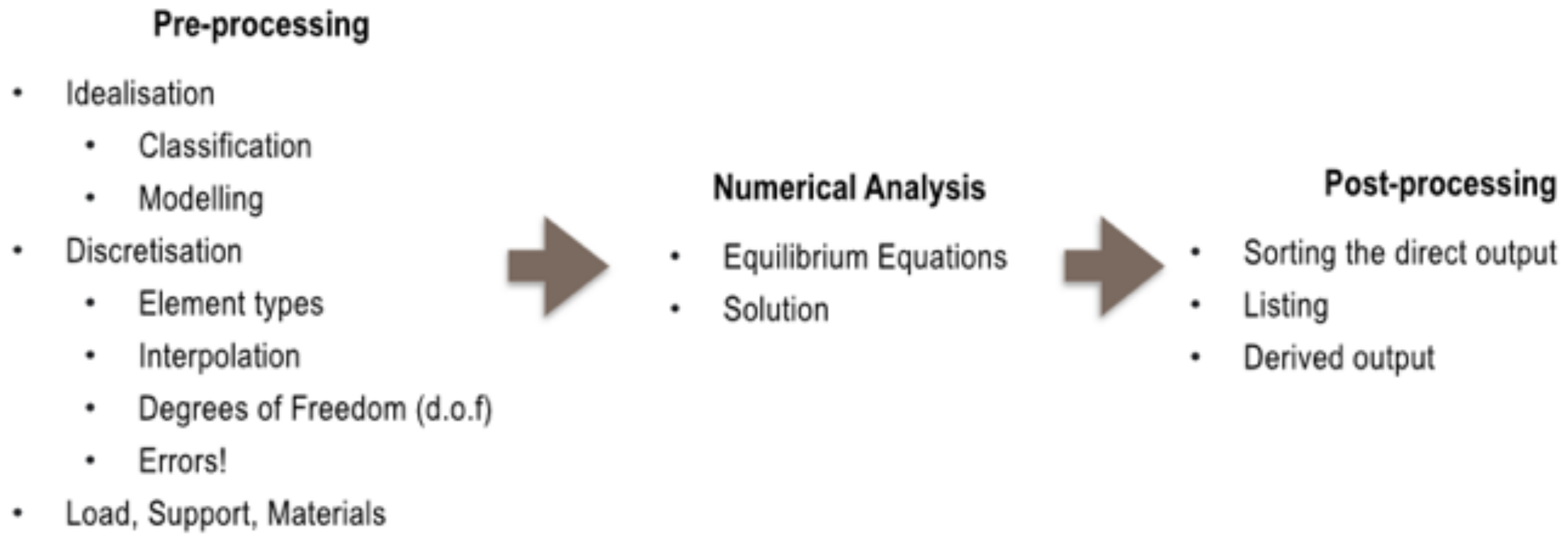
LINEAR MODELLING (INCL. FEM)
AE4ASM003
P1-2015

LECTURE 2
08.09.2015

RECAP

- Solution of a finite element problem
  - Principal of virtual work
  - Physical argument
- Basics of the approximate methods



PHYSICAL ARGUMENT - DIRECT STIFFNESS METHOD

For equilibrium,

$$P_1 + \sigma A = 0$$

$$P_2 - \sigma A = 0 \quad \rightarrow (1)$$

According to Hooke's Law,

$$\sigma = E\epsilon$$

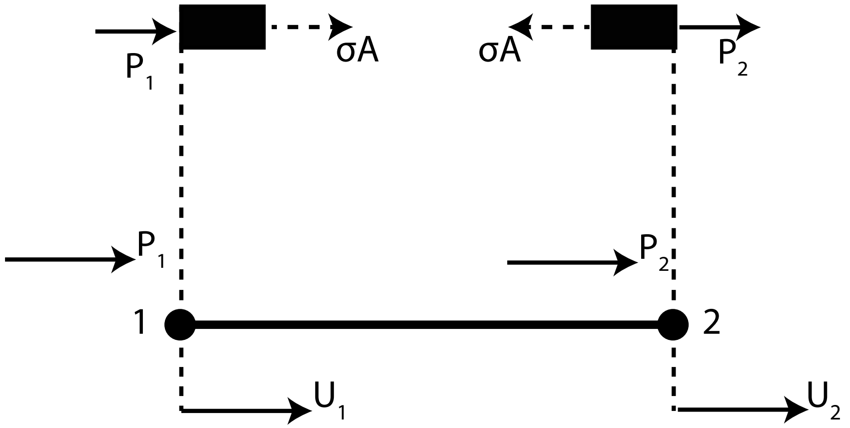
$$\epsilon = \frac{u_2 - u_1}{L} \quad \rightarrow (2)$$

Substituting in (2) in (1)

$$\therefore \begin{aligned} P_1 + EA\frac{u_2 - u_1}{L} = 0 \\ P_2 - EA\frac{u_2 - u_1}{L} = 0 \end{aligned} \quad \Rightarrow \begin{aligned} \frac{EA}{L}(u_1 - u_2) = P_1 \\ \frac{EA}{L}(u_2 - u_1) = P_2 \end{aligned}$$

$$or, \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \quad \rightarrow (3)$$

$$where, k = \frac{EA}{L}$$



TODAY…

- Self Check Quiz and Practice problems - hints and solutions
- Approximate methods of analysis in general
- Variational approach (Rayleigh-Ritz)
- Setting up finite element equations using the Rayleigh-Ritz approach

SELF CHECK QUIZ

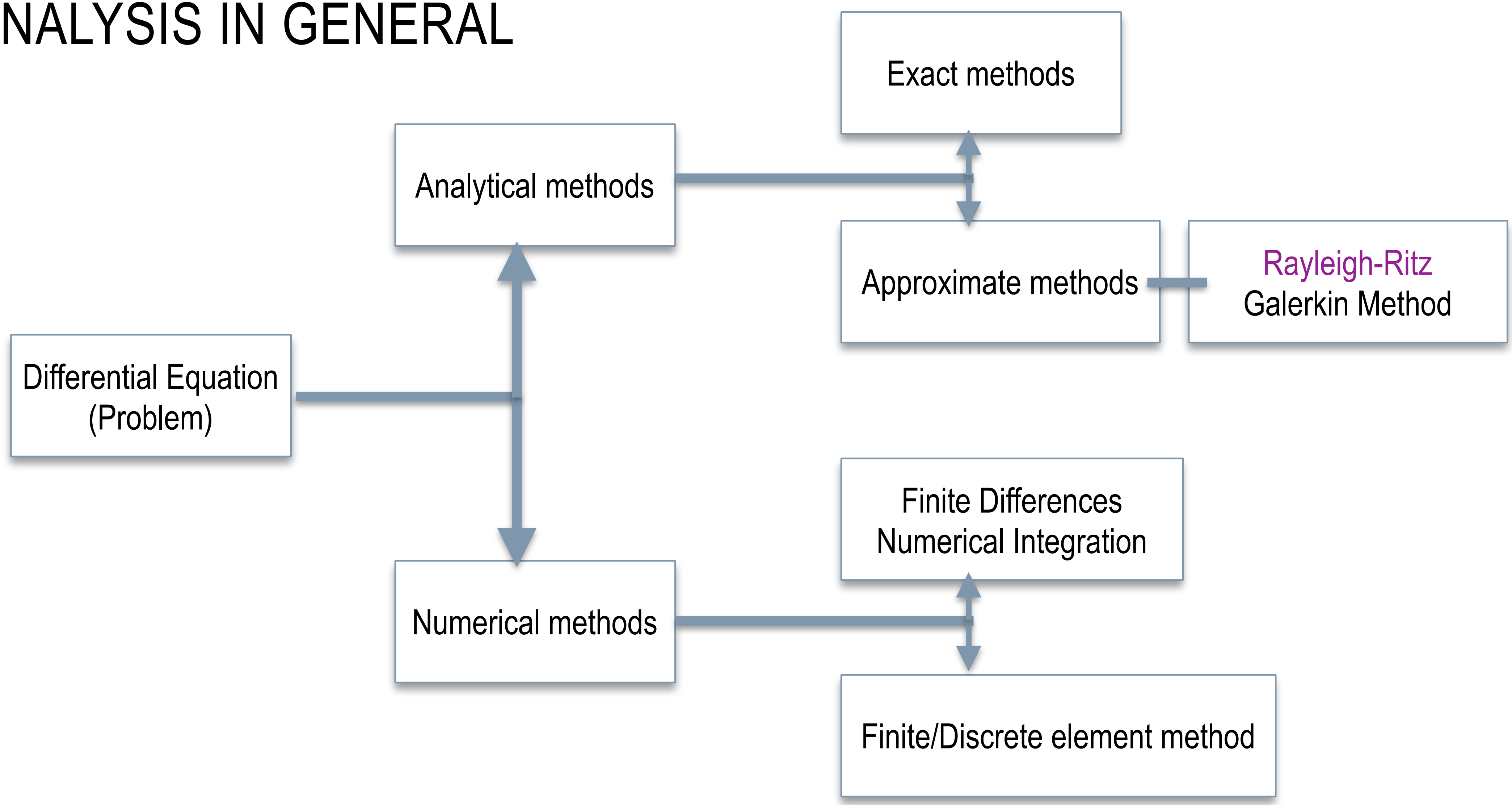
- All information can be gathered from the lectures
- Any questions?

PRACTICE PROBLEMS

- Problem 1
  - Idealisation
  - Global stiffness matrix assembly

- Problem 2
  - Question of minimising the potential energy
  - Global stiffness matrix assembly
  - Global load vector assembly

ANALYSIS IN GENERAL



VARIATIONAL APPROACH - RAYLEIGH RITZ

RECAP

- Calculus of variations
- Advantages of variational formulation

VARIATIONAL METHODS: RAYLEIGH RITZ

Important terms

- **Functional**

  Integral containing the governing differential equation and non-essential boundary conditions
- **Strong form** vs. **Weak form**

  Governing differential equations with boundary conditions vs. Integral containing the differential equation explicitly
- **Essential** boundary conditions

  Eg. Support conditions in structural mechanics
- **Natural** boundary conditions

  Eg. Enforced conditions

THE RAYLEIGH RITZ METHOD

- Weak form
- Avoid solving a differential equation
- Not exact; improved results with higher number of d.o.f.
- Originally used for vibration problems

Again, we are always looking for a field quantity in the continuum/discrete domain

- Begin by approximating this quantity
- In terms of a function of spatial co-ordinates and an unknown amplitude (also called generalised co-ordinates)
- Function must be admissible (essential boundary conditions are enforced)

APPROXIMATE SOLUTION OF A DIFFERENTIAL EQUATION USING RAYLEIGH RITZ METHOD

Equilibrium problem: differential equation formulation

$$A\vec{u} = b \quad in \quad V$$

$$B_j\vec{u} = g_j, \quad j = 1,2,\dots,p \; on \; S$$ (Boundary conditions) → (1)

Variational formulation

$$I = \int_V F(x, u, \dot{u}) \, . \, dx$$ → Minimise → (2)

Approximate solution assumption

$$u(x) = \sum_{i=1}^{n} C_i f_i(x)$$ → (3)

(trial functions: $f_i(x)$; unknown parameters: C_i)

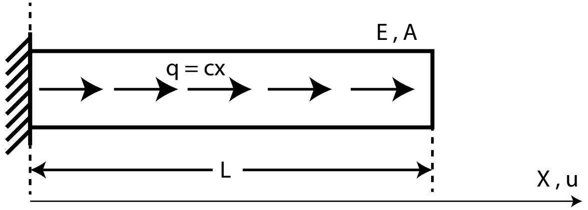
Stationary functional condition

$$\frac{\partial I(u)}{\partial C_i} = 0, \quad i = 1,2,\dots,n$$ → (4)

Solve n set of linear simultaneous equations!

EXAMPLE

- Bar under axial load



q is a linearly distributed load, where c is a constant load value /m[2]

*all calculations are carried out in one spatial co-ordinate (X)

Displacement u

Strain $\epsilon = \dot{u}$ → (a)

Stress $\sigma = E.\epsilon$

Total potential energy $I = \Pi - W_p$

where $\Pi = \frac{1}{2}\int \epsilon^T E\, \epsilon. dV$ & $W_p = \int q.u.dx$

giving $I = \frac{1}{2}\int_0^l E.\dot{u}^2.A.dx - \int_0^l c.u.x.dx$ → (b)

Assumed displacement $u = \sum_{i=1}^{n} a_i f_i = a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ → (c)

Why is there no constant term in there?

Use only first term for the first assumption

Find the first derivative of u and substitute in (X) and simplify $I = \frac{1}{2} EAl.a_1^2 - \frac{cl^3}{3}.a_1$ → (d)

Stationary potential energy $\frac{\partial I}{\partial a_1} = EAl.a_1 - \frac{cl^3}{3} = 0$ → (e)

gives

$$a_1 = \frac{cl^2}{3EA} \qquad u = \frac{cl^2}{3EA}.x \qquad \sigma_x = E.\dot{u} = \frac{cl^2}{3A}$$

OBSERVATIONS

- pick practical approximating fields
  - polynomials, sine, cosine functions
  - admissible
- a sequence of trial solutions is generated
- for the sequence to converge, trial field should be complete
  - lowest order admissible terms have to be included
  - exact displacement and its derivatives that appear in the model must be in the trial function
- Rayleigh Ritz solution too stiff as compared to the mathematical model
  - continuous to discrete!
  - more d.o.f. better the solution

FINITE ELEMENT EQUATIONS USING VARIATIONAL APPROACH - RAYLEIGH RITZ

FINITE ELEMENT FORMULATION USING RAYLEIGH-RITZ

Continuum domain *V* is divided into finite “elements” *e*

Spatial variation of the field quantity along every element $\vec{u} = [N]\vec{u}^e$

Functional is defined as a sum of the functionals for the entire discrete domain $I = \sum_{e=1}^{E} I^e$

Element equilibrium equations generated by extremisation of the functional $\frac{\partial I}{\partial u_i} = \begin{Bmatrix} \partial I/\partial u_1 \\ \partial I/\partial u_2 \\ \cdot \\ \cdot \\ \vdots \\ \partial I/\partial u_e \end{Bmatrix} = 0$ or $\frac{\partial I}{\partial u_i} = \sum_{e=1}^{E} \frac{\partial I^e}{\partial u_i} = 0, \quad i = 1,2,\dots,M$

If the functional is quadratic, the overall system of equations can be written as: $\frac{\partial I^e}{\partial u^e} = [K^e]\vec{u}^e - \vec{P}^e = 0$

FINITE ELEMENT EQUATIONS USING RAYLEIGH-RITZ

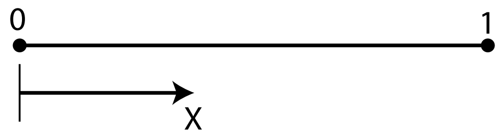
- Overall system of equations is obtained by replacing element characteristic matrix by summation of it for all elements in the system, same for characteristic vector

- Boundary conditions are applied

- Linear simultaneous system of equations is solved

- Field variable for each element can be found out and so can its derivatives

EXAMPLE PROBLEM

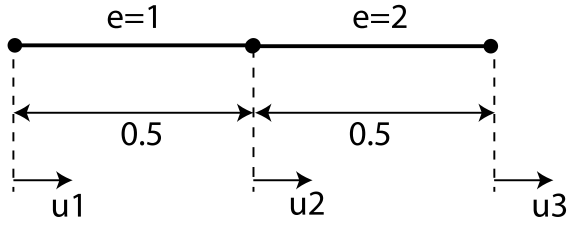


$$\frac{d^2u}{dx^2} + u + x = 0, \quad 0 \leq x \leq 1$$

$$u(0) = u(1) = 0$$

Given that the functional corresponding to the Euler equation is $I = \frac{1}{2}\int_0^1 \left[-\left(\frac{du}{dx}\right)^2 + u^2 + 2ux\right].dx$ → (A)

Idealization - 2 elements



we can assume the linear interpolation model in the element to be $u(x) = [N(x)]u^e = N_i(x).u_i^e + N_j(x).u_j^e$ → (B)

where $N_i(x) = \frac{x_j - x}{l}$ & $N_j(x) = \frac{x - x_i}{l}$

For a multi-element system, the functional is represented as $I = \sum^{E} I^e$

where $I^e = \frac{1}{2}\int_{xi}^{xj}\left[-\left(\frac{du}{dx}\right)^2 + u^2 + 2ux\right].dx$ → (C)

Substituting the linear interpolation model into (C) $I^e = \frac{1}{2}\int_{xi}^{xj}\{-u^{eT}\left[\frac{dN}{dx}\right]^T\left[\frac{dN}{dx}\right]u^e + u^{eT}[N]^T[N]u^e + 2x[N]^T u^e\}.dx$ → (D)

Stationary functional gives $\sum_{e=1}^{E}\frac{\partial I^e}{\partial u^e} = \sum_{e=1}^{E}\int_{xi}^{xj}\{-\left[\frac{dN}{dx}\right]^T\left[\frac{dN}{dx}\right]u^e + [N]^T[N]u^e + x[N]^T\}.dx = \vec{0}$ → (E) or, $\sum_{e=1}^{E}[K^e]\vec{u}^e = \sum_{e=1}^{E}\vec{P}^e$ → (F)

where $[K^e] = \int_{xi}^{xj}\left\{\left[\frac{dN}{dx}\right]^T\left[\frac{dN}{dx}\right] - [N]^T[N]\right\}.dx$ → (G) & $\vec{P}^e = \int_{xi}^{xj} x[N]^T.dx$ → (H)

TUDelft

Substituting the shape functions into the stiffness matrix and load vector, we get,

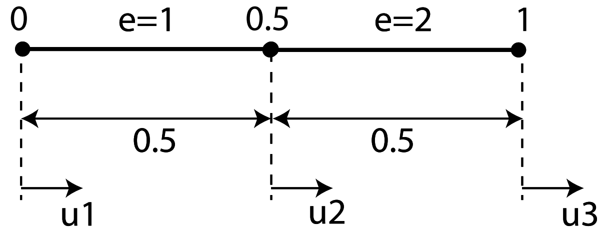
$$[K^e] = \frac{1}{l^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{l^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\vec{P}^e = \frac{1}{6}\begin{Bmatrix} x_j^2 + x_i x_j - 2x_i^2 \\ 2x_j^2 - x_i x_j - x_i^2 \end{Bmatrix}$$

Substitute for the element stiffness matrices and load vector, K=? P=?

Assemble the stiffness matrices and load vectors for the entire system

Apply boundary conditions, and arrive at the unknown displacements u=?



HOMEWORK

- Check blackboard for practice problems on variational approach
- Read revision lecture notes on Interpolation models
- Answer Self-Check questions and discuss on the forum
- **Assignment 1** will be released today. Start working on it!

NEXT WEEK…

- Weighted residual approach
- Setting up a finite element equation using the weighted residual approach

PRACTICALS

- Loads and Boundary Conditions
- Practice Exercises