} &= e_{ij} + \Delta e_{ij} \\
\bar{s}_{ij} &= s_{ij} + \Delta s_{ij}
\end{aligned}
$$

Total Lagrangian (TL) formulation → referred to Ω

Updated Lagrangian (UL) formulation → referred to $\hat{\Omega}$

state variables

$$
\begin{aligned}
\bar{u}_i &= u_i + \Delta u_i \\
\bar{q}_{i0} &= q_{i0} + \Delta q_{i0} \\
\bar{e}_{ij} &= e_{ij} + \Delta e_{ij} \\
\bar{s}_{ij} &= s_{ij} + \Delta s_{ij}
\end{aligned}
$$

Incremental strain-displacement relationship

$$
\begin{aligned}
\bar{e}_{ij} &= \frac{1}{2}(\bar{u}_{i,j} + \bar{u}_{j,i} + \sum_{k=1}^{3} \bar{u}_{k,i}\,\bar{u}_{k,j}) \\
&= \frac{1}{2}(u_{i,j} + \Delta u_{i,j} + u_{j,i} + \Delta u_{j,i} + \sum_{k=1}^{3}(u_{k,i} + \Delta u_{k,i})(u_{k,j} + \Delta u_{k,j})) \\
&= \frac{1}{2}(u_{i,j} + u_{j,i} + \Delta u_{i,j} + \Delta u_{j,i} + \\
&\qquad \sum_{k=1}^{3} u_{k,i}u_{k,j} + u_{k,i}\Delta u_{k,j} + u_{k,j}\Delta u_{k,i} + \Delta u_{k,i}\Delta u_{k,j})
\end{aligned}
$$

incremental equations

state variables

$$
\begin{aligned}
\bar{u}_i &= u_i + \Delta u_i \\
\bar{q}_{i0} &= q_{i0} + \Delta q_{i0} \\
\bar{e}_{ij} &= e_{ij} + \Delta e_{ij} \\
\bar{s}_{ij} &= s_{ij} + \Delta s_{ij}
\end{aligned}
$$

Incremental strain-displacement relationship

$$
\begin{aligned}
\bar{e}_{ij} &= e_{ij} + \Delta e^L_{ij} + \Delta e^N_{ij} \\
e_{ij} &= \frac{1}{2}(u_{i,j} + u_{j,i} + \sum_{k=1}^{3} u_{k,i} u_{k,j}) \\
\Delta e^L_{ij} &= \frac{1}{2}\left(\Delta u_{i,j} + \Delta u_{j,i} + \sum_{k=1}^{3} (u_{k,i} \Delta u_{k,j} + u_{k,j} \Delta u_{k,i})\right) \\
\Delta e^N_{ij} &= \frac{1}{2}\sum_{k=1}^{3} (\Delta u_{k,i} \Delta u_{k,j})
\end{aligned}
$$

state variables

$$\begin{aligned}\bar{u}_i &= u_i + \Delta u_i \\ \bar{q}_{i0} &= q_{i0} + \Delta q_{i0} \\ \bar{e}_{ij} &= e_{ij} + \Delta e_{ij} \\ \bar{s}_{ij} &= s_{ij} + \Delta s_{ij}\end{aligned}$$

Variation of the state of displacements

$$\begin{aligned}\delta\bar{\mathbf{u}} &= \delta(\mathbf{u} + \Delta\mathbf{u}) \\ &= \delta\mathbf{u} + \delta(\Delta\mathbf{u}) \\ &= \delta(\Delta\mathbf{u})\end{aligned}$$

Variation of the state of strain

$$\begin{aligned}\delta\bar{e}_{ij} &= \delta(\Delta e_{ij}) \\ &= \delta(\Delta e_{ij}^{L}) + \delta(\Delta e_{ij}^{N}) \\ \delta(\Delta e_{ij}^{N}) &= \frac{1}{2}\sum_{k=1}^{3}\left(\Delta u_{k,i}\;\delta(\Delta u_{k,j}) + \Delta u_{k,j}\;\delta(\Delta u_{k,i})\right)\end{aligned}$$

Governing equations in vector notation

$$\int_\Omega \sum_i \sum_j \delta(\Delta e_{ij}^L)\,\Delta s_{ij}\,dv + \int_\Omega \sum_i \sum_j \delta(\Delta e_{ij}^N)\,s_{ij}\,dv =$$
$$\Delta r + \int_\Omega \sum_i \delta(\Delta u_i)\,\Delta p_i \rho\,dv + \int_{\Gamma_t} \sum_i \delta(\Delta u_i)\,\Delta t_{i0}\,da$$

with

$$\begin{aligned}\Delta r \;=\; & \int_\Omega \sum_i \delta(\Delta u_i)\,p_i \rho\,dv + \int_{\Gamma_t} \sum_i \delta(\Delta u_i)\,t_{i0}\,da \\ & - \int_\Omega \sum_i \sum_j \delta(\Delta e_{ij}^L)\,s_{ij}\,dv\end{aligned}$$

Governing equations in vector notation

$$\int_\Omega \delta(\Delta\boldsymbol{\epsilon}_C)^T\,\mathbf{C}\,(\Delta\boldsymbol{\epsilon}_C)\,dv + \int_\Omega \delta(\Delta\boldsymbol{\epsilon}_L)^T\,\mathbf{C}\,(\Delta\boldsymbol{\epsilon}_L)\,dv + \int_\Omega \sum_k \delta(\Delta\mathbf{g}_k)^T\,\mathbf{S}\,\Delta\mathbf{g}_k\,dv$$
$$= \Delta r + \int_\Omega \delta(\Delta\mathbf{u})^T\,\Delta\mathbf{p}\rho\,dv + \int_{\Gamma_t} \delta(\Delta\mathbf{u})^T\,\Delta\mathbf{t}_0\,da$$

with

$$\Delta r = \int_\Omega \delta(\Delta\mathbf{u})^T\,\mathbf{p}\rho\,dv + \int_{\Gamma_t} \delta(\Delta\mathbf{u})^T\,\mathbf{t}_0\,da - \int_\Omega \delta(\Delta\boldsymbol{\epsilon}_C + \Delta\boldsymbol{\epsilon}_L)^T\,\boldsymbol{\sigma}\,dv$$

Governing equations in vector notation

$$\Delta\boldsymbol{\epsilon}_C = \begin{bmatrix} \Delta e_{11_C} \\ \Delta e_{22_C} \\ \Delta e_{33_C} \\ \Delta e_{12_C} \\ \Delta e_{23_C} \\ \Delta e_{31_C} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2\,\Delta u_{1,1} \\ 2\,\Delta u_{2,2} \\ 2\,\Delta u_{3,3} \\ (\Delta u_{1,2} + \Delta u_{2,1}) \\ (\Delta u_{2,3} + \Delta u_{3,2}) \\ (\Delta u_{3,1} + \Delta u_{1,3}) \end{bmatrix}$$

$$\Delta\mathbf{g}_k = \begin{bmatrix} \Delta u_{k,1} \\ \Delta u_{k,2} \\ \Delta u_{k,3} \end{bmatrix}$$

$$\Delta\boldsymbol{\epsilon}_L = \begin{bmatrix} \Delta e_{11_L} \\ \Delta e_{22_L} \\ \Delta e_{33_L} \\ \Delta e_{12_L} \\ \Delta e_{23_L} \\ \Delta e_{31_L} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sum_{k=1}^{3} 2\,(u_{k,1}\Delta u_{k,1}) \\ \sum_{k=1}^{3} 2\,(u_{k,2}\Delta u_{k,2}) \\ \sum_{k=1}^{3} 2\,(u_{k,3}\Delta u_{k,3}) \\ \sum_{k=1}^{3}(u_{k,1}\Delta u_{k,2} + u_{k,2}\Delta u_{k,1}) \\ \sum_{k=1}^{3}(u_{k,2}\Delta u_{k,3} + u_{k,3}\Delta u_{k,2}) \\ \sum_{k=1}^{3}(u_{k,3}\Delta u_{k,1} + u_{k,1}\Delta u_{k,3}) \end{bmatrix}$$