LINEAR MODELLING (INCL. FEM)
AE4ASM003
P1-2015

LECTURE 5
29.09.2015

TUDelft

TODAY…

- Isoparametric formulation
- Constant strain triangular elements
- Bending elements

ISOPARAMETRIC FORMULATION

FORMULATION PROPERTIES

- same interpolation functions for geometry and displacement
- tedious/confusing!
- simpler for computer program formulation
- extended application to all element types
- idealisation of elements with curved boundaries

NATURAL OR GENERALISED COORDINATE SYSTEM

Let us assume a polynomial for the geometry

$$x = a_0 + a_1 r \qquad (1)$$

$$at\ x = x1, \quad r = -1$$

$$at\ x = x2, \quad r = +1 \qquad (2)$$

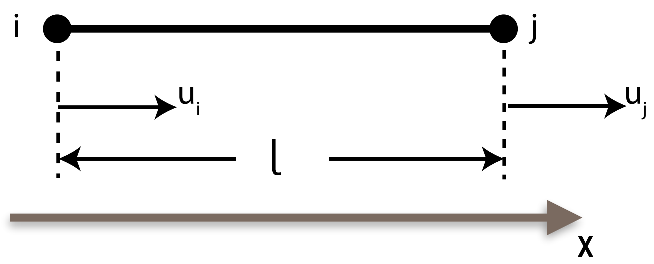
Solving and substituting back into x,

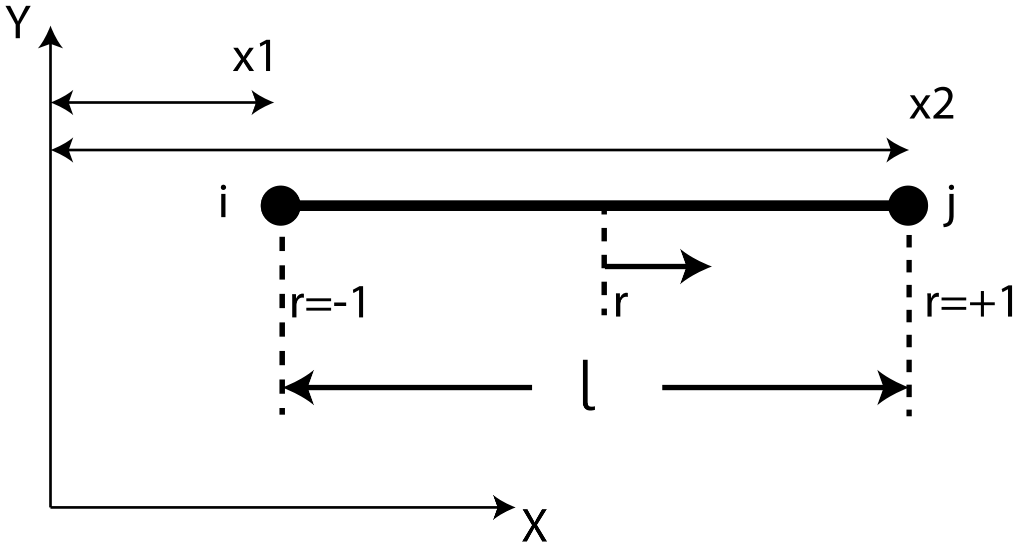
$$x = \frac{x1 + x2}{2} + \frac{x2 - x1}{2}.r \qquad (3)$$

$$= \frac{1 - r}{2}.x1 + \frac{1 + r}{2}.x2 \qquad (4)$$

Therefore,

$$x = N_1 x1 + N_2 x2 \quad \text{where} \quad N_1 = \frac{1 - r}{2}; \quad N_2 = \frac{1 + r}{2}$$

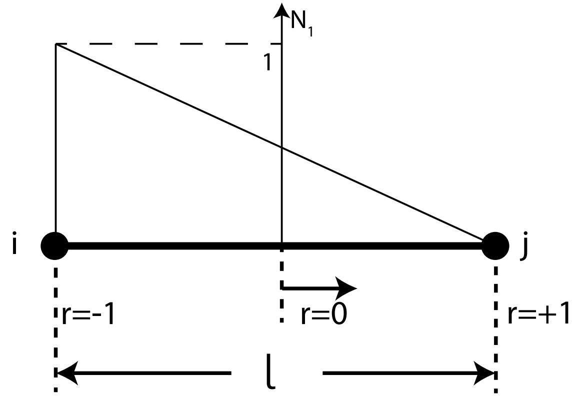




r is the natural coordinate system

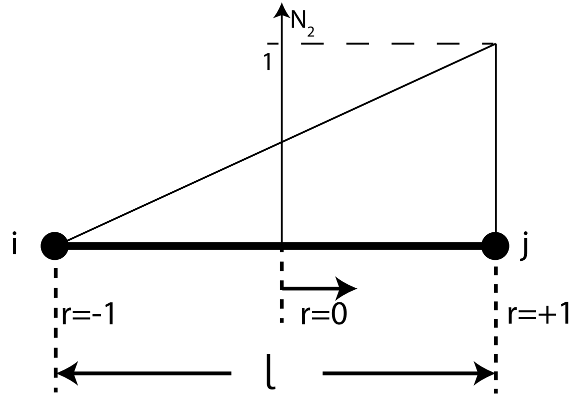
This natural coordinate is attached to the bar no matter how it is oriented or placed in space

Therefore, a transformation to the global coordinate system must be made - Transformation mapping!



So, lets carry out the steps of the finite element analysis using this formulation:

Step 1: interpolation model



Step 2: strain and stress of the system

Step 3: stiffness matrix definition

Step 4: transformation

Step 5: load vector

Step 6: results

Step 2: strain and stress of the system

$$x = N_1 x_1 + N_2 x_2$$
$$u = N_1 u_1 + N_2 u_2 \quad \rightarrow (5)$$

Strain in local coordinate

$$\epsilon_r = \frac{du}{dr} = \frac{du}{dx} . \frac{dx}{dr} \quad \rightarrow (6)$$

Strain in global coordinate

$$\epsilon_x = \frac{du}{dx} = \frac{du}{dr} / \frac{dx}{dr} \quad \rightarrow (7)$$

where $\frac{du}{dr} = \frac{u_2 - u_1}{2}$ and $\frac{dx}{dr} = \frac{x_2 - x_1}{2} = \frac{l}{2}$ → (8)

Substituting in (7),

$$\epsilon_x = \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad \rightarrow (9)$$

Therefore, by definition, $[B] = \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix}$ and, stress $\sigma = E\,[B]\{u\}$ → (10)

In this case, the matrix B is simple because of the choice of the element.

In general, for higher order elements, B will be a function of the natural coordinate.

We will notice this difference for other elements such as higher order beams, triangular or quadrilateral elements.

Step 3: stiffness matrix definition

We know that

$$k = \int_0^l [B]^T E [B] A.\,dx$$ → (11)

Step 4: transformation

From the reasoning above, B is generally a function of r, and therefore, the coordinate must be transformed for integration

$$\int_0^l f(x).\,dx = \int_{-1}^{1} f(r).\,|[J]|\,dr$$ where $|[J]| = J$ for a 1D case → (12)

Jacobian

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So, for the current assumed problem

$$J = \frac{dx}{dr} = \frac{l}{2} \quad \longrightarrow \quad (13)$$

Substituting it back into (11)

$$k = \frac{l}{2}\int_{-1}^{1}[B]^T E[B]A.dr$$

$$= \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \longrightarrow \quad (14)$$

*** For higher order elements, this integration is near-impossible so some numerical integration techniques are used

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Step 5: load vector

Body forces

$$\{f_b\} = \int_V [N]^T \{X_b\} dV$$

$$= \int_x [N]^T \{X_b\} A dx \qquad \rightarrow (15)$$

$$= \frac{ALX_b}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Surface forces

$$\{f_s\} = \int_S [N_s]^T \{T_x\} dS$$

$$= \int_x [N_s]^T \{T_x\} dx \qquad \rightarrow (16)$$

$$= \{T_x\} \frac{L}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

PLATE ANALYSIS

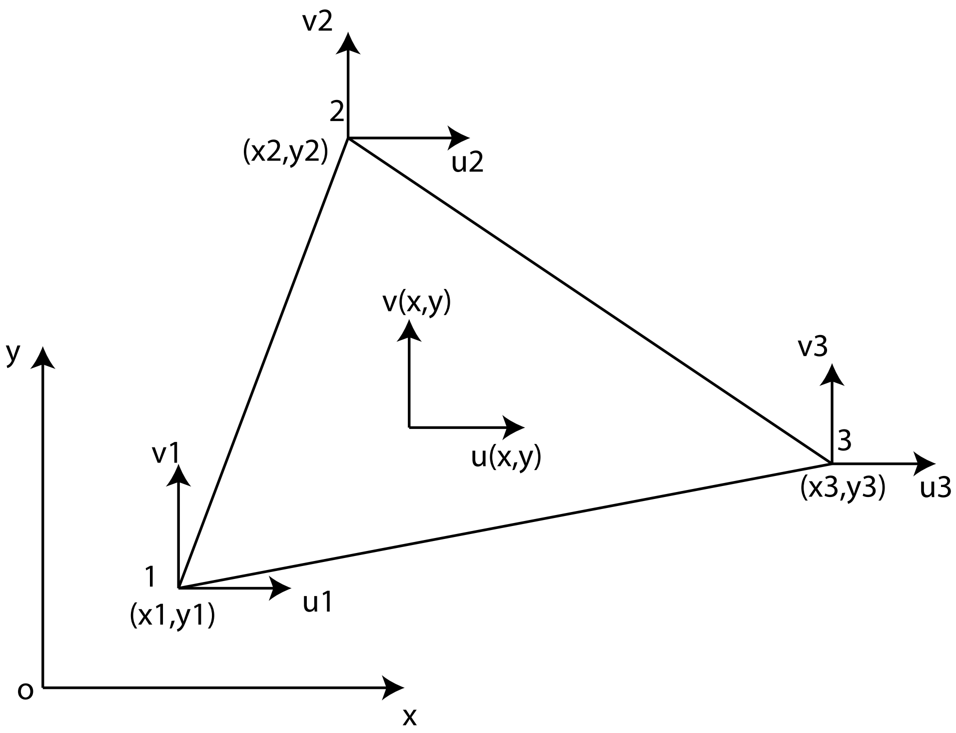
TYPES OF PLATE ELEMENTS

- Any plate under in-plane and transverse loads will deform in 3 spatial coordinates
- Assuming the linear small deflection theory, uncoupling occurs
  - in-plane (membrane) forces
  - transverse (bending) forces

MEMBRANE ELEMENTS
- CONSTANT STRAIN TRIANGLES

The same process has to be followed:

- Step 1: interpolation model
- Step 2: strain and stress of the system
- Step 3: stiffness matrix definition
- Step 4: transformation
- Step 5: load vector
- Step 6: results



Step 1: interpolation model

$$u(x, y) = a_1 + a_2 x + a_3 y$$

$$v(x, y) = a_4 + a_5 x + a_6 y$$

→ (i)

where

$$u(x,y) = u1 \quad \& \quad v(x,y) = v1 \quad at\ (x1, y1)$$

$$u(x,y) = u2 \quad \& \quad v(x,y) = v2 \quad at\ (x2, y2)$$

$$u(x,y) = u3 \quad \& \quad v(x,y) = v3 \quad at\ (x3, y3)$$

→ (ii)

Solving for the constants and substituting and rearranging,

$$\vec{U} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = [N]\vec{u}^e \quad \text{(iii)}$$

where

$$[N(x,y)] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

and,

$$N_1 = \frac{1}{2A}[y_{32}(x - x2) - x_{32}(y - y2)$$

$$N_2 = \frac{1}{2A}[-y_{31}(x - x3) + x_{31}(y - y3)$$

$$N_3 = \frac{1}{2A}[y_{21}(x - x1) - x_{21}(y - y1)$$

$$A = \frac{1}{2}(x_{32}y_{21} - x_{21}y_{32}) = \textit{area of triangle } 123$$

$$x_{ij} = xi - xj$$

$$y_{ij} = yi - yj$$

$$\vec{u}^e = \begin{Bmatrix} u1 \\ v1 \\ u2 \\ v2 \\ u3 \\ v3 \end{Bmatrix}^e \quad \text{(iv)}$$

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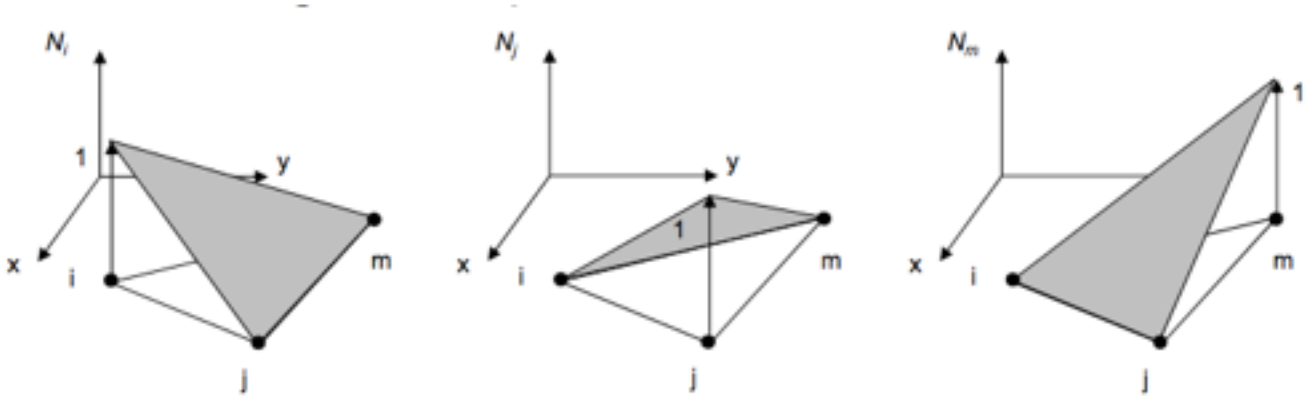
Some requirements for the displacement model

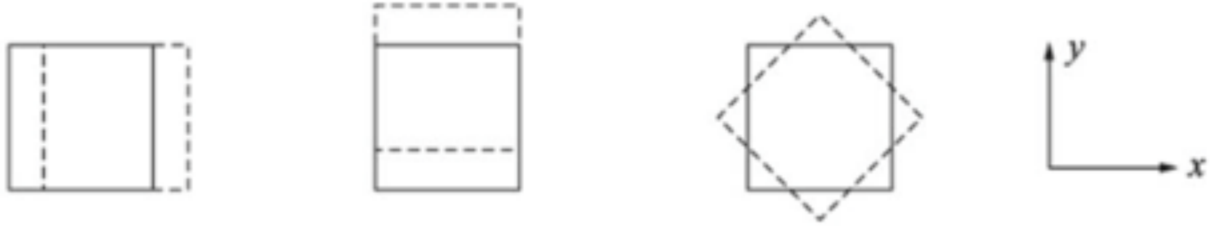


u, v should yield constant value for a rigid body displacement

Therefore,

$$N_1 + N_2 + N_3 = 1$$

→ (v)



for all x, y locations

From solid mechanics,

$$\vec{\epsilon} = \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial x \\ \partial u/\partial y \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \qquad \text{(vi)}$$

which can be written in terms of nodal displacements as,

$$\vec{\epsilon} = [B]\vec{u}^e$$

where $[B] = \dfrac{1}{2A}\begin{bmatrix} y_{32} & 0 & -y_{31} & 0 & y_{21} & 0 \\ 0 & -x_{32} & 0 & x_{31} & 0 & -x_{21} \\ -x_{32} & y_{32} & x_{31} & -y_{31} & -x_{21} & y_{21} \end{bmatrix}$ (vii)

For a plane stress condition,

$$\vec{\sigma} = [D]\vec{\epsilon}$$

where $\vec{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}$ and, $[D] = \dfrac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$ (viii)

Stiffness matrix

$$[k^e] = \iiint_V [B]^T [D][B] dV$$

Since in this case, the matrices B and D are constants and are independent of spatial coordinates, the thickness can be extracted to give,

$$[k^e] = [B]^T [D][B] t \iint_A dA = tA[B]^T [D][B] \quad \longrightarrow \text{(ix)}$$

Transformation?

In-plane

$$[\lambda] = \begin{bmatrix} \underline{\lambda} & \underline{0} & \underline{0} \\ \underline{0} & \underline{\lambda} & \underline{0} \\ \underline{0} & \underline{0} & \underline{\lambda} \end{bmatrix} \quad \longrightarrow \text{(x)}$$

Stiffness matrix, load vector, displacements!

where $\underline{\lambda} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$ and, $\underline{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

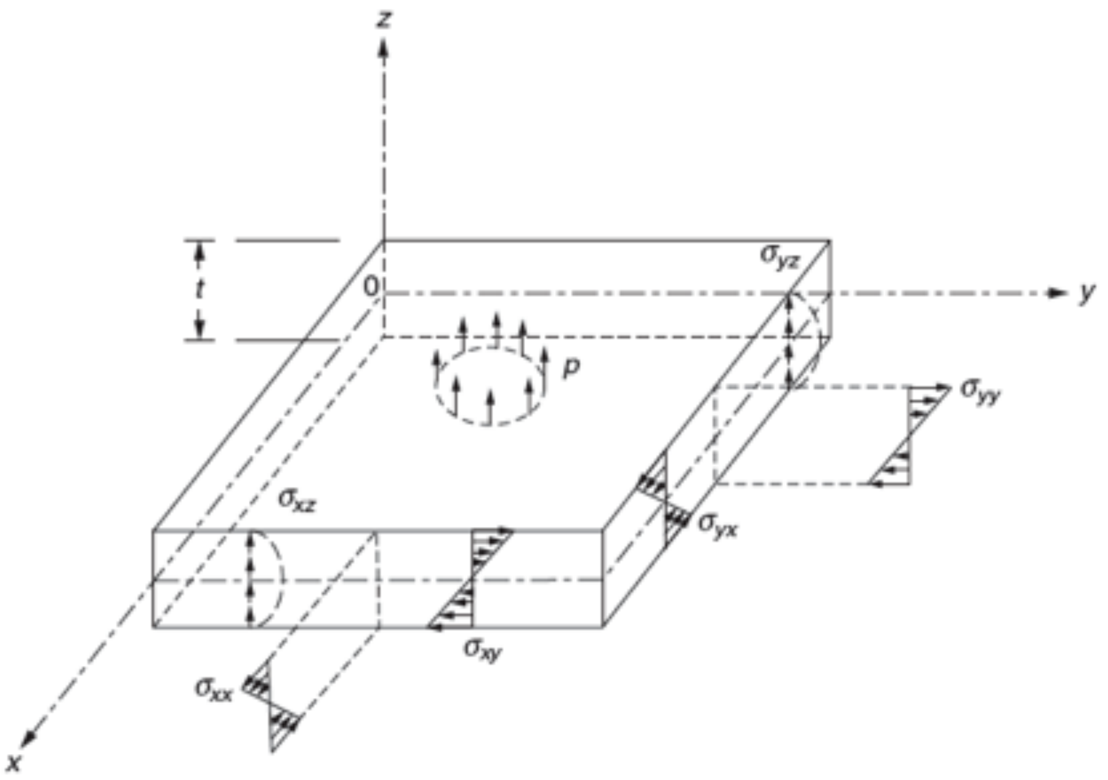
CHARACTERISTICS

- displacement model should be continuous
- why is this a constant strain triangle?

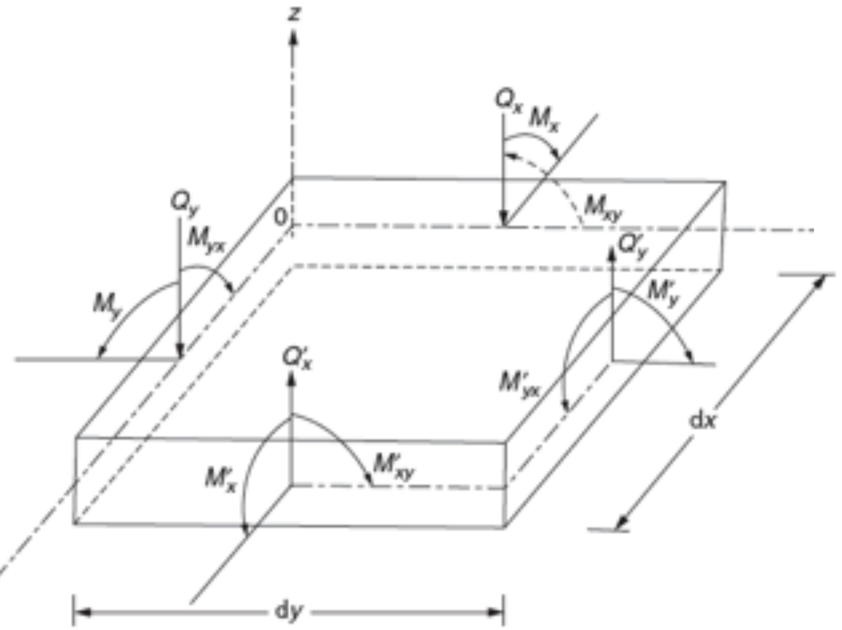
PLATE BENDING

PLATE BENDING BEHAVIOUR

- thickness is small as compared to other dimensions
- deflections are small
- mid-plane does not undergo in-plane deformation
- transverse shear deformation is zero



(a) Stresses in a plate



(b) Forces and moments in a plate

FINITE ELEMENT ANALYSIS OF PLATE BENDING

- special elements - transverse deflection of middle surface only
- other elements - include transverse shear

- displacement model
  - ‘w’ should be continuous
  - derivative should be continuous
  - for convergence, assumed displacement model will have constant curvature states

TRIANGULAR PLATE BENDING ELEMENT

9 displacement degrees of freedom

Displacement model:

$$w(x,y) = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_yx^3 + a_8(x^2y + xy^2) + a_9y^3$$

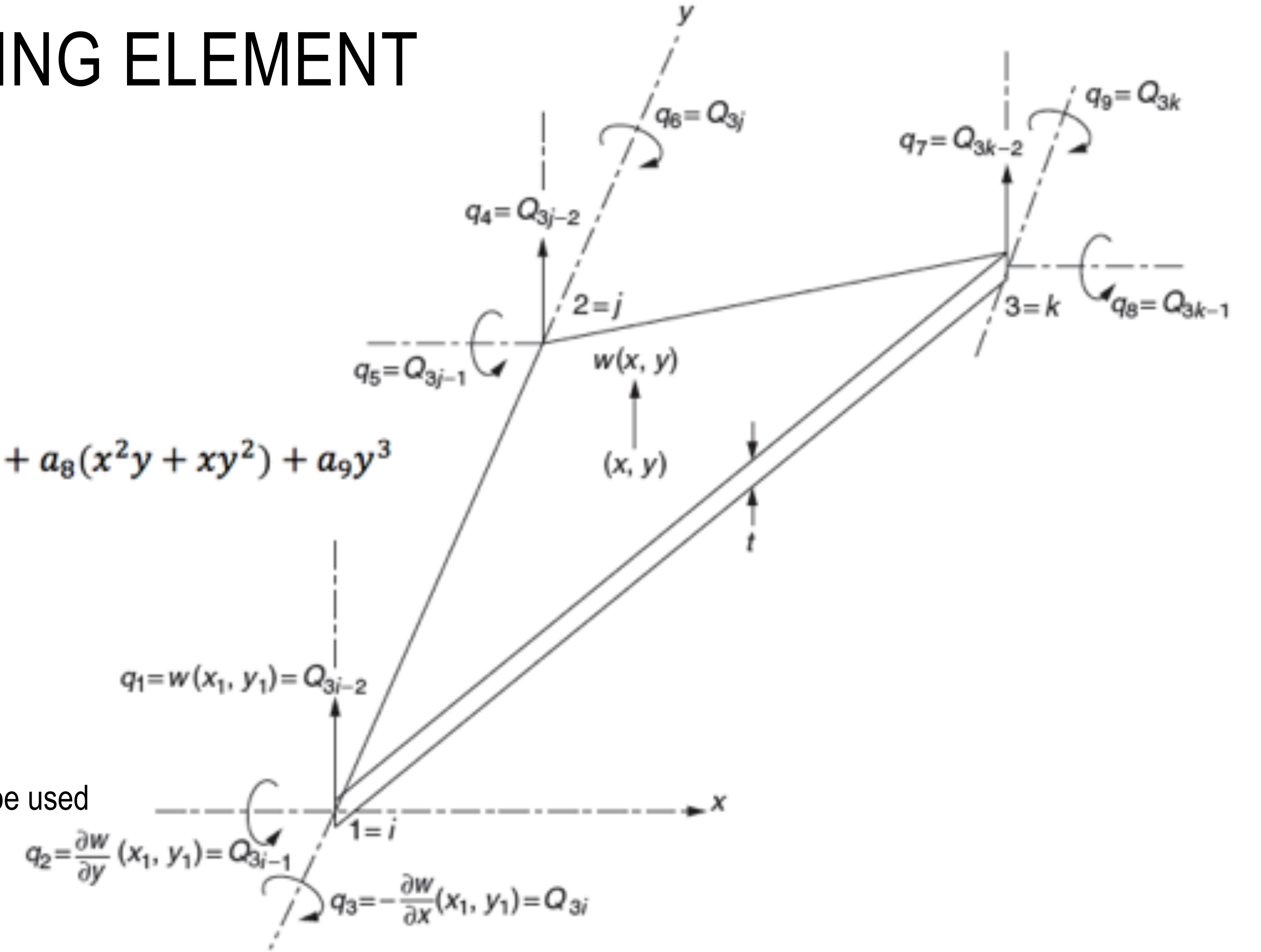
The same process is followed

Differences?

B matrix is a function of x, y and z!

thickness t cannot be assumed to be constant, so integral must be used

Transformation matrix will be 9x9



HOMEWORK

- Check blackboard for practice problems on membrane and bending plates
- Answer Self-Check questions and discuss on the forum
- Continue working on Homework assignment 2.

NEXT WEEK…

- Post processing
- Convergence
- Strain energy error