LINEAR MODELLING (INCL. FEM)
AE4ASM003
P1-2015

LECTURE 6
06.10.2015

TU Delft

TODAY…

- Post-processing
- Convergence
- Strain energy error

NUMERICAL INTEGRATION BY GAUSS RULES

GAUSS INTEGRATION

Partial evaluation of integrals over isoparametric elements

- *gauss integration*
- *minimal number of sample points*
- *high level of accuracy*
- *higher computational efficiency*

GAUSS INTEGRATION

One-dimensional rule

$$\int_{-1}^{1} F(\xi)\, d\xi \approx \sum_{i=1}^{p} w_i F(\xi_i)$$

- *w are the integration weights*
- *p is the number of gauss points*
- *transformation can be easily done using the Jacobian*

Two-dimensional rule

$$\int_{-1}^{1}\int_{-1}^{1} F(\xi,\eta)\, d\xi\, d\eta = \int_{-1}^{1} d\eta \int_{-1}^{1} F(\xi,\eta)\, d\xi$$

each integral is processed numerically to give:

$$\int_{-1}^{1}\int_{-1}^{1} F(\xi,\eta)\, d\xi\, d\eta = \int_{-1}^{1} d\eta \int_{-1}^{1} F(\xi,\eta)\, d\xi \approx \sum_{i=1}^{p_1}\sum_{j=1}^{p_2} w_i w_j F(\xi_i,\eta_j)$$

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Stiffness matrix definition

$$\mathbf{K}^e = \int_{\Omega^e} h\,\mathbf{B}^T\mathbf{E}\mathbf{B}\,d\Omega^e$$

We can also represent it as

$$\mathbf{K}^e = \int_{-1}^{1}\int_{-1}^{1} \mathbf{F}(\xi,\eta)\,d\xi\,d\eta.$$

where $\mathbf{F}(\xi,\eta) = h\,\mathbf{B}^T\mathbf{E}\mathbf{B}\,\det\mathbf{J}.$

This can be easily solved using a Gauss rule!

POST PROCESSING

POST PROCESSING

Displacement - direct results

$$\mathbf{Ku} = \mathbf{f}$$

- *processing!*
- *calculated at nodes*
- *high accuracy*

Strain and Stresses - derived results

$$\mathbf{e} = \mathbf{Bu}^{(e)}$$

$$\sigma = \mathbf{Ee} = \mathbf{EBu}$$

- *post processing!*
- *calculated at nodes or gauss points*
- *lower accuracy than direct results*

STRESS CALCULATION

- at *element* nodal points
  - corners
  - mid-points
- stresses do not have to be continuous across elements
  - nodal points shared between two elements
  - not the same stress
- so, stresses need to be averaged!
  - these are nodal point stresses

STRESS CALCULATION

- two ways of calculating average nodal point stresses
  - substitution of natural coordinates into strain, stress relations
  - stress evaluation at gauss integration points
    - ✤ element stiffness integration rule
    - ✤ extrapolate to the element node points
- second method is better…

STRESS CALCULATION

- direct stress evaluation at nodes
- extrapolation of gauss points

INTERELEMENT AVERAGING

- unweighted
- weighted

CONVERGENCE

CONVERGENCE REQUIREMENTS

- Completeness
  - approximation power
- Compatibility
  - displacement continuity
- Stability
  - non physical zero energy modes
  - no excessive distortion
  - …

DISCUSSION WITH EXAMPLE

- Beam problem from the practical assignment
- Whats important?
    - location of nodes
    - type of element
    - number of elements
    - directional mesh biasing
    - …

STRAIN ENERGY ERROR

DISCUSSION WITH EXAMPLE

- Beam problem from the practical assignment
- Whats important?
  - location of nodes
  - type of element
  - number of elements
  - directional mesh biasing
  - …

PROOF OF CONVERGENCE

- measurement of quality
  - comparison between fem and exact solution
  - discretisation error reduced to minimum

exact strain energy of the body

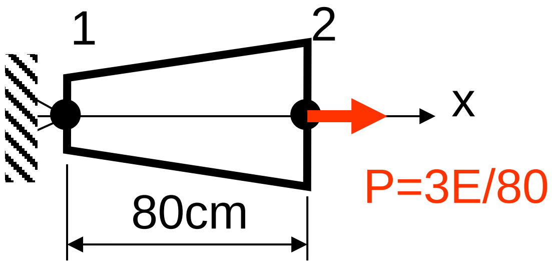
$$U = \frac{1}{2}\int_V \underline{\sigma}^T \underline{\varepsilon}\, dV$$

fe strain energy of the body (with element size h)

$$U_h = \frac{1}{2}\int_V \underline{\sigma}_h^T \underline{\varepsilon}_h\, dV$$

EXAMPLE

- Linear elastic bar



variable area

$$A(x) = \left(1 + \frac{x}{40}\right)^2 \ cm^2$$

Boundary conditions

$$u(x=0) = 0$$

$$EA\left.\frac{du}{dx}\right|_{x=80cm} = P = \frac{3E}{80}$$

The governing differential (equilibrium) equation

$$E\frac{d}{dx}\left(A(x)\frac{du}{dx}\right) = 0 \quad for \quad x \in (0,80)$$

Analytical solution

$$u^{exact}(x) = \frac{3}{2}\left(1 - \frac{1}{1+\frac{x}{40}}\right)$$

Exact strain energy

$$U = \frac{1}{2}\int_{x=0}^{80} \sigma\varepsilon \; Adx = \frac{1}{2}\int_{x=0}^{80} EA\left(\frac{du^{exact}(x)}{dx}\right)^2 dx = \frac{3E}{160} = \frac{39E}{2080}$$

If we discretize the problem using a single linear finite element, the stiffness matrix is

$$\underline{K} = \frac{E\int_{x=0}^{80} A(x)dx}{80^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{13E}{240}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The strain energy of the FE system is

$$U_h = \frac{1}{2}\int_{x=0}^{80} \sigma_h \varepsilon_h A dx = \frac{1}{2}\underline{d}^T \underline{K}\,\underline{d} = \frac{27E}{2080}$$

where $\underline{d}^T = \begin{bmatrix} 0 & 9/13 \end{bmatrix}$

convergence in strain energy

$$U \rightarrow U_h \quad as \quad h \rightarrow 0$$

convergence in displacement

$$\left\| \underline{u} - \underline{u}_h \right\|_0 \equiv \sqrt{\int_V \left[\left(u - u_h \right)^2 + \left(v - v_h \right)^2 \right] dV} \rightarrow 0 \; as \; h \rightarrow 0$$

convergence rate

- measure of discretization error tending to zero
- dependent on the order of polynomial assumed as displacement model

HOMEWORK

- Check blackboard for practice problems
- Start working on Homework assignment 3.

NEXT WEEK…

- Quadrilateral elements