LINEAR MODELLING (INCL. FEM)
AE4ASM003
P1-2015

LECTURE 4
22.09.2015

TODAY…

- Element equations of 1-D structural elements
    - 3-D truss
    - Beam element
    - Applications

PRIOR KNOWLEDGE

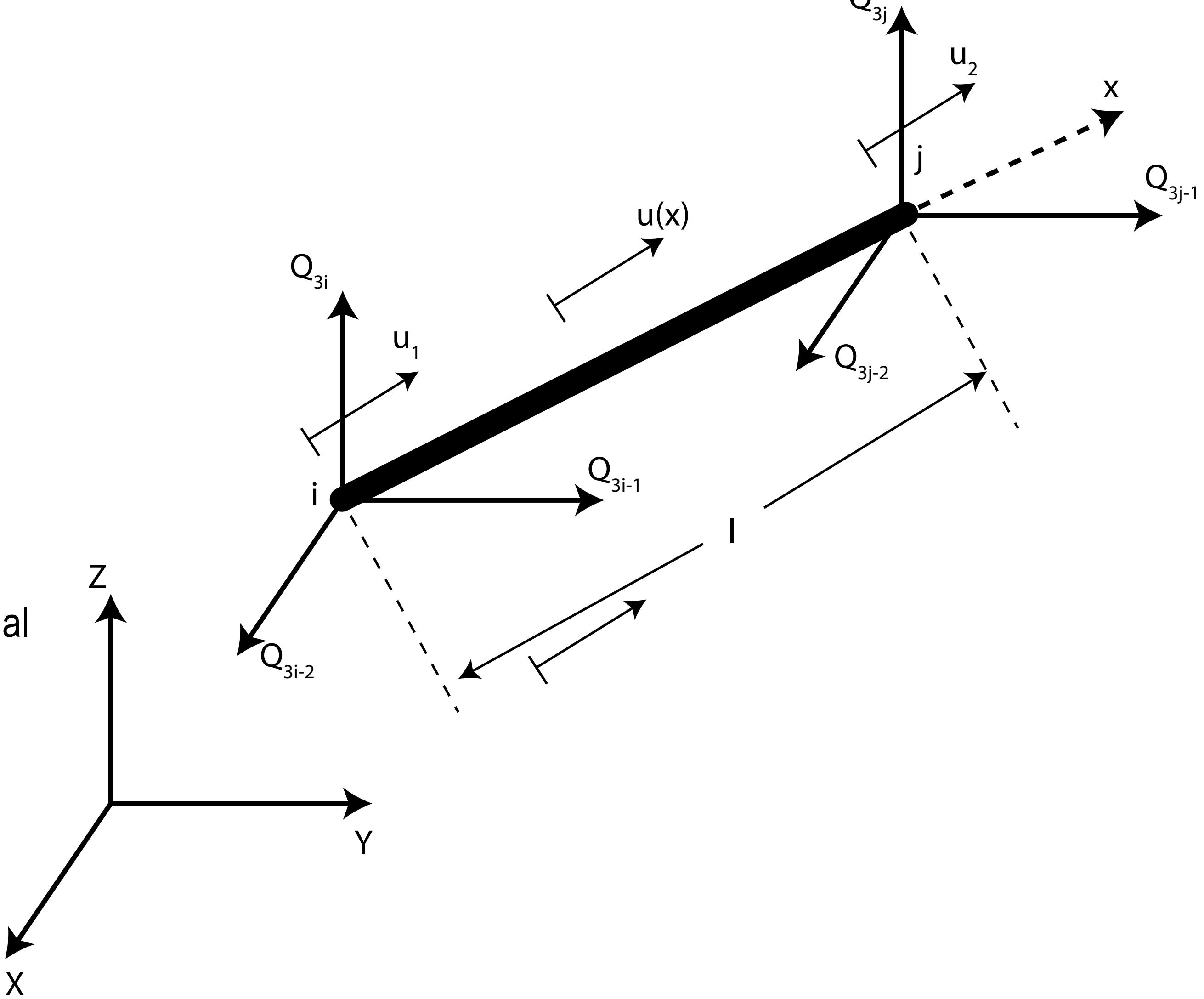
- basic solid mechanics
- knowledge from lecture 1
    - 1-D interpolation function
    - element stiffness matrix
    - in general, the finite element analysis procedure

3-D TRUSS ELEMENT

3-D TRUSS OR SPACE TRUSS

- each node has 3 displacement d.o.f
- local node 1 is global node i
- local node 2 is global node j

- Steps?
  1. displacement model
  2. strain and stress of the system
  3. stiffness matrix definition
  4. transformation from local to global
  5. load vector
  6. compute results



FINITE ELEMENT EQUATION FORMULATION

Steps 1 and 2

Linear displacement model

$$u(x) = u_1 + (u_2 - u_1)\frac{x}{l} \qquad \longrightarrow (1)$$

or,

$$\underset{1\,x\,1}{\{u(x)\}} = \underset{1\,x\,2}{[N]} \; \underset{2\,x\,1}{\vec{u}^e} \qquad \longrightarrow (2)$$

where

$$[N] = \left[\left(1 - \frac{x}{l}\right) \;\; \frac{x}{l}\right]$$

and,

$$\vec{u}^e = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Axial strain is written as

$$\epsilon_{xx} = \frac{\partial u(x)}{\partial x} = \frac{u_2 - u_1}{l} \qquad \longrightarrow (3)$$

or,

$$\underset{1\,x\,1}{\{\epsilon_{xx}\}} = \underset{1\,x\,2}{[B]} \; \underset{2\,x\,1}{\vec{u}^e} \qquad \longrightarrow (4)$$

where

$$[B] = \left[-\frac{1}{l} \;\; \frac{1}{l}\right]$$

Stress-strain relation is given by $$\sigma_{xx} = E\epsilon_{xx} \longrightarrow (5)$$

or, $$\underset{1\,x\,1}{\{\sigma_{xx}\}} = \underset{1\,x\,1}{[D]}\ \underset{1\,x\,1}{\{\epsilon_{xx}\}} \longrightarrow (6)$$

Step 3

Principal of minimum potential energy

potential energy $$I = \Pi - W_p \longrightarrow (A)$$

where, $$\Pi = \frac{1}{2}\iiint_V \vec{\epsilon}^T\vec{\sigma}\,dV \longrightarrow (B)$$

and with an initial strain, $$\Pi = \frac{1}{2}\iiint_V \vec{\epsilon}^T[D]\vec{\epsilon}\,dV - \iiint_V \vec{\epsilon}^T[D]\overrightarrow{\epsilon_0}\,dV \longrightarrow (C)$$

work done is given by $$W_p = \iiint_V \vec{f_b}^T\,\vec{U}\,dV + \iint_{S_1} \vec{f_S}^T\,\vec{U}\,dS_1 \longrightarrow (D)$$

Therefore,

$$I^e = \frac{1}{2}\iiint_V \vec{U}^{e^T}[B]^T[D][B]\vec{U}^e dV - \iiint_V \vec{U}^{e^T}[B]^T[D]\overrightarrow{\epsilon_0} dV - \iiint_V \vec{U}^{e^T}[N]^T\overrightarrow{f_b} dV - \iint_{S_1} \vec{U}^{e^T}[N]^T\vec{f}_S\, dS_1$$

→ (E)

or,

$$I^e = \frac{1}{2}\vec{U}^T\left[\sum_{e=1}^{E}\iiint_V [B]^T[D][B]dV\right]\vec{U} - \vec{U}^T\sum_{e=1}^{E}\left(\iiint_{V^e}[B]^T[D]\overrightarrow{\epsilon_0}dV + \iiint_{V^e}[N]^T\overrightarrow{f_b}dV + \iint_{S_1^e}[N]^T\vec{f}_S\, dS_1\right)$$

→ (F)

Minimising,

$$\frac{\partial I}{\partial \vec{U}} = \vec{0}$$

→ (G)

we get,

$$\left[\sum_{e=1}^{E}\iiint_V [B]^T[D][B]dv\right]\vec{U} = \sum_{e=1}^{E}\left(\iiint_{V^e}[B]^T[D]\overrightarrow{\epsilon_0}dV + \iiint_{V^e}[N]^T\overrightarrow{f_b}dV + \iint_{S_1^e}[N]^T\vec{f}_S\, dS_1\right)$$

→ (H)

Therefore,

$$[k^e] = \iiint\limits_{V^e} [B]^T [D][B] dV = A \int_{x=0}^{l} \begin{Bmatrix} -\frac{1}{l} \\ \frac{1}{l} \end{Bmatrix} E \left\{ -\frac{1}{l} \quad \frac{1}{l} \right\} dx$$

→ (7)

$$= \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

→ (8)

Step 4

Using transformation matrix λ

$$\vec{u}^e = [\lambda] \vec{U}^e$$

→ (9)

where, $$[\lambda] = \begin{bmatrix} l_{ij} & m_{ij} & n_{ij} & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{ij} & m_{ij} & n_{ij} \end{bmatrix}$$

→ (10)

and, $$\vec{U}^e = \begin{Bmatrix} Q_{3i-2} \\ Q_{3i-1} \\ Q_{3i} \\ Q_{3j-2} \\ Q_{3j-1} \\ Q_{3j} \end{Bmatrix}$$

and, $$l_{ij} = \frac{X_j - X_i}{l}$$

$$m_{ij} = \frac{Y_j - Y_i}{l}$$

$$n_{ij} = \frac{Z_j - Z_i}{l}$$

$$l = \left\{ (X_j - X_i)^2 + (Y_j - Y_i)^2 + (Z_j - Z_i)^2 \right\}^{\frac{1}{2}}$$

Therefore, stiffness matrix in global coordinates is given by

$$\underset{6\,x\,6}{[K^e]} = \underset{6\,x\,2}{[\lambda]^T}\ \underset{2\,x\,2}{[k^e]}\ \underset{2\,x\,6}{[\lambda]} \qquad \longrightarrow \text{(11)}$$

$$[K^e] = \frac{AE}{l}\begin{bmatrix} l_{ij}^2 & l_{ij}m_{ij} & l_{ij}n_{ij} & -l_{ij}^2 & -l_{ij}m_{ij} & -l_{ij}n_{ij} \\ l_{ij}m_{ij} & m_{ij}^2 & m_{ij}n_{ij} & -l_{ij}m_{ij} & -m_{ij}^2 & -m_{ij}n_{ij} \\ l_{ij}n_{ij} & m_{ij}n_{ij} & n_{ij}^2 & -l_{ij}n_{ij} & -m_{ij}n_{ij} & -n_{ij}^2 \\ -l_{ij}^2 & -l_{ij}m_{ij} & -l_{ij}n_{ij} & l_{ij}^2 & l_{ij}m_{ij} & l_{ij}n_{ij} \\ -l_{ij}m_{ij} & -m_{ij}^2 & -m_{ij}n_{ij} & l_{ij}m_{ij} & m_{ij}^2 & m_{ij}n_{ij} \\ -l_{ij}n_{ij} & -m_{ij}n_{ij} & -n_{ij}^2 & l_{ij}n_{ij} & m_{ij}n_{ij} & n_{ij}^2 \end{bmatrix} \qquad \longrightarrow \text{(12)}$$

Step 5

Load vector,

$$\vec{P}^e = [\lambda]^T\vec{p}^e \qquad \longrightarrow \text{(13)}$$

Step 6

Stress

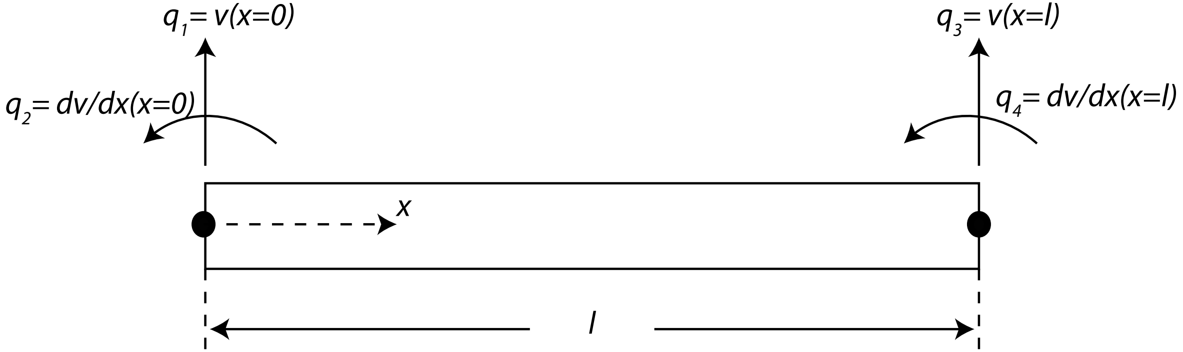
$$\sigma_{xx} = E[B][\lambda]\vec{U}^e \qquad \longrightarrow \text{(14)}$$

BEAM ELEMENT - SIMPLE AND 3D

SIMPLE BEAM ELEMENT

- straight bar
- transverse displacement
- rotation (slope)
- four unknown d.o.f

- Steps?
  1. displacement model
  2. strain and stress of the system
  3. stiffness matrix definition
  4. transformation from local to global
  5. load vector
  6. compute results



FINITE ELEMENT EQUATION FORMULATION

Steps 1 and 2

Cubic displacement model

$$v(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \qquad \longrightarrow \text{(i)}$$

or,

$$\underset{1x1}{v(x)} = \underset{1x4}{[N]} \quad \underset{4x1}{\vec{q}} \qquad \longrightarrow \text{(ii)}$$

where

$$\begin{aligned} v(x) &= q_1 \quad \text{and,} \quad \frac{dv}{dx}(x) = q_2 \quad at\ x = 0 \\ v(x) &= q_3 \quad \text{and,} \quad \frac{dv}{dx}(x) = q_4 \quad at\ x = l \end{aligned} \qquad \longrightarrow \text{(iii)}$$

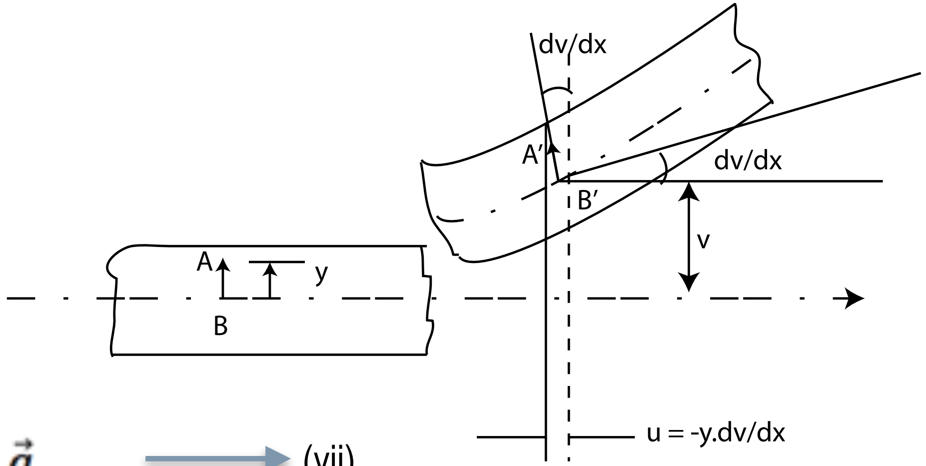
And,

$$[N] = [N_1 \quad N_2 \quad N_3 \quad N_4] \quad \text{and,} \quad \vec{q} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} \qquad \longrightarrow \text{(iv)}$$

where

$$\begin{aligned} N_1(x) &= (2x^3 - 3lx^2 + l^3)/l^3 \qquad & N_3(x) &= -(2x^3 - 3lx^2)/l^3 \\ N_2(x) &= (x^3 - 2lx^2 + l^2x)/l^2 \qquad & N_4(x) &= (x^3 - lx^2)/l^2 \end{aligned} \qquad \longrightarrow \text{(v)}$$

Simple beam theory states: “plane sections of the beam remain plane after deformation”



Therefore, axial displacement can be written as:

$$u = -y\frac{\partial v}{\partial x}$$

→ (vi)

And, strain can be written as:

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = -y\frac{\partial^2 v}{\partial x^2} = [B]\,\vec{q}$$

→ (vii)

where,

$$[B] = -\frac{y}{l^3}\{(12x-6l) \quad l(6x-4l) \quad -(12x-6l) \quad l(6x-2l)\}$$

→ (viii)

So, the stiffness matrix can be written as:

$$[k^e] = \iiint_{V^e} [B]^T[D][B]dV = E\int_0^l dx \iint_A [B]^T[B]dA$$

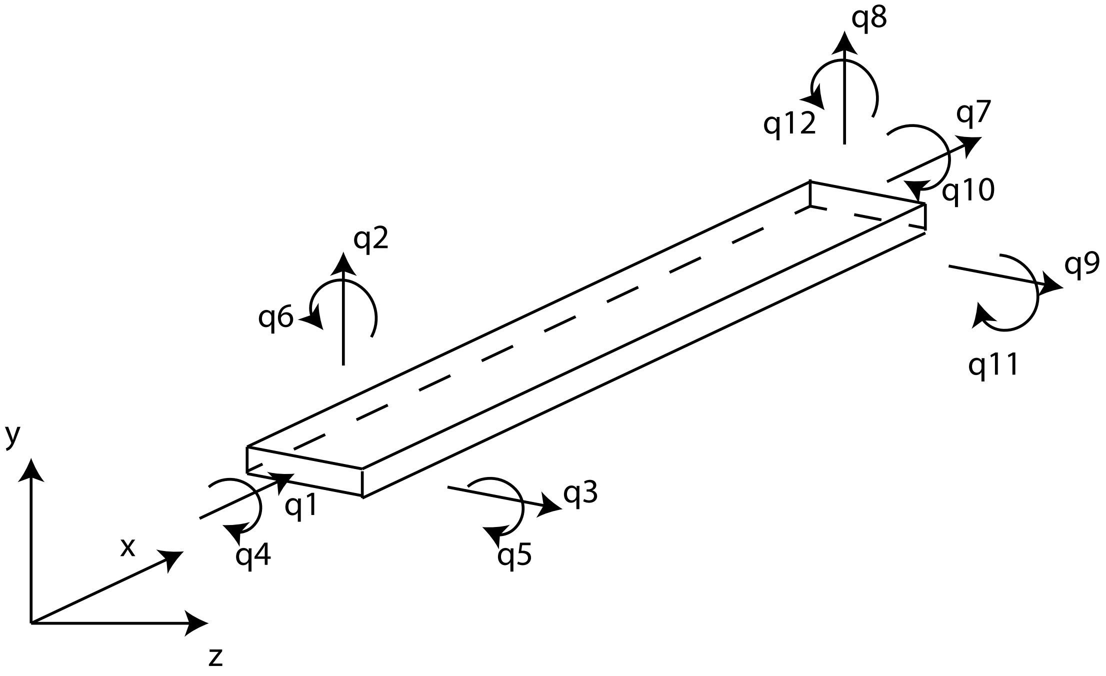
→ (ix)

$$= \frac{EI_{xx}}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

→ (x)

EXTENSION - SPACE FRAME ELEMENT

- resists
    - axial forces
    - bending moments
    - twisting moment
- 12 x 12 stiffness matrix!
    - constructed using known stiffness matrices
- theoretically
    - axial disp. depend on axial loads
    - torsional disp. depends on torsional loads
    - bending disp. in one plane depends on bending moments **if and only if** t*he planes coincide with the principle axes of cross-section*



CONSTRUCTION OF THE STIFFNESS MATRIX

Separate the displacements into four groups

Group 1: Axial displacements

$$[k_a^e] = \iiint_{V^e} [B]^T[D][B]dV = \frac{AE}{l}\begin{array}{cc|c} q_1 & q_7 & \\ 1 & -1 & q_1 \\ -1 & 1 & q_7 \end{array} \longrightarrow \text{(a)}$$

Group 2: Torsional displacements

Twist angle with a linear variation is given by

$$\theta(x) = [N]\vec{q_t} \longrightarrow \text{(b)}$$

where $[N] = \left[\left(1-\frac{x}{l}\right) \quad \frac{x}{l}\right]$ and $\vec{q_t} = \begin{Bmatrix} q_4 \\ q_{10} \end{Bmatrix}$

For a circular cross-section frame $\epsilon_{\theta x} = r\frac{d\theta}{dx}$ where r is the distance of the fiber from the centroidal axis → (c)

We can write: $\vec{\epsilon} = [B]\vec{q_t}$ where $\vec{\epsilon} = \{\epsilon_{\theta x}\}$ and $[B] = \left[-\frac{r}{l} \quad \frac{r}{l}\right]$ → (d)

From Hooke's law: $\vec{\sigma} = [D]\vec{\epsilon}$ where, $\vec{\sigma} = \{\sigma_{\theta x}\}$ and $[D] = [G]$ $\longrightarrow$ (e)

So, the stiffness matrix can be written as:

$$[k_t^e] = \iiint_{V^e} [B]^T[D][B]dV = G\int_0^l dx \iint_A r^2 dA \begin{Bmatrix} -\frac{1}{l} \\ \frac{1}{l} \end{Bmatrix} \begin{Bmatrix} -\frac{1}{l} & \frac{1}{l} \end{Bmatrix} \longrightarrow \text{(f)}$$

$$= \frac{GJ}{l} \begin{array}{c} \\ \\ \end{array} \begin{array}{cc|c} q_4 & q_{10} & \\ 1 & -1 & q_4 \\ -1 & 1 & q_{10} \end{array} \longrightarrow \text{(g)}$$

Group 3: Bending displacements (in xy)

$$[k_{xy}^e] = \frac{EI_{zz}}{l^3} \begin{array}{ccccc} q_2 & q_6 & q_8 & q_{12} & \\ 12 & 6l & -12 & 6l & q_2 \\ 6l & 4l^2 & -6l & 2l^2 & q_6 \\ -12 & -6l & 12 & -6l & q_8 \\ 6l & 2l^2 & -6l & 4l^2 & q_{12} \end{array} \longrightarrow \text{(h)}$$

Group 4: Bending displacements (in xz)

$$[k_{xz}^e] = \frac{EI_{yy}}{l^3} \begin{array}{ccccc} q_3 & q_5 & q_9 & q_{11} & \\ 12 & 6l & -12 & 6l & q_3 \\ 6l & 4l^2 & -6l & 2l^2 & q_5 \\ -12 & -6l & 12 & -6l & q_9 \\ 6l & 2l^2 & -6l & 4l^2 & q_{11} \end{array} \longrightarrow \text{(i)}$$

Putting (a), (g), (h), and (i) together for assembly, you will arrive at a 12x12 symmetric stiffness matrix

This stiffness matrix is the total element stiffness matrix in the local coordinate system!

A transformation matrix must be used to arrive at the global stiffness matrix, such that

$$\vec{q} = [\lambda]\vec{Q} \qquad \longrightarrow \text{(j)}$$

where

$$[\lambda] = \begin{bmatrix} [\underline{\lambda}] & [0] & [0] & [0] \\ [0] & [\underline{\lambda}] & [0] & [0] \\ [0] & [0] & [\underline{\lambda}] & [0] \\ [0] & [0] & [0] & [\underline{\lambda}] \end{bmatrix} \qquad \longrightarrow \text{(k)}$$

and

$$[\underline{\lambda}] = \begin{bmatrix} l_{ox} & m_{ox} & n_{ox} \\ l_{oy} & m_{oy} & n_{oy} \\ l_{oz} & m_{oz} & n_{oz} \end{bmatrix} \qquad \longrightarrow \text{(l)}$$

EXTENSION - OTHER SPECIAL CASES

- Planar frame element
    - 2 translation d.o.f
    - 1 in-plane rotation d.o.f
    - what prior stiffness matrices can be used?
    - how does the **transformation matrix** change?

TRANSFORMATION MATRIX

2-STEP PROCESS

- Assumptions
  - x,y,z are local coordinates
  - x’,y’,z’ are intermediate local coordinates
  - X,Y,Z are global coordinates
- Step 1
  - transform x’,y’,z’ to X,Y,Z
  - z’ is parallel to XZ plane
  - y,z coincide with y’,z’
- Step 2
  - transform x,y,z to x’,y’,z’
  - system is rotated about the x’ axis by an angle

HOMEWORK

- Check blackboard for practice problems on bar, beam and truss systems
- Follow mini video on transformation matrix derivation
- Answer Self-Check questions and discuss on the forum
- Homework assignment 2 will be released today. Start working on it!

NEXT WEEK…

- 2D interpolation models
- Iso-parametric elements
- Triangular elements