LINEAR MODELLING (INCL. FEM)
AE4ASM003
P1-2015

LECTURE 3
15.09.2015

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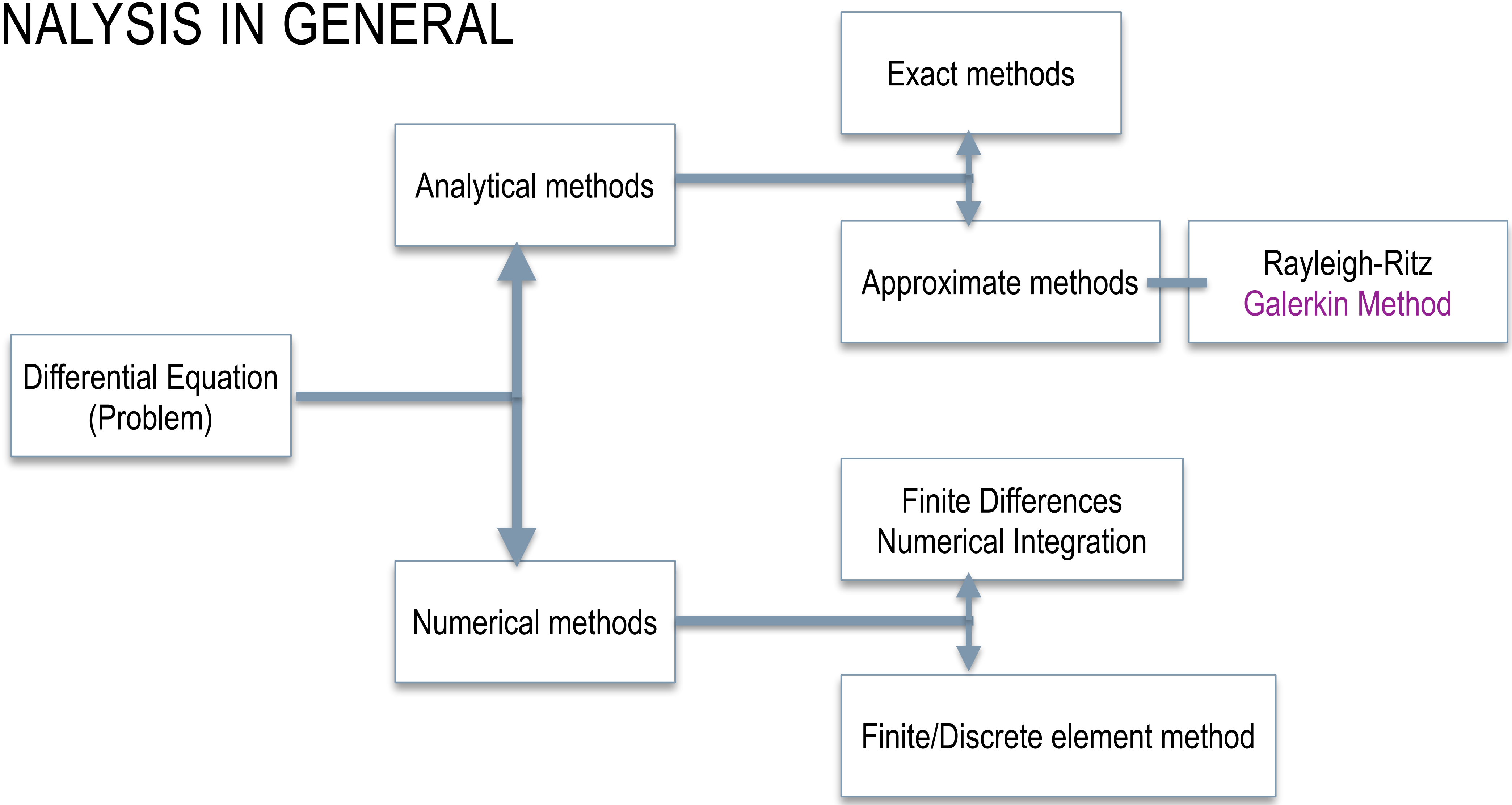
TODAY…

- Weighted residual approach (Galerkin)
- Setting up finite element equations using the Galerkin approach
- Co-ordinate tranformations

ANALYSIS IN GENERAL



WEIGHTED RESIDUAL APPROACH

WEIGHTED RESIDUAL VS VARIATIONAL APPROACH

- Both approximate
- No “functional” required for the weighted residual

APPROXIMATE SOLUTION OF A DIFFERENTIAL EQUATION USING WEIGHTED RESIDUAL METHOD

Equilibrium problem: differential equation formulation

Differential operator

$$Au = b \qquad in \ \ V$$

$$B_j u = g_j, \quad j = 1,2,\dots p \ on \ S \qquad \text{(Boundary conditions)} \qquad \rightarrow (1)$$

Equilibrium equation can be expressed as:

$$F(u) = G(u) \quad in\ V \qquad \rightarrow (2)$$

Residual or Error can be defined as: $R = G(u) - F(u)$ → (3)

where the field variable in weighted residual method can be described as:

$$u = \sum_{i=1}^{n} C_i f_i(x) \qquad \rightarrow (4)$$

So, the function $f(R)$ is chosen such that it must be zero when the field variable u is exact!

A weighted function of the residual must now be taken to be a minimum or satisfy the "smallness criterion" such that

$$\int_V w f(R).dV = 0 \quad \longrightarrow (5)$$

Various methods are available to solve this using the weighted residual approach, such as

(1) Least squares method (2) Collocation method (3) Galerkin method (gives the best approximation)

What differs? —— the weights!

In the Galerkin approach,

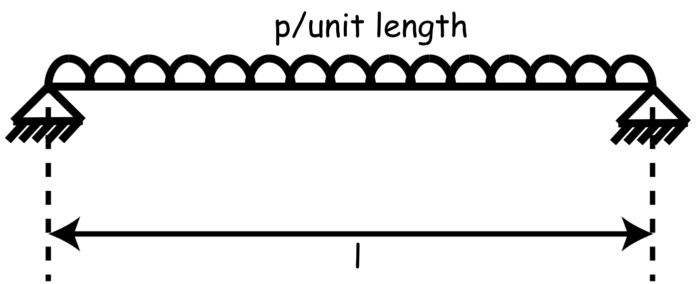
$$w_i = f_i(x) \quad \text{(known functions of the trial solution)} \quad \longrightarrow (6)$$

So, for "n" unknowns, n integrals of weighted residuals are

$$\int_V f_i R.dV = 0 \quad i = 1,2,\dots,n \quad \longrightarrow (7)$$

EXAMPLE

- Simple supported bar under uniformly distributed load



Equilibrium problem: differential equation formulation

$$EI\frac{d^4w}{dx^4} - p = 0, \quad 0 \le x \le l \qquad \text{(A)}$$

$$w(x=0) = w(x=l) = 0$$

$$EI\frac{d^2w}{dx^2}(x=0) = EI\frac{d^2w}{dx^2}(x=l) = 0 \qquad \text{(B)}$$

The trial function for the field variable can be assumed to be

$$w(x) = C_1 sin\left(\frac{\pi x}{l}\right) + C_2 sin\left(\frac{3\pi x}{l}\right)$$

$$= C_1 f_1(x) + C_2 f_2(x) \qquad \text{(C)}$$

Residual can be written as:

$$R = EI\frac{d^4w}{dx^4} - p \qquad \longrightarrow \text{(D)}$$

$$= EI\frac{\pi^4}{l^4}\left[C_1 \sin\left(\frac{\pi x}{l}\right) + 3^4 C_2 \sin\left(\frac{3\pi x}{l}\right)\right] - p \qquad \longrightarrow \text{(E)}$$

where $\frac{d^4w}{dx^4} = \frac{\pi^4}{l^4}\left[C_1 \sin\left(\frac{\pi x}{l}\right) + 3^4 C_2 \sin\left(\frac{3\pi x}{l}\right)\right]$

We have

$$f_1(x) = \sin\left(\frac{\pi x}{l}\right) \quad \& \quad f_2(x) = \sin\left(\frac{3\pi x}{l}\right) \qquad \longrightarrow \text{(F)}$$

Following the Galerkin approach,

$$\int_0^l f_1(x)R\, dx = 0 \qquad \longrightarrow \text{(G)}$$

$$\int_0^l f_2(x)R\, dx = 0 \qquad \longrightarrow \text{(H)}$$

Finally, you will arrive at

$$EIC_1\left(\frac{\pi}{l}\right)^4\frac{l}{2}-p\frac{2l}{\pi}=0 \quad \longrightarrow \text{(I)}$$

$$EIC_2\left(\frac{3\pi}{l}\right)^4\frac{l}{2}-p\frac{2l}{3\pi}=0 \quad \longrightarrow \text{(J)}$$

yielding,

$$C_1=\frac{4pl^4}{\pi^5EI} \quad \& \quad C_2=\frac{4pl^4}{243\pi^5EI} \quad \longrightarrow \text{(K)}$$

FINITE ELEMENT EQUATIONS USING WEIGHTS RESIDUAL APPROACH - GALERKIN

FINITE ELEMENT EQUATIONS USING GALERKIN APPROACH

Equilibrium problem: differential equation formulation

$$Au = b \qquad in \;\; V$$

$$B_j u = g_j, \quad j = 1,2, \dots p \; on \; S \quad \text{(Boundary conditions)} \quad \rightarrow \text{(i)}$$

Galerkin method yield's the integral to satisfy smallness criterion as:

$$\int_V [A(u) - b] f_i dV = 0, \quad i = 1,2, \dots, n \quad \rightarrow \text{(ii)}$$

with f_i being the trial functions of the assumed approximate solution with unknowns C_i

For an elemental volume,

$$\int_{V^e} [A(u^e) - b^e] N_i^e dV = 0\,, \quad i = 1,2, \dots, n \quad \rightarrow \text{(iii)}$$

with N_i^e replacing the trial function f_i

such that an approximate solution is assumed to be the interpolation model given by, $u^e = [N^e] u^e$

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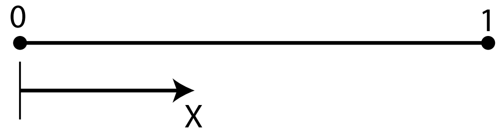
EXAMPLE PROBLEM



$$\frac{d^2u}{dx^2} + u + x = 0, \quad 0 \le x \le 1$$

$$u(0) = u(1) = 0$$

Residual can be written as:

$$R = \frac{d^2u}{dx^2} + u + x \qquad \text{(a)}$$

Galerkin method yield's the integral to satisfy smallness criterion as:

$$\int_0^1 \left[\frac{d^2u}{dx^2} + u + x\right] N_k(x)dx = 0; \quad k = i, j \qquad \text{(b)}$$

Or, for a discretised domain,

$$\sum_{e=1}^{E} \int_{x_i}^{x_j} [N^e]^T \left[\frac{d^2u^e}{dx^2} + u^e + x\right] dx = 0 \qquad \text{(c)}$$

The linear interpolation model is assumed to be

$$u^e(x) = N_i(x)u_i^e + N_j(x)u_j^e \qquad \text{(d)}$$

So $[N^e] = [N_i(x) \quad N_j(x)]$ (e)

And, $N_i(x) = \frac{x_j - x}{l^e}$ & $N_j(x) = \frac{x - x_i}{l^e}$ (f)

The first term on the left can be integrated by parts, to yield

$$\int_{x_i}^{x_j} [N]^T \frac{d^2u}{dx^2} dx = [N]^T \frac{du}{dx}\Big|_{x_i}^{x_j} - \int_{x_i}^{x_j} \frac{d[N]^T}{dx}\frac{du}{dx} dx \quad \text{(g)}$$

Substituting this back into (c) for a single element,

$$[N]^T \frac{du}{dx}\Big|_{x_i}^{x_j} - \int_{x_i}^{x_j} \left\{ \frac{d[N]^T}{dx}\frac{du}{dx} - [N]^T u - [N]^T x \right\} dx = 0 \quad \text{(h)}$$

Leading back to the finite element equation containing stiffness matrix and load vector,

$$[K^e]\vec{u}^e = \vec{P}^e \quad \text{(i)}$$

where, upon substitution of

$$\frac{d}{dx}[N]^T = ? \quad \& \quad \frac{du}{dx} = ?$$

we can arrive at

$$[K^e] = \frac{1}{l^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{l^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \& \quad P^e = \frac{1}{6}\begin{Bmatrix} (x_j^2 + x_i x_j - 2x_i^2) \\ (2x_j^2 - x_i x_j - x_i^2) \end{Bmatrix} \quad \text{(j)}$$

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WRAP UP OF FINITE ELEMENT FORMULATION BY VARIOUS METHODS

RECAP

Direct Stiffness Approach

Physical Argument   Principle of Virtual Work / Potential Energy

Variational Approach - Rayleigh-Ritz

Functional (Integral)

Weighted Residuals - Galerkin Approach

Differential Equation to form residual

CO-ORDINATE TRANSFORMATIONS

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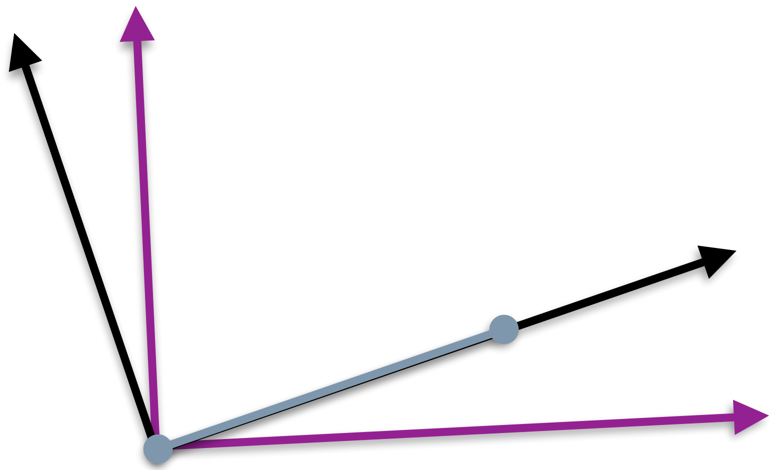
CO-ORDINATE SYSTEMS

- Local
- Global



- Elements may be aligned in varied local co-ordinate axes
- Global characteristics therefore, cannot be compared

Consistent co-ordinate system for all elements!

LOCAL TO GLOBAL TRANSFORMATION

- All lower case characters refer to local co-ordinates
- All upper case characters refer to global co-ordinates

Let's say the characteristic equilibrium equation is written in the local co-ordinate system as

$$[k^e]\vec{u}^e = \vec{p}^e \qquad \text{(I)}$$

If a transformation matrix λ^e exists between the local and global coordinate systems,

$$\vec{u}^e = [\lambda^e]\vec{U}^e$$
$$\vec{p}^e = [\lambda^e]\vec{P}^e \qquad \text{(II)}$$

Substituting this back into (I), we get,

$$[k^e][\lambda^e]\vec{U}^e = [\lambda^e]\vec{P}^e \quad x\ [\lambda^e]^{-1} \qquad \text{(III)}$$

we get,

$$[\lambda^e]^{-1}[k^e][\lambda^e]\vec{U}^e = \vec{P}^e \qquad \text{(IV)}$$

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From (IV), we can say,

$$[K^e]\vec{U}^e = \vec{P}^e \qquad \text{where} \qquad [K^e] = [\lambda^e]^{-1}[k^e][\lambda^e] \longrightarrow \text{(V)}$$

Because, the transformation matrix has a great property called orthogonality,

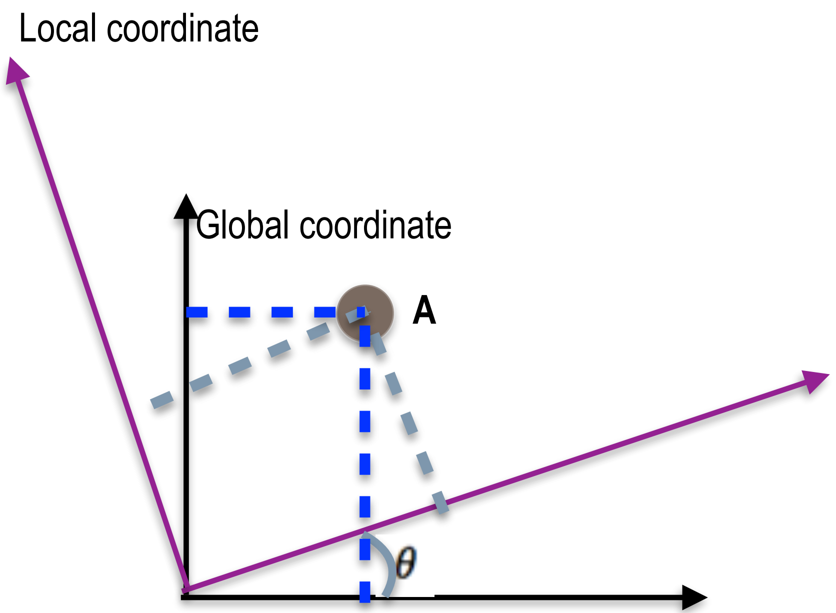
$$\lambda^T\lambda = \lambda\lambda^T = I$$
$$\therefore\ \lambda^T = \lambda^{-1} \longrightarrow \text{(VI)}$$

So, relation (V) can be re-written as,

$$[K^e] = [\lambda^e]^T[k^e][\lambda^e] \longrightarrow \text{(VII)}$$

SOME EXAMPLES

- Point transformation



$$\begin{bmatrix} u_A \\ u_B \end{bmatrix} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} U_A \\ U_B \end{bmatrix}$$

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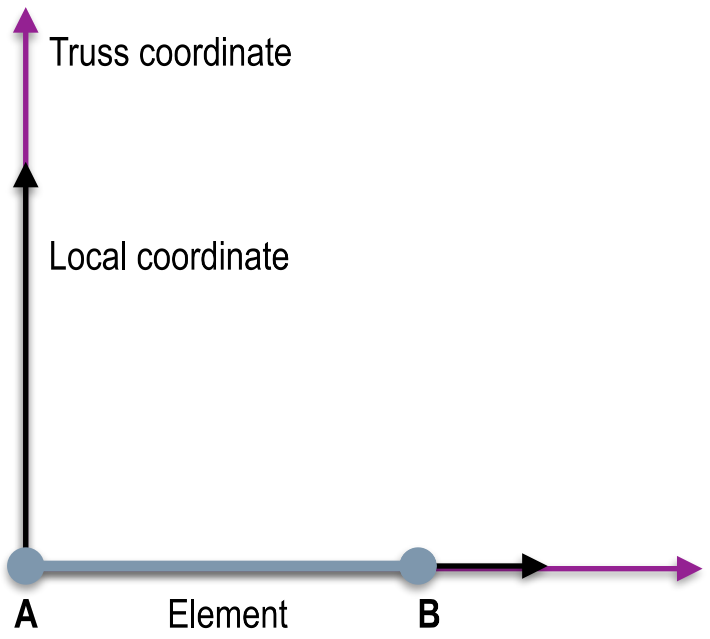
- Truss transformation
  - Truss element is locally 1D
  - Truss co-ordinate system is 2D
  - Truss co-ordinate system requires 2 d.o.f to be defined, both in x and y
  - So far we have done only 1, in x

So lets align the truss to a 2D truss co-ordinate system



$$u = [T]^T U$$

$$\begin{bmatrix} u_A \\ u_B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} U_{1A} \\ U_{2A} \\ U_{1B} \\ U_{2B} \end{bmatrix}$$

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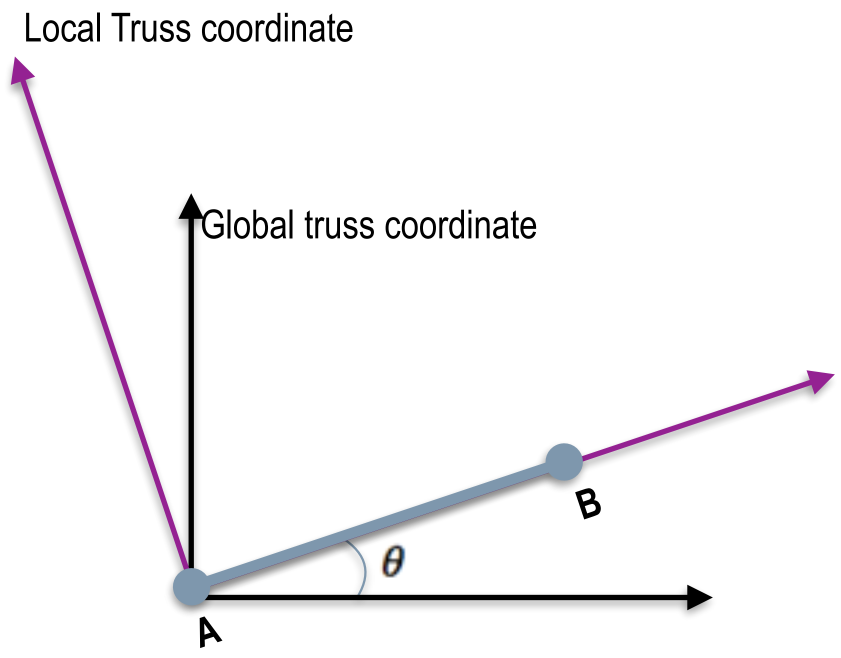
And, what if we rotate the truss element in the truss 2D coordinate system?



$$U = [\lambda]u$$

$$\begin{bmatrix} U_{1A} \\ U_{2A} \\ U_{1B} \\ U_{2B} \end{bmatrix} = \begin{bmatrix} \cos\theta & -sin\theta & 0 & 0 \\ sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -sin\theta \\ 0 & 0 & sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_{1A} \\ u_{2A} \\ u_{1B} \\ u_{2B} \end{bmatrix}$$

So, in one step from truss local co-ordinate to truss global co-ordinate system:

$$u = [T]^T[\lambda]^T U$$

$$\begin{bmatrix} u_A \\ u_B \end{bmatrix} = \begin{bmatrix} cos\theta & sin\theta & 0 & 0 \\ 0 & 0 & cos\theta & sin\theta \end{bmatrix} \begin{bmatrix} U_{1A} \\ U_{2A} \\ U_{1B} \\ U_{2B} \end{bmatrix}$$

HOMEWORK

- Check blackboard for practice problems on Galerkin approach
- Answer Self-Check questions and Practice problems and discuss on the forum
- Continue working on Assignment 1 and don't forget to submit on time!
- Watch flipped classroom video on Interpolation functions for next week

NEXT WEEK…

- Formulation of the FE equations (solid mechanics)
- Beam element

PRACTICALS…

- Continue working on application of different load types
- Material and Property definition
- Plane Strain/Stress Elements