LINEAR MODELLING (INCL. FEM)
AE4ASM003
P1-2015

LECTURE 1
01.09.2015

TU Delft

OUR TEAM

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LOCATION AND TIMINGS

Theory Lectures

Lecture 1

Tuesday, September 1, 1545 to 1730

EWI-Lecture hall Chip

All other lectures

Tuesdays, 1345 to 1530

(September 8, 15, 20, 29;

October 6, 13)

AULA CZ A

Practicals

Thursdays, 0845 to 1230

LR-PC 007

4 hours! Do not miss!

COURSE SETUP

- Blended Online Learning
- Flipped Classroom
- Theory / Practical
- Assignment-based
- NO FINAL EXAM

“Homework” is not graded! Its for your practice and understanding!
“Assignment” is graded!

- 7 weeks (1 theory + 1 practical)
- Homework: video lecture/lecture notes
- 3 theory based assignments (Weeks 2, 4 and 6)
- 2 practical assignments (Weeks 4 and 6) → Not take home!

- Weekly topics: Blackboard (under course information/course setup)

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COURSE SETUP

- Contact
  - Discussion board (All questions must be posted here first!)
  - 2nd point: TA; TA will post questions on the discussion board anyway; so do it yourself to avoid delays
- Peer2Peer/Interaction
  - Use your own knowledge database: your peers
  - Respond to questions and discussions on the discussion board
  - Debate: you are all engineers, share your experiences
  - Our team will wait for some of you to answer questions on discussion board first before stepping in
  - Help us to Help you!

  - There are no open office hours!

COURSE SETUP

- Study material
  - Slides and Recorded lectures on Blackboard
  - Support videos and lecture notes on Blackboard
- Reference books
  - **Finite Element Procedures**, K.J. Bathe, 1995 (Prentice Hall)
  - **Concepts and Applicational of Finite Element Analysis**, R.D. Cook, D.S. Malkus, M.E. Alesha and R.J. Witt, 2002 (John Wiley & Sons)
  - **The Finite Element Method in Engineering**, S.S. Rao, 2005 (Elsevier Inc.)

WHY STUDY THIS?

- One of the most widely used methods for numerical solutions, both in research and industry
- Become a wise FE user; avoid worthless results
- Don't just push buttons on a commercial code; Oil platforms have collapsed due to insufficient FE analysis *(Sleipner A)*
- Increase your skill set and be a better engineer
- Be prepared for a Master thesis assignment

The sinking of the Sleipner A offshore platform

The investigation into the accident is described in 16 reports...

The conclusion of the investigation was that the loss was caused by a failure in a cell wall, resulting in a serious crack and a leakage that the pumps were not able to cope with. The wall failed as a result of a combination of a serious error in the finite element analysis and insufficient anchorage of the reinforcement in a critical zone.

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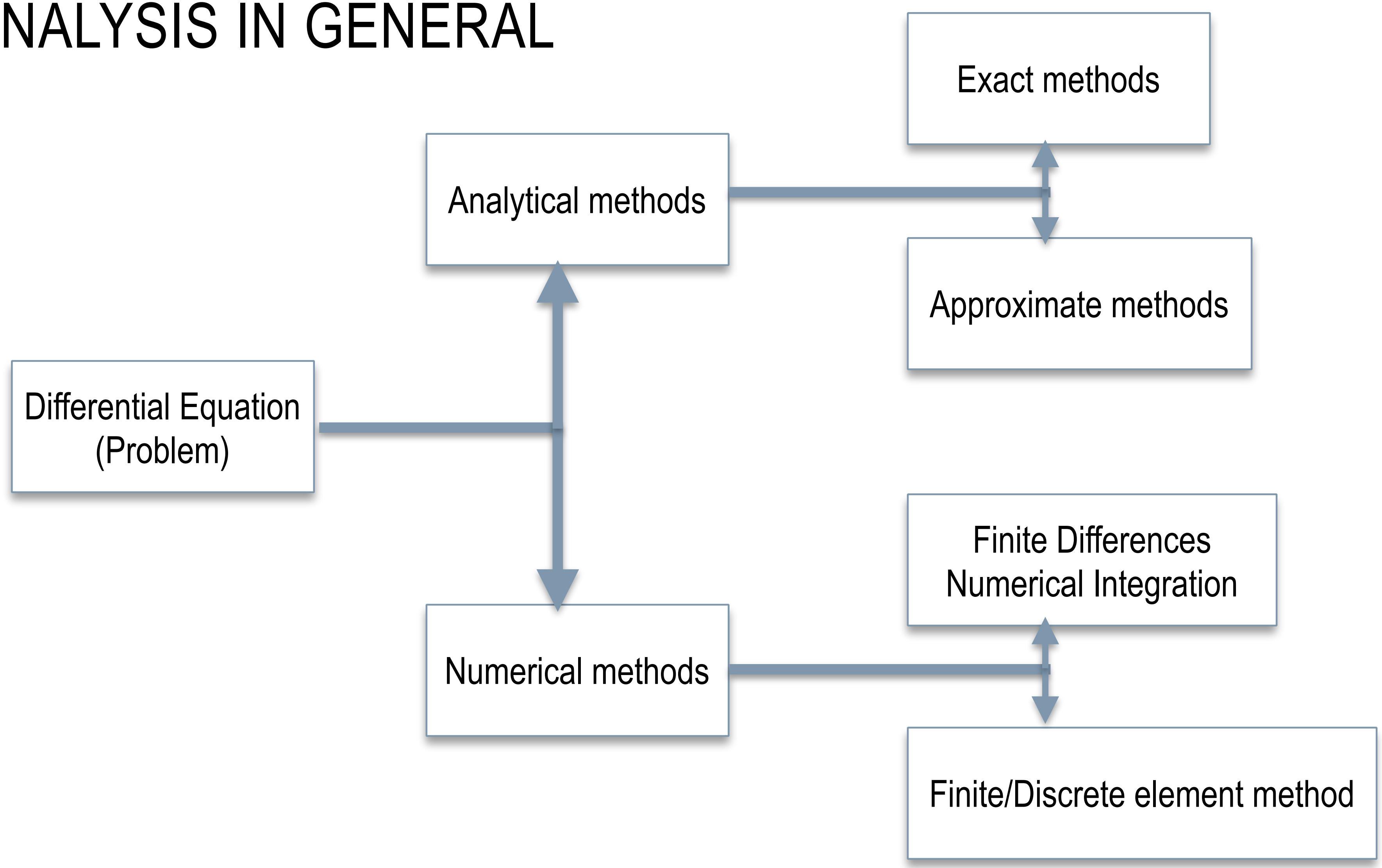
DESIRED SKILL SET

- Linear Algebra
- Calculus I
- Calculus II
- Aerospace Mechanics of Materials
- Applied Numerical Analysis
- Computational Modelling

TODAY…

- What is the Finite Element Method?
- A “bar” element
- Direct Stiffness method

ANALYSIS IN GENERAL



FINITE ELEMENT METHOD

- ..or Finite Element (FE) Analysis is a method for numerical solution of problems where a *field quantity* is sought.
  - ✓ versatility : displacement field, temperature field, stream function, etc.
  - ✓ approximate solution : except simple problems where an exact formula already exists
- Definition

Structure is cut into several *elements* to form a mesh

Behaviour of these *elements* is defined in a simple way

These *elements* are *glued* together at *nodes*

Result: several simultaneous equations

Piecewise polynomial interpolation

Field quantity is allowed to have a simple spatial variation over an *element*

These *elements* are *glued* together at *nodes*

Equilibrium equation is setup for the entire *Mesh.* Minimised for all unknowns

Mesh is represented by a system of algebraic equations, solved at nodes for unknowns

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FINITE ELEMENT METHOD

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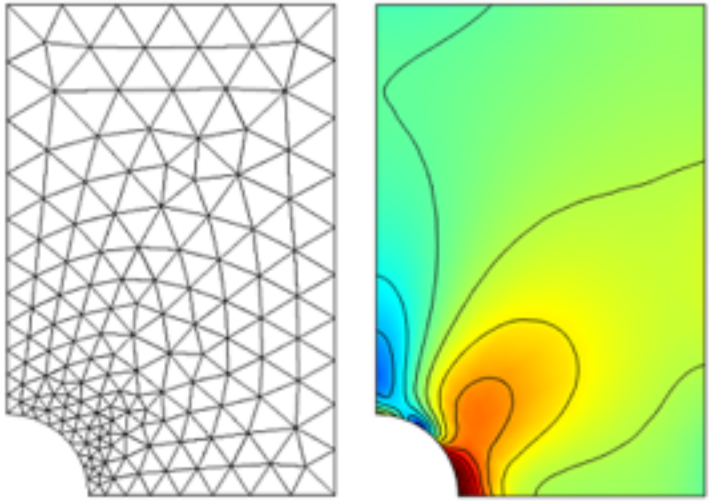

Picture courtesy: Comsol

Piecewise polynomial interpolation

Field quantity is allowed to have a simple spatial variation over an *element*

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Equilibrium equation is setup for the entire *Mesh.* Minimised for all unknowns

Mesh is represented by a system of algebraic equations, solved at nodes for unknowns

Minimization of a function

Nodal field quantities

=

Algebraic equations

$$K\,U = F$$

K: Characteristic Matrix (Stiffness)

U: Nodal unknowns (Displacement)

F: Applied/unknown loads

An FEA solution is not exact! But it can be improved!

STEPS IN A FINITE ELEMENT ANALYSIS

Pre-processing

- Idealisation
  - Classification
  - Modelling
- Discretisation
  - Element types
  - Interpolation
  - Degrees of Freedom (d.o.f)
  - Errors!
- Load, Support, Materials

Numerical Analysis

- Equilibrium Equations
- Solution

Post-processing

- Sorting the direct output
- Listing
- Derived output

THE ESSENTIAL…

- Discretization
- Selection of suitable interpolation/displacement model
- Element stiffness and load calculations followed by solving equilibrium equations
- *Meshing* and solution for unknown nodal displacements
- Computation of derived results, i.e., stress and strain

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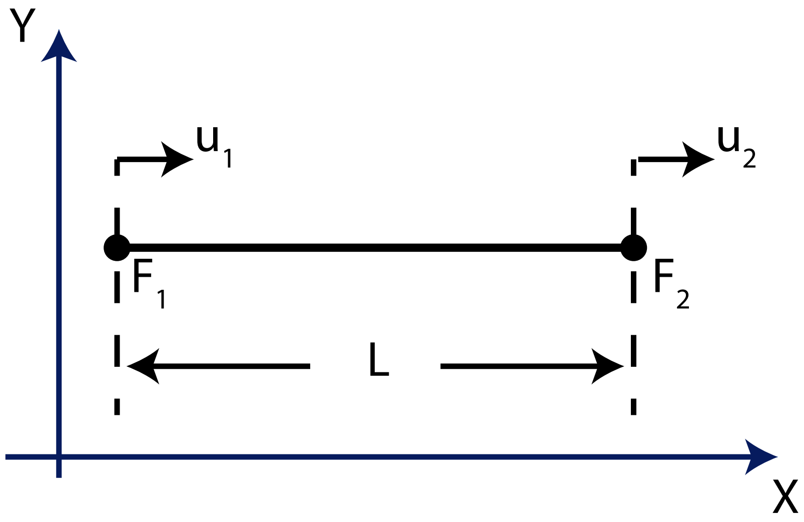
BAR ELEMENT - TRUSS SYSTEM

BAR ELEMENT - STIFFNESS MATRIX

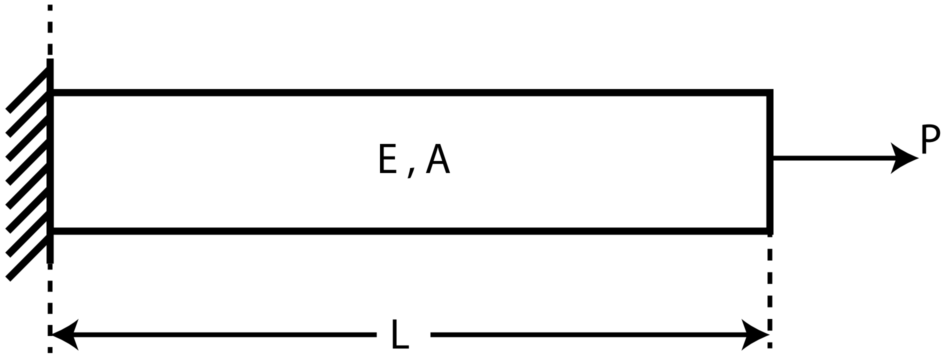
- Direct Method
- Formal Procedure

- 1-D element
- Rod!

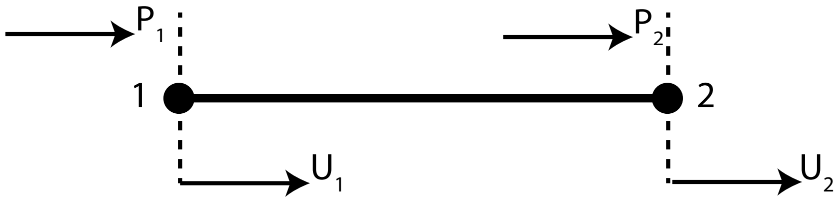


A = Area of Cross-section
E = Modulus of Elasticity

EXAMPLE PROBLEM - SIMPLE BAR WITH AXIAL LOAD



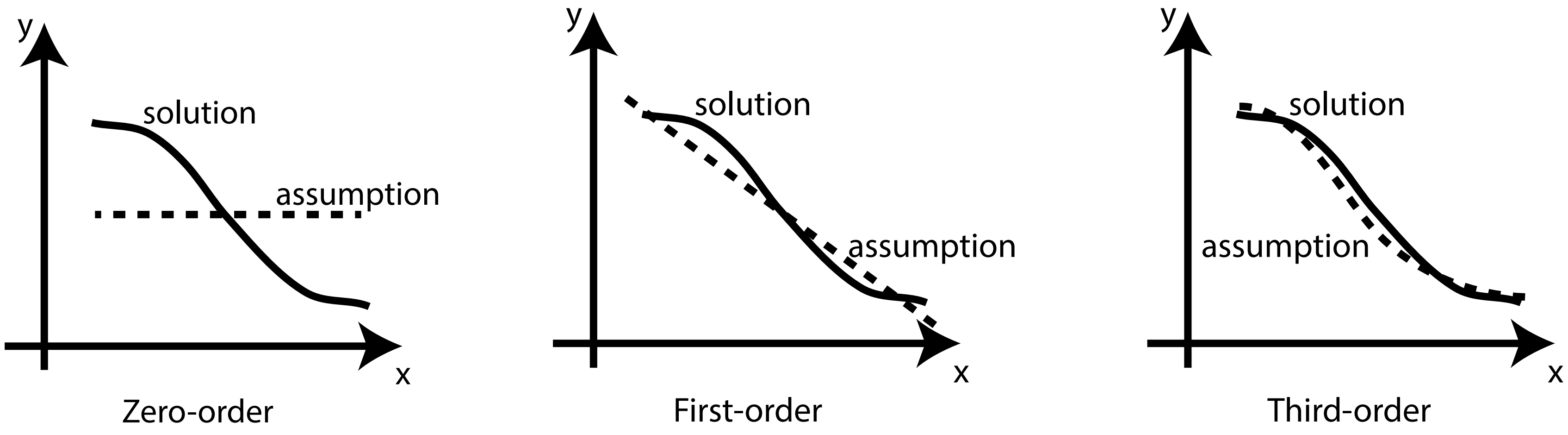
1. Idealization and Discretisation



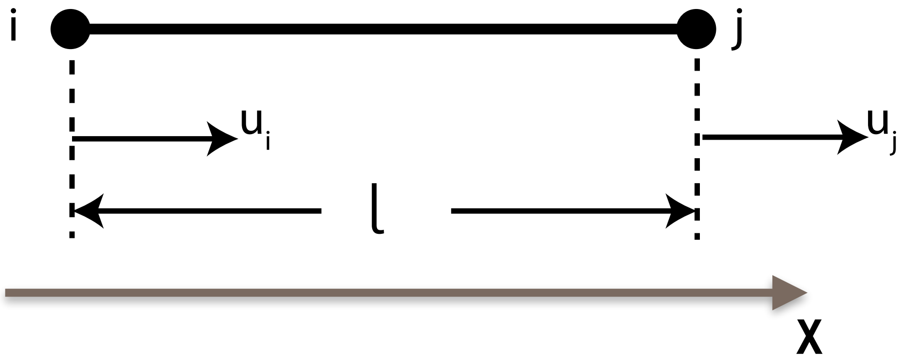
- Elements
- Nodes
- Degrees of Freedom

2. Displacement function/Interpolation Model/ Interpolation function

- Polynomials



- Linear displacement model



Let us assume

$$u = \alpha_1 + \alpha_2 . x \qquad \longrightarrow (1)$$

We know that

$$u(x) = u_i \quad at \quad x = x_i$$
$$u(x) = u_j \quad at \quad x = x_j \qquad \longrightarrow (2)$$

Substituting (2) in to (1)

$$u_i = \alpha_1 + \alpha_2 . x_i$$
$$u_j = \alpha_1 + \alpha_2 . x_j \qquad \longrightarrow (3)$$

Solving for unknowns

$$\alpha_1 = \frac{u_i . x_j - u_j . x_i}{l} \quad \& \quad \alpha_2 = \frac{u_j - u_i}{l} \qquad \longrightarrow (4)$$

So, displacement model (1) can be written as

$$u = \frac{u_i . x_j - u_j . x_i}{l} + \frac{u_j - u_i}{l} . x \qquad \longrightarrow (5)$$

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Rearranging $$u = \frac{u_i.x_j - u_j.x_i}{l} + \frac{u_j - u_i}{l}.x$$

$$= N_i(x).u_i + N_j(x).u_j$$

$$= [N(x)]\vec{u}^e \longrightarrow (6)$$

where $N(x) = [N_i(x) \quad N_j(x)]$ & $\vec{u}^e = \begin{bmatrix} u_i \\ u_j \end{bmatrix}$

And,

$$N_i(x) = \frac{x_j - x}{l}$$

$$N_j(x) = \frac{x - x_i}{l}$$

Shape functions!

For the current case,

$$At\ x = 0, \quad u = u_1$$

$$\therefore u = u_1 + \frac{u_2 - u_1}{l}.x$$

Linear displacement model

3. Equilibrium equation

Principal of minimum potential energy

$$Potential\ Energy\ I = \Pi - W_p$$

Strain Energy (Π) External Work (W_p)

For an element,

$$\Pi^e = \int_0^{l_e} \frac{1}{2} A^e \sigma^e \epsilon^e . dx \qquad \boxed{\sigma = E.\epsilon}$$

$$= \frac{E^e A^e}{2} \int_0^{l_e} \epsilon^{e2} . dx \longrightarrow (7)$$

From the interpolation model derivation, we know that

$$u = u_1 + \frac{u_2 - u_1}{l^e} . x$$

Therefore,

$$\frac{\partial u}{\partial x} = \frac{u_2 - u_1}{l^e} \longrightarrow (8)$$

Substituting (8) in to (7)

$$\Pi^e = \frac{E^e A^e}{2} \int_0^{l^e} \left(\frac{u_2 - u_1}{l^e}\right)^2 . dx$$

$$= \frac{E^e A^e}{2} \int_0^{l^e} \left(\frac{u_1^2 + u_2^2 - 2u_1 u_2}{l^{e2}}\right) . dx$$

$$= \frac{E^e A^e}{2} \left(\frac{u_1^2 + u_2^2 - 2u_1 u_2}{l^{e2}}\right) (l^e - 0)$$

$$= \frac{E^e A^e}{2l^e} (u_1^2 + u_2^2 - 2u_1 u_2)$$

Rearranging in matrix format

$$\Pi^e = [u_1 \ \ u_2] . \boxed{\frac{E^e A^e}{2l^e} . \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\Pi^e = \frac{1}{2} \vec{u}^{e^T} [\bar{\bar{K}}] \vec{u}^e \longrightarrow (9)$$

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Work done due to external forces

$$W_p = u_1 P_1 + u_2 P_2$$

$$= \vec{u}^{e^T} \vec{P} \longrightarrow (10)$$

Minimum Potential Energy

$$\frac{\partial I}{\partial u_n} = 0, \quad i = 1,2, \ldots.$$

$$\frac{\partial I}{\partial u_n} = \frac{\partial}{\partial u_i}\left(\sum_{e=1}^{n} \Pi^e - W_p\right) = 0 \longrightarrow (11)$$

$$= \frac{\partial}{\partial u_i}\left(\sum_{e=1}^{1}\left[\frac{1}{2} u^{e^T}.K.u^e - u^{e^T}.P^e\right]\right)$$

$$\Rightarrow \overline{\overline{K}}^e \vec{u}^e - \vec{P}^e = 0 \longrightarrow (12)$$

4. Solution of unknown displacements

Let us assume

$$\frac{E^e A^e}{l^e} = 4x10^6$$

Given that,

$$P_1 = R \quad \& \quad P_2 = 1$$

And, $\overline{\overline{K}}\vec{u} = \vec{P}$

$$\Rightarrow 4x10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} R \\ 1 \end{bmatrix}$$

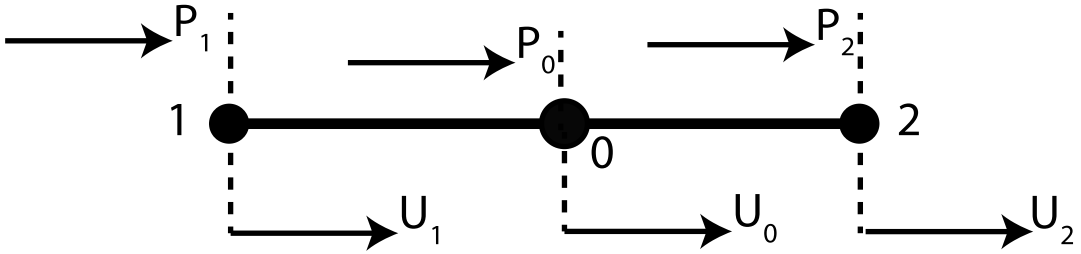
$$u_1 = 0 \ !$$

$$u_2 = ?; R = ?$$

LETS MAKE THIS FUN!

Same problem; different idealisation!

Two elements!



What changes?

$$Potential\ Energy\ I = \Pi^1 + \Pi^2 - W_p \longrightarrow \text{(A)}$$

$$= \frac{EA}{2l}(u_1^2 + u_0^2 - 2u_1u_0) + \frac{EA}{2l}(u_0^2 + u_2^2 - 2u_0u_2) - P_1u_1 - P_0u_0 - P_2u_2$$

Minimum Potential Energy

$$\frac{\partial I}{\partial u_0} = \frac{EA}{2l}[2u_0 - 2u_1] + \frac{EA}{2l}[2u_0 - 2u_2] - P_0$$

$$\frac{\partial I}{\partial u_1} = \frac{EA}{2l}[2u_1 - 2u_0] - P_1 \quad \& \quad \frac{\partial I}{\partial u_2} = \frac{EA}{2l}[2u_2 - 2u_0] - P_2$$

→ (B)

Rearranging equations (B)

$$K\begin{bmatrix} u_1 & u_0 & 0 \\ -u_1 & 2u_0 & -u_2 \\ 0 & -u_0 & u_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_0 \\ P_2 \end{bmatrix} \longrightarrow \text{(C)}$$

(C) resembles equilibrium equation

$$\bar{\bar{K}}\vec{u} = \vec{P}$$

Substituting material properties,

$$K^{e1} = 4x10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} u_1 \\ u_0 \end{matrix} \text{ x 2}$$

$$K^{e2} = 4x10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} u_0 \\ u_2 \end{matrix} \text{ x 2}$$

Assembling element matrices,

$$\bar{\bar{K}} = 4x10^6 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_0 \\ u_2 \end{bmatrix} \text{ x 2}$$

$$\bar{\bar{K}} = 4x10^6 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_0 \\ u_2 \end{bmatrix} \text{ x 2} \longrightarrow \text{(D)}$$

Solution of (C) gives unknown field quantities

** x 2 ; because the length of the element is also halved!

5. Derived results

Strain

$$\epsilon = \frac{\partial u}{\partial x}$$

Stress

$$\sigma = E.\epsilon$$

RECAP

- Idealization
- Displacement methods
    - Interpolation models
    - Interpolation functions/ Shape functions
- Element Stiffness matrix
- Assembly
- Load vectors
- Solution for displacements
- Element strains and stresses

HOMEWORK

- Check blackboard for practice problems on direct stiffness approach
- Read revision lecture notes on Calculus of Variations
- Answer Self-Check questions and discuss on the forum

NEXT WEEK…

- Variational approach
- Setting up a finite element equation using the variational approach

PRACTICALS

- Tutorials
- Discretization

☑ If you will use your own laptop for the practicals, please install the software you will use prior to the practical session tomorrow!