Physical Nonlinearity

phenomena which require consideration of material nonlinearity, e.g.

- analyses of loss of load-bearing capacity
- time-dependent bearing behavior due to time-dependent material properties
- non-elastic energy absorption from alternating loads due to impulse loading or earthquake loading
- ...

nonlinear incremental solution considers constitutive relations

- state of stress for the computation of
  - nodal forces
  - geometric stiffness → $\ldots + \int_\Omega \sum_i \sum_j \delta(\Delta e_{ij}^N)\, s_{ij}\, dv = \ldots$
- stress-strain relation for remaining inner work integrals

linear elastic material

$$\sigma = E\,(\epsilon - \epsilon_0) + \sigma_0$$

ϵ_0 prescribed strain
σ_0 prescribed stress

E modulus of elasticity
G modulus of rigidity
ν Poisson ratio

$$G = \frac{E}{2\,(1+\nu)}$$

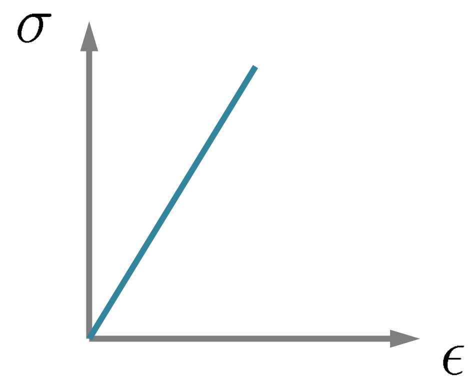
$$N = \frac{E}{(1+\nu)(1-2\,\nu)}$$

e.g. isotropic material

$$E = \begin{bmatrix} (1-\nu)N & \nu\,N & \nu\,N & 0 & 0 & 0 \\ \nu\,N & (1-\nu)N & \nu\,N & 0 & 0 & 0 \\ \nu\,N & \nu\,N & (1-\nu)N & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix}$$

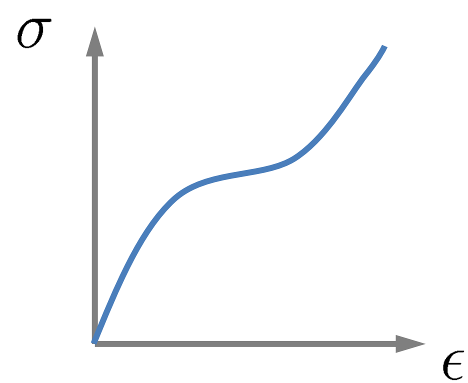
- **linear elastic material**

  stresses are
  - exclusively a function of the current state of strain
  - independent of the load history
    - for small strains: metals, wood, glass, ...
    - before yielding or failure



- **nonlinear elastic material**

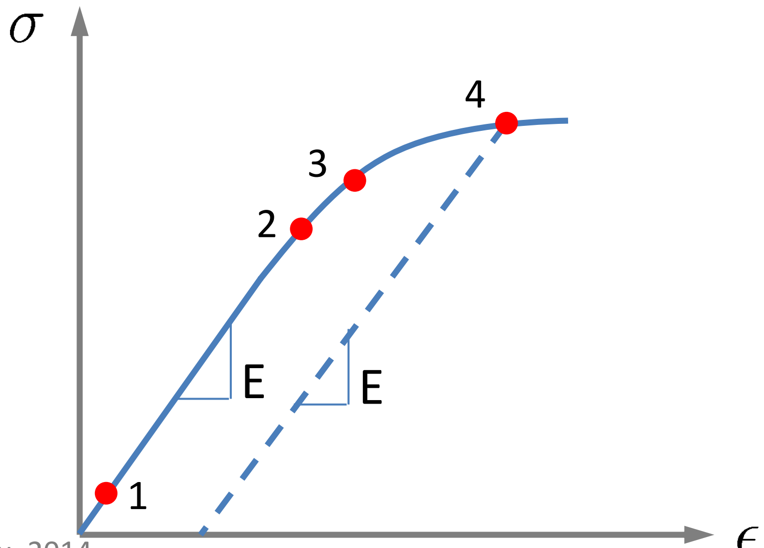
  stresses are
  - a function of the state of strain



$$\hat{s}_{ij} = \hat{C}_{ijkl}\,\hat{e}_{kl} \qquad \text{linear elastic}$$

$$\hat{s}_{ij}(\hat{\mathbf{e}}) = \hat{C}_{ijkl}(\hat{\mathbf{e}})\,\hat{e}_{kl} \qquad \text{nonlinear elastic}$$

- **elasto-plasticity**
  - elastic up the yield strength = elastic limit (3)
  - ... then plastic → irreversible deformation
  - requires
    - yield condition
    - flow rule
    - hardening rule
  - yield condition, e.g. Mohr-Coulomb, von Mises, Tresca, Rankine ...

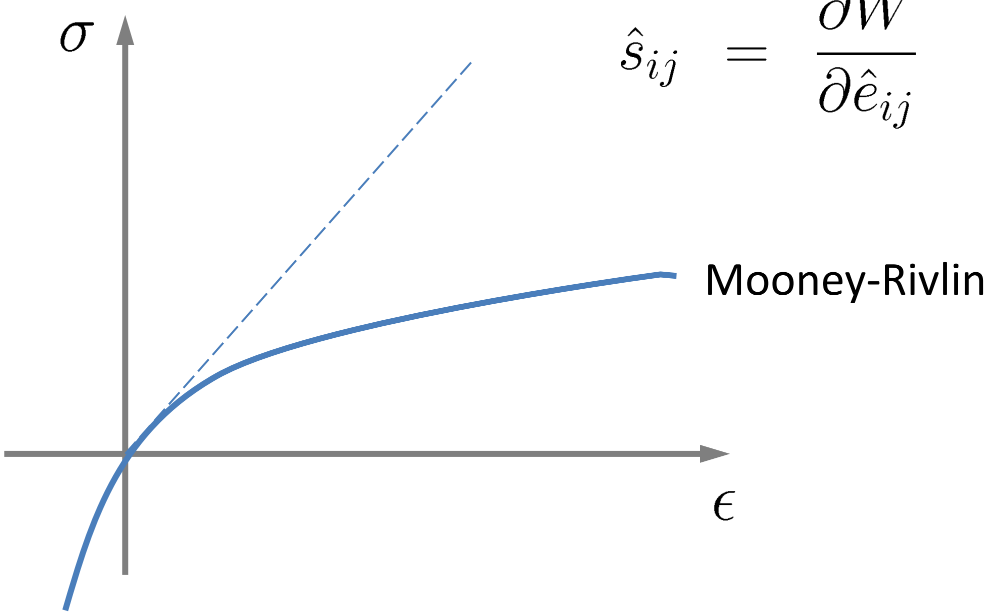


1. true elastic limit (crystal dislocation starts)
2. proportionality limit (Hooke!)
3. elastic limit (yielding starts)
4. offset yield strength

- **hyper-elasticity**
  - rubber-like material
  - models: Mooney-Rivlin, Ogden
  - stresses derived from strain energy functional

$$W = \frac{1}{2} \hat{C}_{ijkl} \, \hat{e}_{ij} \hat{e}_{kl} \qquad \text{strain energy}$$

$$\hat{s}_{ij} = \frac{\partial W}{\partial \hat{e}_{ij}}$$



- **visco-plasticity**
  - rate-dependent inelastic behavior
  - considers
    - time-dependent inelastic strains
    - yield stresses may change with time
    - velocity effects, includes fluidity parameters
  - e.g. metals, polymers

- **hypo-elastic**
- **...**

- **invariants are important in nonlinear constitutive relations**
  - invariant w.r.t. the chosen coordinate system
  - invariants are functions of stress components
  - → properties of isotropic material are independent of coordinate system

- **state of stress**

$$\mathbf{S} = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix}$$

s_{ij} stress component w.r.t. global coordinate system

- **state of stress**

$$\mathbf{S} = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix}$$

s_{ij} stress component w.r.t. global coordinate system

- **stress tensor invariants**

$$\begin{aligned} I_1 &= s_{11} + s_{22} + s_{33} = tr(\mathbf{S}) \\ &= p_1 + p_2 + p_3 \end{aligned}$$

$$\begin{aligned} I_2 &= s_{11}\, s_{22} + s_{22}\, s_{33} + s_{33}\, s_{11} - s_{12}^2 - s_{23}^2 - s_{31}^2 \\ &= p_1 p_2 + p_2 p_3 + p_3 p_1 \end{aligned}$$

$$\begin{aligned} I_3 &= det(\mathbf{S}) \\ &= p_1\, p_2\, p_3 \end{aligned}$$

- **stress tensor invariants**

$$
\begin{aligned}
I_1 &= s_{11} + s_{22} + s_{33} = tr(\mathbf{S}) \\
&= p_1 + p_2 + p_3 \\
I_2 &= s_{11}\, s_{22} + s_{22}\, s_{33} + s_{33}\, s_{11} - s_{12}^2 - s_{23}^2 - s_{31}^2 \\
&= p_1 p_2 + p_2 p_3 + p_3 p_1 \\
I_3 &= det(\mathbf{S}) \\
&= p_1\, p_2\, p_3
\end{aligned}
$$

p_i $\quad i^{th}$ principal strain

- **composed stress tensor representation**

$$\mathbf{S} = \mathbf{S}_{hyd} + \mathbf{S}_{dev}$$

$$\mathbf{S}_{hyd} = p\,\mathbf{I} = \frac{1}{3}\left(s_{11} + s_{22} + s_{33}\right)\mathbf{I}$$

$$p = \frac{1}{3}\left(s_{11} + s_{22} + s_{33}\right)$$

$$\mathbf{S}_{dev} = \begin{bmatrix} s_{11} - p & s_{12} & s_{13} \\ s_{21} & s_{22} - p & s_{23} \\ s_{31} & s_{32} & s_{33} - p \end{bmatrix}$$

stress invariants

- **hydrostatic stress tensor**

  → **causes change in volume**

$$\mathbf{S}_{hyd} = p\,\mathbf{I} = \frac{1}{3}\left(s_{11} + s_{22} + s_{33}\right)\mathbf{I}$$

- **deviatoric stress tensor**

  → **causes change of the shape**

$$\mathbf{S}_{dev} = \begin{bmatrix} s_{11} - p & s_{12} & s_{13} \\ s_{21} & s_{22} - p & s_{23} \\ s_{31} & s_{32} & s_{33} - p \end{bmatrix}$$

  - ... provides the following invariants

$$J_1 = 0$$

$$J_2 = \frac{1}{6}\left(\left(s_{11} - s_{22}\right)^2 + \left(s_{22} - s_{33}\right)^2 + \left(s_{33} - s_{11}\right)^2\right) + s_{12}^2 + s_{23}^2 + s_{31}^2$$

$$J_3 = det(\mathbf{S}_{dev})$$

- **mechanical interpretation of I_1 & J_2 possible**

$$
\begin{aligned}
I_1 &= s_{11} + s_{22} + s_{33} = tr(\mathbf{S}) \\
&= p_1 + p_2 + p_3
\end{aligned}
$$

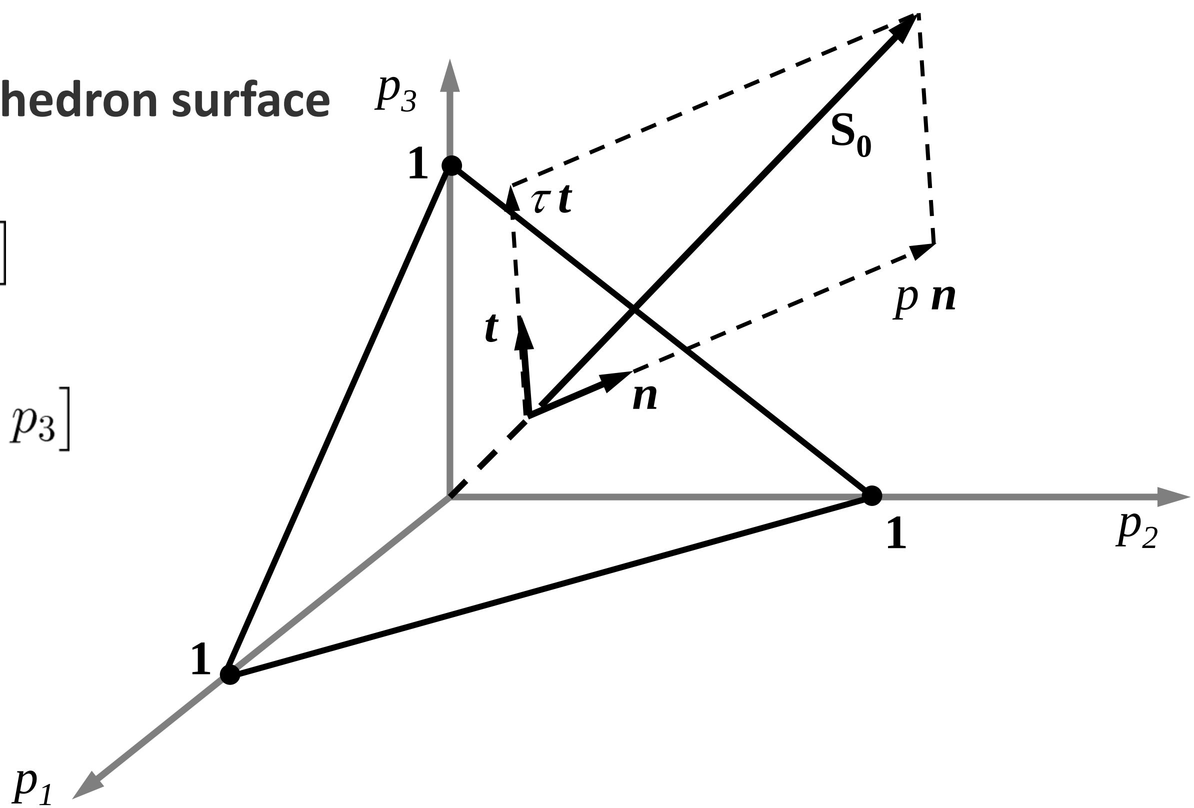
$$
J_2 = \frac{1}{6}\left((s_{11} - s_{22})^2 + (s_{22} - s_{33})^2 + (s_{33} - s_{11})^2\right) + s_{12}^2 + s_{23}^2 + s_{31}^2
$$

- **stress vector on octahedron surface**

$$
\mathbf{n}^T = \frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\end{bmatrix}
$$

$$
\mathbf{S}_0^T = \frac{1}{\sqrt{3}}\begin{bmatrix}p_1 & p_2 & p_3\end{bmatrix}
$$

$$
\mathbf{S}_0 = p\,\mathbf{n} + \tau\,\mathbf{t}
$$



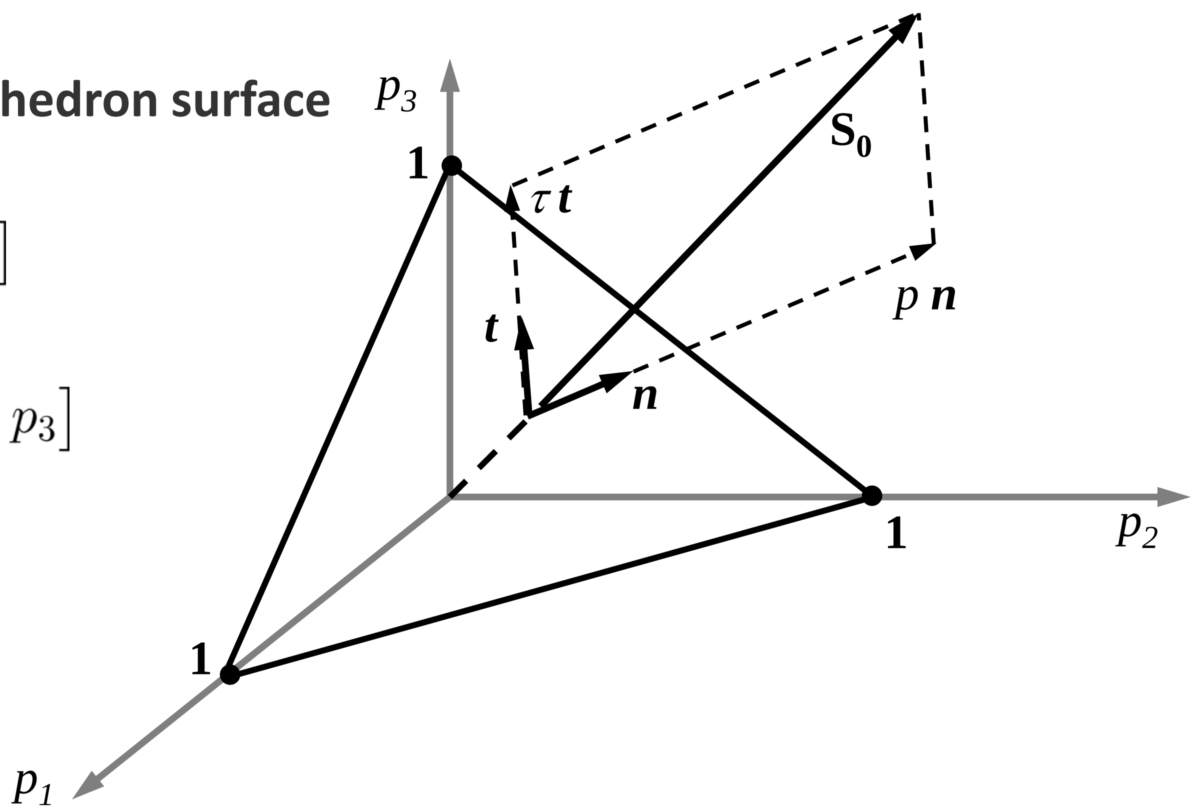
- **mechanical interpretation of I_1 & J_2 possible**
  - octahedron *normal stresses* depend only on 1st invariant I_1
  - octahedron *shear stresses* depend only on 2nd invariant J_2
  - → constitutive laws represented w.r.t. invariants or octahedron stresses

- **stress vector on octahedron surface**

$$\mathbf{n}^T = \frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\end{bmatrix}$$

$$\mathbf{S}_0^T = \frac{1}{\sqrt{3}}\begin{bmatrix}p_1 & p_2 & p_3\end{bmatrix}$$

$$\mathbf{S}_0 = p\,\mathbf{n} + \tau\,\mathbf{t}$$



total stress is not uniquely related to total strain!

→ description of the stress-strain relation requires

- response history
- differential stress / strain increments (including temperature effects)

$$d\,s_{ij} \;=\; C^{el}_{ijkl}\,(d\,e_{kl} - d\,e^{ie}_{kl}) - d\,e^{th}_{ij}$$

$d\,e_{kl}$	total differential strain increment
$d\,e^{ie}_{kl}$	inelastic differential strain increment
$d\,e^{th}_{ij}$	thermal differential strain increment

- **inelastic behavior considering time gives**
  - plasticity : inelastic strains occur immediately with loading
    → negligible time span between loading and response
  - creep : inelastic strains are a function of time
    → time window considered (hours, days, years)

derivative of a stress-strain relation of form

$$\boxed{d\,\mathbf{s} \;=\; \mathbf{C}^{ep}\, d\,\mathbf{e}}$$

$\mathbf{C}^{ep}$	elastic-plastic material relation matrix
$d\,\mathbf{e}$	total strain increment
$d\,\mathbf{s}$	stress increment

$$\mathbf{s}^T \;=\; \begin{bmatrix} s_{11} & s_{22} & s_{33} & s_{12} & s_{23} & s_{31} \end{bmatrix}$$

$$\mathbf{e}^T \;=\; \begin{bmatrix} e_{11} & e_{22} & e_{33} & (e_{12}+e_{21}) & (e_{32}+e_{23}) & (e_{13}+e_{31}) \end{bmatrix}$$

(1) initial loading → material is linear elastic

$$s_{ij} = C_{ijkl}\, e_{kl}$$

C_{ijkl} stress independent elasticity matrix

(2) maximum stresses that may occur in the material are restricted by the yield condition (→ at yielding F = 0)

$$F(\sigma_y) = 0$$

$$s_{ij} \leq \sigma_y$$

σ_y yield stress

(3) stress increment between adjacent stress states, **s** and (**s**+d **s**), is orthogonal to the gradient **f** of the potential F

$$\mathbf{f}^T d\,\mathbf{s} = 0$$

$$\mathbf{f} = \left[\frac{\partial F}{\partial s_{11}}\ \frac{\partial F}{\partial s_{22}}\ \frac{\partial F}{\partial s_{33}}\ \frac{\partial F}{\partial s_{12}}\ \frac{\partial F}{\partial s_{23}}\ \frac{\partial F}{\partial s_{31}}\right]$$

(4) any change in stresses that satisfies orthogonality condition results in a elastic and plastic strain increment

$$d\,e_{ij} \;=\; d\,e_{ij}^{e} + d\,e_{ij}^{p}$$

e_{ij}	total strain
e_{ij}^{e}	elastic strain part
e_{ij}^{p}	plastic strain part

(5) elastic strain increment follows from the elasticity law

$$d\,s_{ij} \;=\; C_{ijkl}\, e_{kl}^{e}$$

(6) the plastic strain increment is found from the experimentally derived flow rule

$$\begin{aligned} d\,e_{ij}^{p} &= \lambda\, q_{ij} \\ q_{ij} &= \left[\frac{\partial Q}{\partial s_{11}}\; \frac{\partial Q}{\partial s_{22}}\; \frac{\partial Q}{\partial s_{33}}\; \frac{\partial Q}{\partial s_{12}}\; \frac{\partial Q}{\partial s_{23}}\; \frac{\partial Q}{\partial s_{31}} \right] \end{aligned}$$

$Q(\mathbf{s})$	plastic potential of the material
λ	yield parameter

(7) for metallic materials the elastic potential Q is often chosen as the potential F → associated plasticity / associated flow rule

(8) flow parameter is chosen such that orthogonality condition is satisfied

$$\begin{aligned} \mathbf{f}^T d\,\mathbf{s} &= \mathbf{f}^T \mathbf{C}\,(d\,\mathbf{e} - d\,\mathbf{e}^p) \\ &= \mathbf{f}^T \mathbf{C}\,(d\,\mathbf{e} - \lambda \mathbf{q}) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lambda &= \frac{1}{a} \mathbf{f}^T \mathbf{C}\; d\,\mathbf{e} \\ a &= \mathbf{f}^T \mathbf{C}\; \mathbf{q} \end{aligned}$$

(9) final material matrix follows from relations (4)—(8)

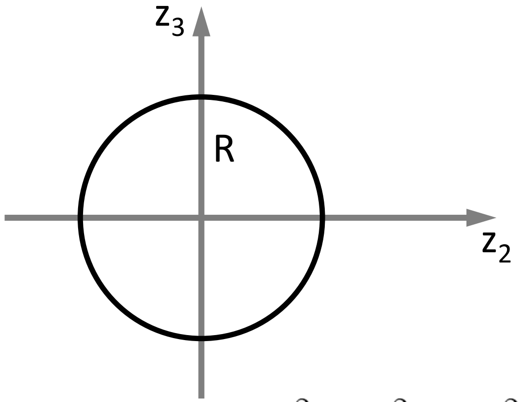
$$d\,s_{ij} = C^{ep}_{ijkl}\; d\,e_{kl}$$

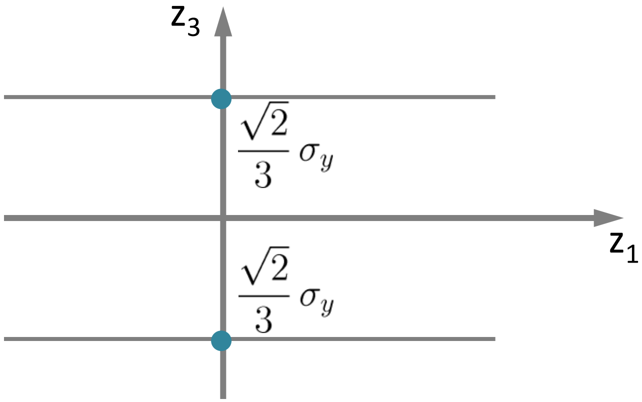
C^{ep}_{ijkl} elasto-plastic material matrix

$$\boxed{\begin{aligned} \mathbf{C}^{ep} &= \mathbf{C}\,(\mathbf{I} - \frac{1}{a} \mathbf{q} \mathbf{f}^T \mathbf{C}) \\ a &= \mathbf{f}^T \mathbf{C}\, \mathbf{q} \end{aligned}}$$

ideal elasto-plastic material

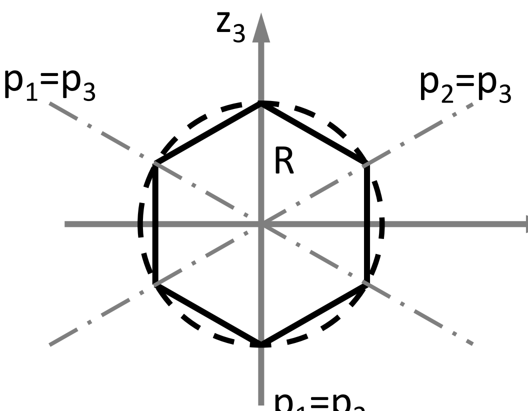
Huber – von Mises

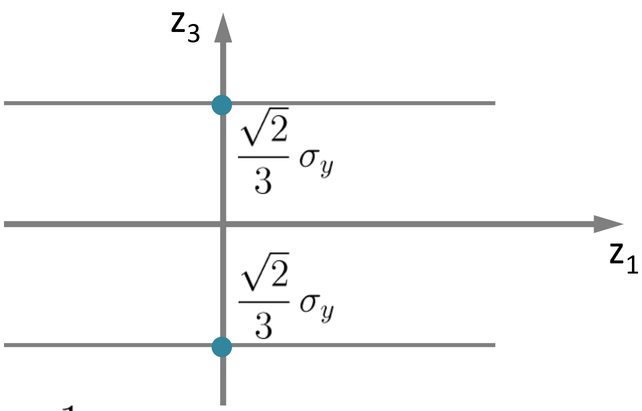




$$\begin{aligned} F(\mathbf{s}) &= z_2^2 + z_3^2 - R^2 &= 0 \\ &= (p_1 - p_2)^2 + (p_2 - p_3)^2 + (p_3 - p_1)^2 - 2\sigma_y^2 &= 0 \end{aligned}$$

Tresca

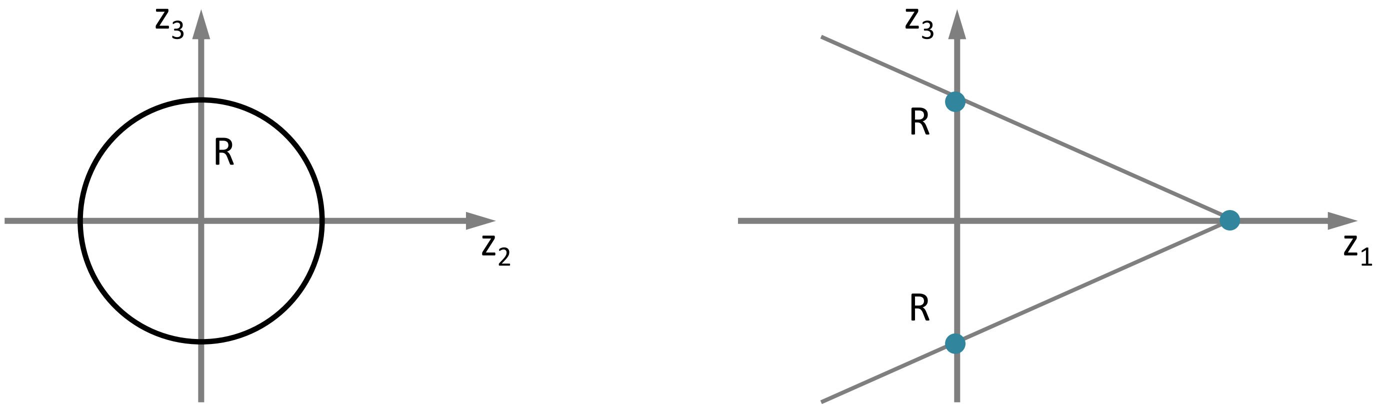




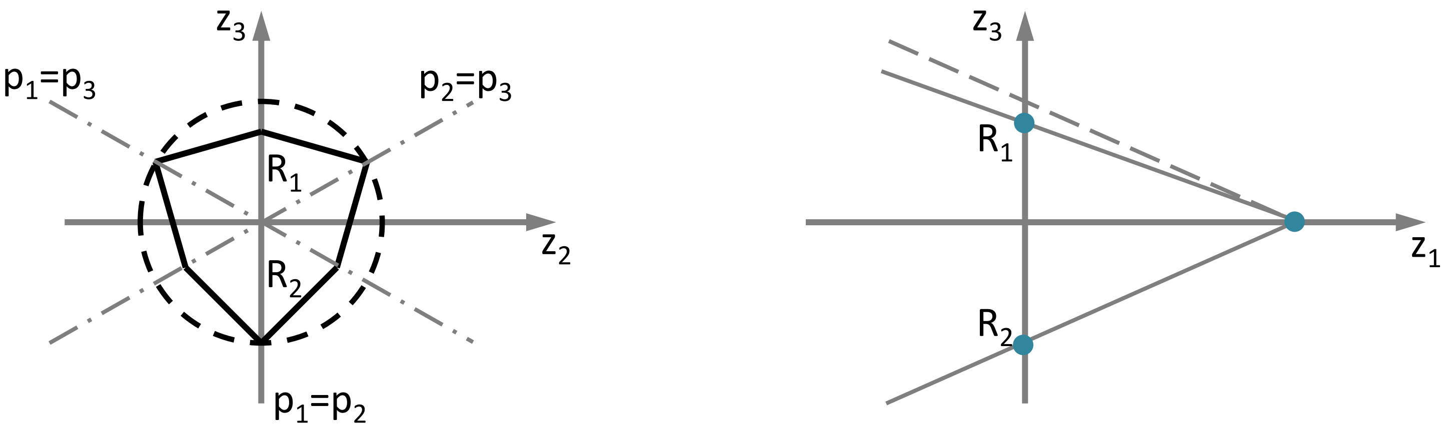
$$\begin{aligned} F(\mathbf{s}) &= z_3 + \frac{1}{\sqrt{3}} z_2 + R &= 0 \\ &= p_1 - p_3 - \sigma_y &= 0 \end{aligned}$$

ideal elasto-plastic material

Drucker – Prager

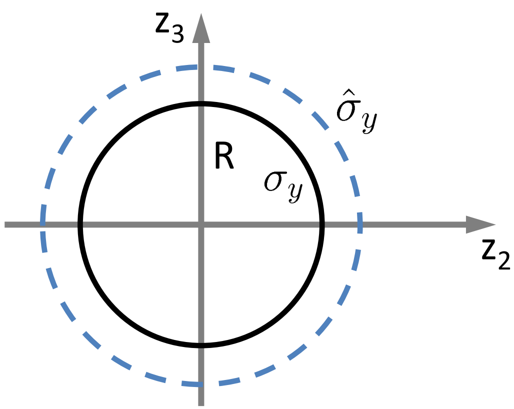


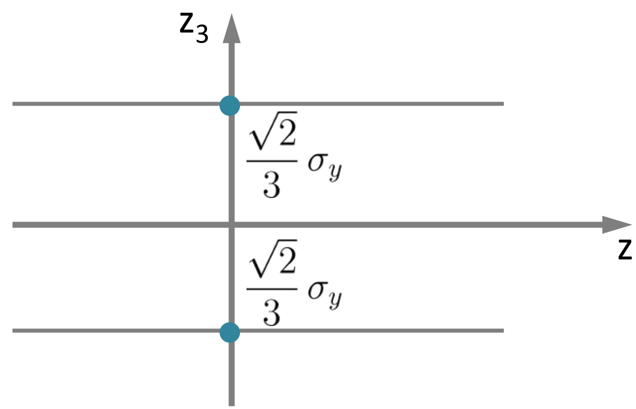
Mohr – Coulomb



Huber – von Mises

- model, used for metals
- R increases uniformly with isotropic hardening σ_y → $\hat{\sigma}_y$





$$\begin{aligned} F(\mathbf{s}) &= z_2^2 + z_3^2 - R^2 &&= 0 \\ &= (p_1 - p_2)^2 + (p_2 - p_3)^2 + (p_3 - p_1)^2 - 2\sigma_y^2 &&= 0 \end{aligned}$$

hardening

- ***true*** yield surface depends on stress curve history of a specific point
- beyond the yield point each stress increment causes
  - a plastic strain increment $d\,e_{ij}^{p}$
  - change of the yield surface
- law for the change of the yield surface is derived experimentally

  → hardening effect
- hardening effect is expressed with scale factors changing the material strength and/or plastic potential
- scale factors follow from
  - stress related material laws OR
  - work related material laws

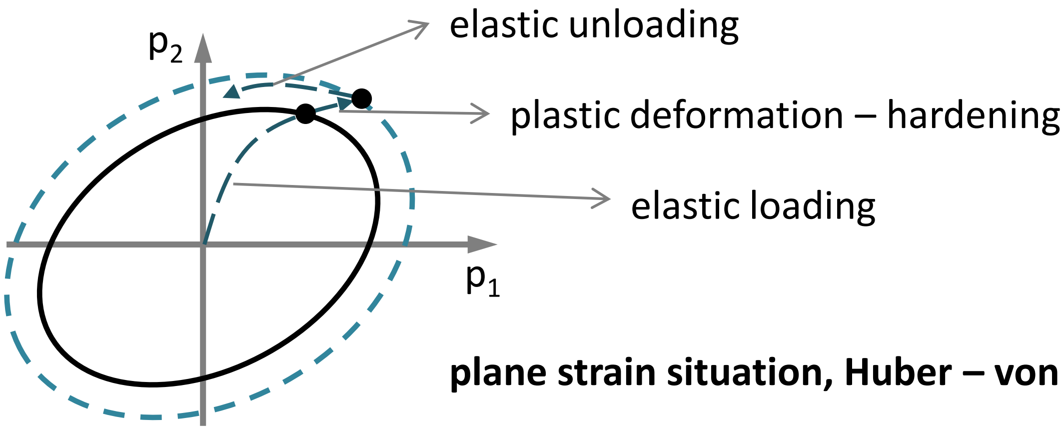
isotropic hardening

- size of the yield surface changes
- shape of the yield surface remains similar
- yield surface parameters are multiplied with the same factor

  → isotropic scaling

$$\bar{\sigma}_{TS} = c_1\,\sigma_{TS}$$
$$\bar{\sigma}_{CS} = c_1\,\sigma_{CS}$$

σ_{TS}, σ_{CS}	tensile/compression strength before initial loading
$\bar{\sigma}_{TS}, \bar{\sigma}_{CS}$	tensile/compression strength after hardening
c_1	isotropic scale factor

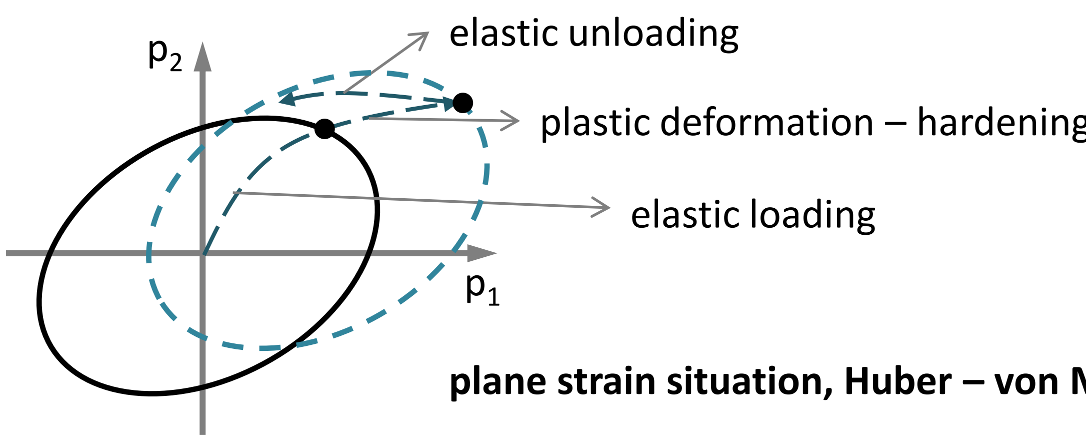


plane strain situation, Huber – von Mises

- size of the yield surface remains unchanged
- displacement of the yield surface
- every point of the yield surface is displaced by the same increment
- direction of displacement often parallel to the plastic strain increment

$$\Delta\sigma_k \quad = \quad c_2\, q$$

$\Delta\sigma_k$	kinematic displacement of the yield surface
c_2	kinematic scale factor
q	gradient of the plastic potential



plane strain situation, Huber – von Mises

- assumption (here):

  hardening and strength of the material depends mainly on the deviatoric state of stresses

- behavior of such materials is often expressed as function of J_2

$$J_2 \;=\; \frac{1}{6}\left[(p_1 - p_2)^2 + (p_2 - p_3)^2 + (p_3 - p_1)^2\right]$$

p_i $\quad i^{th}$ principal stress

steps for the computation of hardening and failure:

(1) initial yield strength is represented as function of stresses

$$F^I(\sigma_y) \;=\; 0$$

σ_y $\quad$ yield stress

(2) stress increment after yielding causes plastic strain increment

$$d\,e^p_{ij} = \lambda\, q_{ij}$$

λ	yield parameter
q_{ij}	gradient of plastic potential

(3) plastic strain increment changes yield surface. Stress increment does not satisfy the orthogonality condition thus yield parameter λ requires own material law

$$\lambda = \frac{s_n}{H}$$

$$s_n = \frac{\mathbf{f}^T\, d\,\mathbf{s}}{\sqrt{\mathbf{f}^T\mathbf{f}}}$$

$\mathbf{f}$	gradient of the flow potential F
s_n	component of $d\,\mathbf{s}$ normal to the yield surface
H	hardening rate of the material, derived as function of J_2

(4) strain increment follows from substitution of λ into the stress-strain relation

$$
\begin{aligned}
d\,\mathbf{s} &= \mathbf{C}_h\, d\,\mathbf{e} \\
\mathbf{C}_h^{-1} &= \mathbf{C}^{-1} + \frac{\mathbf{q}\,\mathbf{f}^T}{H\,\sqrt{f^T\mathbf{f}}}
\end{aligned}
$$

$\mathbf{C}$ elastic material matrix

$\mathbf{C}_h$ strain-hardened material matrix

(5) the failure surface is derived in analogy to the yield surface, replacing the yield stress with the failure stress

$$F(\sigma_f) = 0$$

σ_f failure stress

nonlinear computation with finite elements

- *material* nonlinear analysis follows from a sequence of linear steps
- step size is adjusted such that approximation is sufficiently good
- the nonlinearity has the following influence
  - the material matrix $\mathbf{C}$ depends on stresses thus element stiffness $\mathbf{K}_\mathrm{e}$ depends on state of stresses
  - plastic deformation results in prescribed strains → e_{ij}^p

    depending on state of stresses, the inelastic work W and time t

$$d\,\mathbf{p}^p = \int_{\Omega_e} \mathbf{B}_e^T \mathbf{C}_e \, d\,\mathbf{e}^p$$

$d\,\mathbf{e}^p$	plastic strain increment
$d\,\mathbf{p}^p$	element load increment
$\mathbf{C}_e$	elasticity matrix of the load increment

- development of separate methods for inelasticity and plasticity necessary since only inelastic deformation is fully reversible
- both methods can be used in combination if both effects are present

- introduction of load steps
- each load step increases the external loading by a fixed increment
- each load increment changes displacements and state of stresses
- displacement and stress increments depend on secant matrix
- secant matrix of step *m* is not known a priori and requires an iterative scheme within each load step

iteration cycle 1 in load step *m*

… based on a known deformation state

- at the beginning of load step 1
  - state of stresses at the integration points is known
  - displacements at the element nodes are zero
- at the beginning of load steps m=2,3,…,M
  - deformation state of the previous load step are known

$s_{ij}^{(m-1)}$ stress component at integration point

$e_{ij}^{(m-1)}$ strain component at integration point

$v_k^{(m-1)}$ displacements at element nodes

iteration cycle 1 in load step *m*

- at the beginning of load step *1* …
- at the beginning of load steps *m=2, 3, …, M* …
- the kinematic and static increments of the load step are unknown

$$\begin{array}{ll} \Delta\, s_{ij}^{(m)} & \text{stress increment} \\ \Delta\, e_{ij}^{(m)} & \text{strain increment} \\ \Delta\, v_k^{(m)} & \text{displacement increment} \\ \Delta\, p_k^{(m)} & \text{load step increment} \end{array}$$

- determine constitutive relations in load step *m* dependent on
  - hydrostatic stress p
  - octahedron shear stress τ
- choose initial guess for cycle *1* from load step *m-1*

$$\begin{aligned} p_1^{(m)} &= p^{(m-1)} \\ \tau_1^{(m)} &= \tau^{(m-1)} \end{aligned}$$

iteration cycle 1 in load step *m*

- at the beginning of load step *1* …
- at the beginning of load steps *m=2, 3, …, M* …
- the kinematic and static increments of the load step are unknown …
- determine constitutive relations in load step *m* dependent on …
- choose initial guess for cycle *1* from load step *m-1 …*
- compute approximation of the stress strain relation from initial guess

$$\Delta\, s^{(m)}_{ij_{(1)}} = \mathbf{C}_{T_{(1)}}\, \Delta\, e^{(m)}_{ij_{(1)}}$$

$\Delta\, s^{(m)}_{ij_{(1)}}$ stress increment in cycle 1 of load step m

$\Delta\, e^{(m)}_{ij_{(1)}}$ strain increment in cycle 1 of load step m

$\mathbf{C}_{T_{(1)}}$ tangential matrix for stress state $\mathbf{s}^{(m-1)}$

iteration cycle 1 in load step *m*

- at the beginning of load step *1* ...
- at the beginning of load steps *m=2, 3, ..., M* ...
- the kinematic and static increments of the load step are unknown ...
- determine constitutive relations in load step *m* dependent on ...
- choose initial guess for cycle *1* from load step *m-1 ...*
- compute approximation of the stress strain relation from initial guess
- governing equations in load step *m* follow as

$$\mathbf{K}_T \, \Delta \, \mathbf{v}_1^{(m)} = \Delta \, \mathbf{p}^{(m)}$$

$$\mathbf{K}_T = \sum_e \int_{\Omega_e} \mathbf{B}_e^T \mathbf{C}_T \, \mathbf{B}_e \, dv$$

$\mathbf{K}_T$ stiffness matrix, cycle 1, load step m

$\Delta \, \mathbf{v}_1^{(m)}$ displacement increment, cycle 1, load step m

$\Delta \, \mathbf{p}^{(m)}$ increment of external load, step m

iteration cycles 2, 3, … in load step *m*

- use displacement increment of cycle *1* to compute

$$\begin{aligned}
\Delta\,\mathbf{e}_{(1)}^{(m)} &= \mathbf{B}_e\,\Delta\,\mathbf{v}_{e,(1)}^{(m)} \\
\Delta\,\mathbf{s}_{(1)}^{(m)} &= \mathbf{C}_T\,\Delta\,\mathbf{e}_{e,(1)}^{(m)} \\
\mathbf{s}_{(1)}^{(m)} &= \mathbf{s}^{(m-1)} + \Delta\,\mathbf{s}_{(1)}^{(m)}
\end{aligned}$$

- using $\mathbf{s}_{(1)}^{(m)}$ allows the computation of octahedron normal and shear stresses

$$\begin{aligned}
p_1^{(m)} &= p^{(m-1)} \\
\tau_1^{(m)} &= \tau^{(m-1)}
\end{aligned}$$

- from this follows a new stress-strain relation

$$\Delta\,s_{ij_{(2)}}^{(m)} = \mathbf{C}_{S_{(2)}}\,\Delta\,e_{ij_{(2)}}^{(m)}$$

$\Delta\,s_{ij_{(2)}}^{(m)}$ stress increment in cycle 2 of load step m

$\Delta\,e_{ij_{(2)}}^{(m)}$ strain increment in cycle 2 of load step m

$\mathbf{C}_{S_{(2)}}$ secant matrix for stress states $(\mathbf{s}^{(m-1)}, \mathbf{s}_{(1)}^{(m)})$

iteration cycles 2, 3, … in load step *m*

- use displacement increment of cycle *1* to compute …
- using $\mathbf{s}_{(1)}^{(m)}$ allows the computation of octahedron normal …
- continuation of the iteration cycle finally gives …
- improved displacement increments follow as

$$\mathbf{K}_S \, \Delta \, \mathbf{v}_2^{(m)} = \Delta \, \mathbf{p}^{(m)}$$

$$\mathbf{K}_S = \sum_e \int_{\Omega_e} \mathbf{B}_e^T \mathbf{C}_{S_{(2)}} \mathbf{B}_e \, dv$$

$\mathbf{K}_S$ stiffness matrix, cycle 2, load step m

$\Delta \, \mathbf{v}_2^{(m)}$ displacement increment, cycle 2, load step m

$\Delta \, \mathbf{p}^{(m)}$ increment of external load, step m

iteration cycles 2, 3, … in load step *m*

- use displacement increment of cycle *1* to compute …
- using $\mathbf{s}^{(m)}_{(1)}$ allows the computation of octahedron normal …
- continuation of the iteration cycle finally gives …
- improved displacement increments follow as …
- finally the results of load step m follow as

$$
\begin{aligned}
\mathbf{v}^{(m)} &= \mathbf{v}^{(m-1)} + \Delta\,\mathbf{v}^{(m)}_{(N)} \\
\mathbf{e}^{(m)} &= \mathbf{e}^{(m-1)} + \Delta\,\mathbf{e}^{(m)}_{(N)} \\
\mathbf{s}^{(m)} &= \mathbf{s}^{(m-1)} + \Delta\,\mathbf{s}^{(m)}_{(N)} \\
N & \quad \text{number of iteration cycles in load step } m
\end{aligned}
$$

load increment in load step m+1

- incremental computations produce accumulated errors
- the errors are compensated by considering fictive nodal forces at the end of load step m

$$\begin{aligned}
\Delta\, p^{(m+1)} &= p^{(m+1)} - \sum_e \int_{\Omega_e} \mathbf{B}_e^T \mathbf{s}_e^{(m)}\, dv \\
\mathbf{s}_e^{(m)} &= \mathbf{s}(\mathbf{e}_e^{(m)}) \\
\mathbf{e}_e^{(m)} &= \mathbf{B}_e\, \mathbf{v}_e^{(m)}
\end{aligned}$$

$\Delta\, p^{(m+1)}$	load increment for load step m+1
$p^{(m+1)}$	loading at the end of step m+1
$\mathbf{B}_e$	element strain interpolation
$\mathbf{v}_e^{(m)}$	displacement vector at the end of load step m
$\mathbf{e}_e^{(m)}$	strains at the end of load step m
$\mathbf{s}_e^{(m)}$	stresses, computed as a function of strains

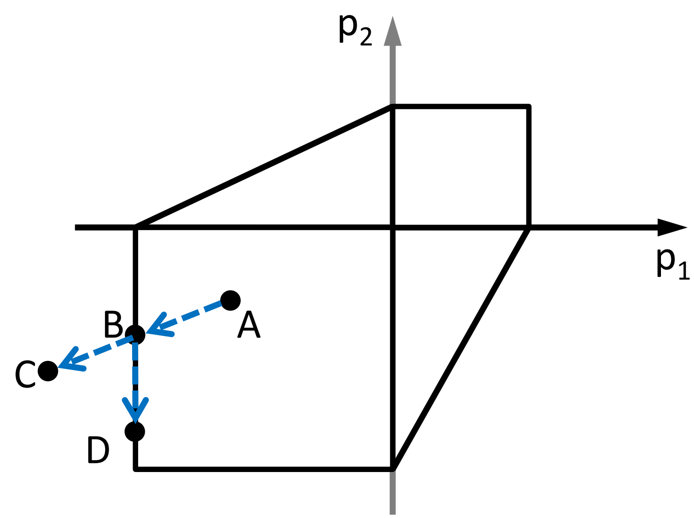
stress path in load step m

- at the beginning of load step 1
  - state of stresses and plastic strains at the integration points is known
  - displacements at the element nodes are zero
- at the beginning of load steps m=2,3,…,M
  - deformation state of the previous load step are known

$$\begin{aligned} \mathbf{e}^{p,(m-1)} &\quad \text{plastic strains} \\ \mathbf{e}^{(m-1)} &\quad \text{total strains} \\ \mathbf{s}^{(m-1)} &\quad \text{total stresses} \\ \mathbf{v}^{(m-1)} &\quad \text{total displacements} \end{aligned}$$

- at the beginning of each load step all stresses are inside or on the yield surface

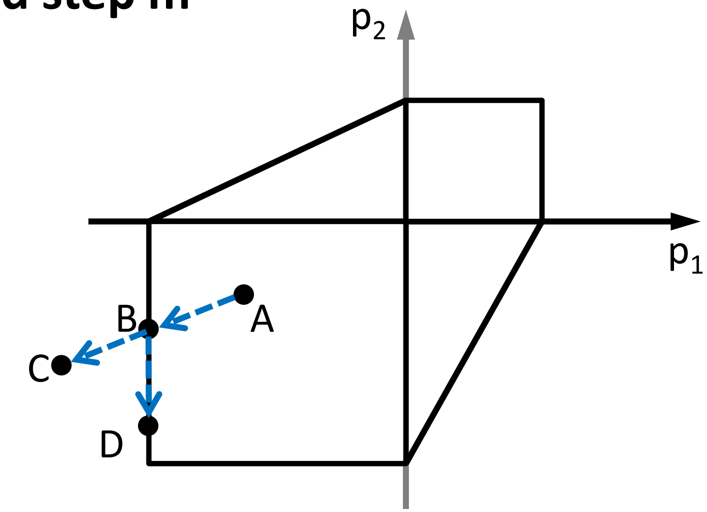
stress path in load step m



- stress path from A to B to C → computed stress path
- stress path from A to B to D → physical stress path
- discontinuity at B
- discontinuity is in general not captured → ***smeared*** plastic strain increment is used instead

elasto-plastic computation scheme

stress path in load step m



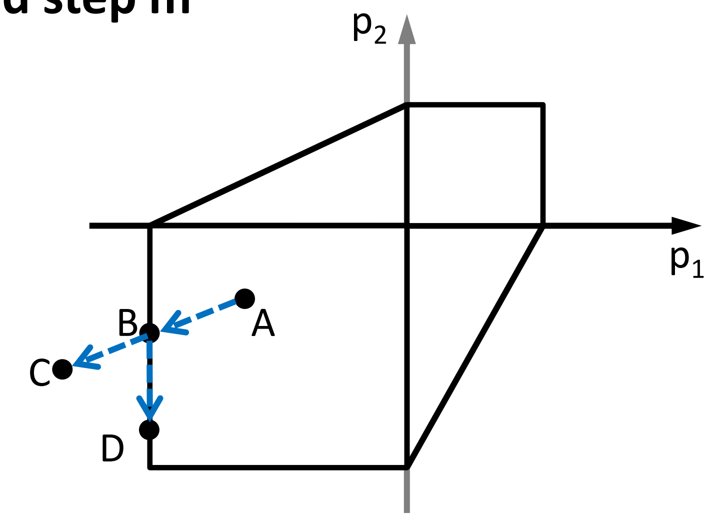
discontinuity is in general not captured → ***smeared*** plastic strain increment is used instead

$$d\,\mathbf{s}^{(m)} \;=\; \mathbf{C}\,(\mathbf{B}_e\,d\,\mathbf{v}_e^{(m)} - d\,\mathbf{e}_e^{p,(m)})$$

$d\,\mathbf{s}^{(m)}$	stress increment in load step m
$d\,\mathbf{v}_e^{(m)}$	displacement increment in load step m
$d\,\mathbf{e}_e^{p,(m)}$	plastic strain increment in load step m
$\mathbf{C}$	constant elasticity matrix
$\mathbf{B}_e$	element strain interpolation matrix

elasto-plastic computation scheme

stress path in load step m



- for the computation of the plastic strain increment $d\,\mathbf{e}_e^{p,(m)}$ stresses at B are required
- additional assumption is used: displacements are linear within load step m

→ for path AB follows

$$
\begin{aligned}
\mathbf{s}_B &= \mathbf{s}_A + \mathbf{b}\,\mathbf{C}\,\mathbf{B}_e\,d\,\mathbf{v}_e^{(m)} \\
F(\mathbf{s}_B) &= F(\mathbf{b}) = 0 \\
F & \quad \text{flow potential of the material}
\end{aligned}
$$

→ determine parameter b such that stresses $\mathbf{s}_B$ are on the yield surface

iteration cycle 1 in load step m

- constitutive relations for cycle 1 are

$$d\,\mathbf{s}_{(1)}^{(m)} \;=\; \mathbf{C}\,(\mathbf{B}_e\,d\,\mathbf{v}_{e,(1)}^{(m)} - d\,\mathbf{e}_{(1)}^{p,(m)})$$

$d\,\mathbf{s}_{(1)}^{(m)}$	stress increment in iteration cycle 1
$d\,\mathbf{v}_{(1)}^{(m)}$	displacement increment in iteration cycle 1
$d\,\mathbf{e}_{(1)}^{p,(m)}$	plastic strain increment in iteration cycle 1

- plastic strain increment is unknown a priori → material law is approximated

$$d\,\mathbf{s}_{(1)}^{(m)} \;=\; \mathbf{C}_{(1)}^{p,(m)}\;\;\mathbf{B}_e\,d\,\mathbf{v}_{e,(1)}^{(m)}$$

$\mathbf{C}_{(1)}^{p}$	elasto-plastic material matrix of step m

iteration cycle 1 in load step m

- in cycle 1 the material behavior is **elastic**
- in the following cycles 2,3,… the plastic material matrix $\mathbf{C}^{p,(m)}$ is set to matrix of the previous step
- the governing equations follow as

$$\mathbf{K}^{p,(m)}_{(1)}\, d\,\mathbf{v}^{(m)}_{e,(1)} = d\,\mathbf{p}^{(m)}$$

$$\mathbf{K}^{p,(m)}_{(1)} = \sum_e \int_{\Omega_e} \mathbf{B}_e^T \mathbf{C}^p_{(1)}\, \mathbf{B}_e\, dv$$

$$\mathbf{K}^{p,(m)}_{(1)} \quad \text{elasto-plastic stiffness matrix of step m}$$

- from which the displacement increment follows

iteration cycles 2,3,… in load step m

- improvement of the material law from cycle 1
- limit state B is approximated with

$$\mathbf{s}_{B,(2)} \;=\; \mathbf{s}^{m-1} + b\,\mathbf{C}\,\mathbf{B}_e\,d\,\mathbf{v}_{e,(1)}^{(m)}$$

$\mathbf{s}_{B,(2)}$ yield surface stress in iteration cycle 2

$\mathbf{s}^{m-1}$ stress at the end of load step m-1

- parameter b is determined such that flow condition is satisfied
- an estimate of the plastic strains is found from an estimate of the plastic strains at state D → consider

$$d\,\mathbf{e}_{AD} \;=\; d\,\mathbf{e}_{AD}^{e} + d\,\mathbf{e}_{BD}^{p}$$

$d\,\mathbf{e}_{AD}$ change of the total strain from A to D

$d\,\mathbf{e}_{AD}^{e}$ change of the elastic strain from A to D

$d\,\mathbf{e}_{BD}^{p}$ change of the plastic strain from B to D

iteration cycles 2,3,… in load step m

- the strain increments follow from

$$
\begin{aligned}
d\,\mathbf{e}^e_{AD} &= \mathbf{C}^{-1}\,d\,\mathbf{s}^e_{AD} = \mathbf{C}^{-1}\,(d\,\mathbf{s}_{AB} + d\,\mathbf{s}_{BD}) \\
d\,\mathbf{e}^p_{BD} &= \lambda\,\mathbf{q}_B
\end{aligned}
$$

$d\,\mathbf{s}_{AB}$	change of elastic stresses
$d\,\mathbf{s}_{BD}$	change of elasto-plastic stresses
$\mathbf{q}_B$	gradient of the plastic potential for state B
λ	yield parameter in step m

- substitution gives for the strain increment AD

$$
\mathbf{C}\,d\,\mathbf{e}_{AB} = d\,\mathbf{s}_{AB} + d\,\mathbf{s}_{BD} + \lambda\,\mathbf{C}\,\mathbf{q}_B
$$

iteration cycles 2,3,… in load step m

- stress increment $d\,\mathbf{s}_{BD}$ must be on the limit surface
- the yield parameter λ follows from

$$\mathbf{f}_B^T\, d\,\mathbf{s}_{BD} = \mathbf{f}_B^T\,(\mathbf{C}\;d\,\mathbf{e}_{AD} - d\,\mathbf{s}_{AB} - \lambda\,\mathbf{C}\,\mathbf{q}_B) \;=\; 0$$

- an improved estimate for the plastic strain increment in step m follows as

$$\begin{aligned} d\,\mathbf{e}^p_{BD} &= \frac{1}{a}\,(\mathbf{f}_B^T\mathbf{C}\;d\,\mathbf{e}_{AD} - \mathbf{f}_B^T d\,\mathbf{s}_{AB})\,\mathbf{q}_B \\ a &= \mathbf{f}_B^T\mathbf{C}\,\mathbf{q}_B \end{aligned}$$

- this improves the constitutive relation to

$$\begin{aligned} d\,\mathbf{s}_{AD} &= \frac{1}{a}\,(\mathbf{f}_B^T d\,\mathbf{s}_{AB})\,\mathbf{C}\,\mathbf{q}_B + \mathbf{C}^{ep}\,.\mathbf{e}_{AD} \\ \mathbf{C}^{ep} &= \mathbf{C}(\mathbf{I} - \frac{1}{a}\mathbf{q}_B\mathbf{f}_B^T\mathbf{C}) \end{aligned}$$

iteration cycles 2,3,… in load step m

- a new displacement increment is computed with the elasto-plastic law

$$\begin{aligned} \mathbf{K}^{ep}_{(2)}\, d\,\mathbf{v}^{(m)}_{(2)} &= d\,\mathbf{q}^{(m)} + d\,\mathbf{q}^{s}_{(2)} \\ d\,\mathbf{q}^{s}_{(2)} &= -\sum_{e} \int_{\Omega_e} \frac{1}{a} \left(\mathbf{f}^T_B d\,\mathbf{s}_{AB}\right) \mathbf{B}^T_e \mathbf{C}\,\mathbf{q}_B\, dv \\ d\,\mathbf{q}^{s}_{(2)} & \quad \text{stress correction of the load vector of cycle 2} \end{aligned}$$

- iteration is continued until stress state D is sufficient close to the limit surface for all plastic integration points