LINEAR MODELLING (INCL. FEM)
AE4ASM003
P1-2015

LECTURE 7
13.10.2015

TODAY…

- Quadrilateral Elements
  - Bilinear
  - Quadratic
  - Isoparametric formulation

QUAD ELEMENTS

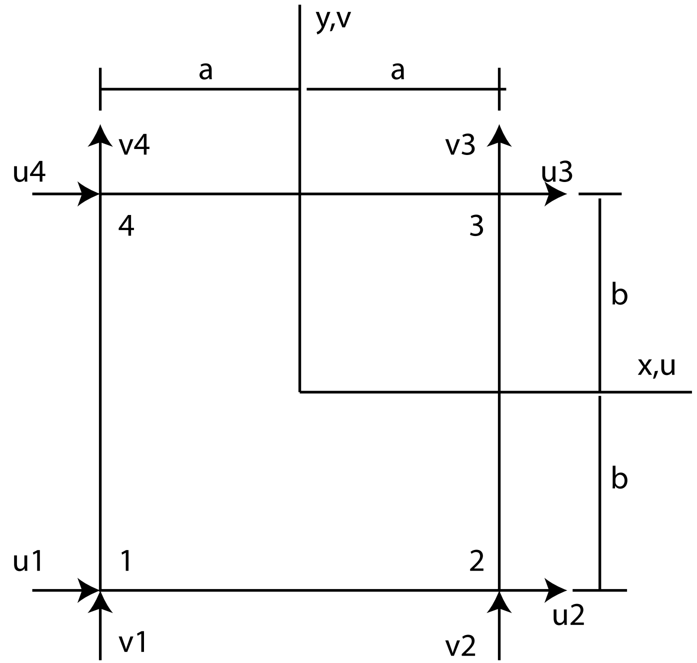
BI-LINEAR QUAD ELEMENTS

Also called, Q4 element, has four nodes, and eight nodal d.o.f

The linear displacement model is given as:

$$u = a_1 + a_2 x + a_3 y + a_4 xy$$
$$v = a_5 + a_6 x + a_7 y + a_8 xy \qquad (1)$$

Bilinearity is due to the displacement function being a product of two linear functions!!



The elemental strain field can be found out to be:

$$\epsilon_x = a_2 + a_4 y$$
$$\epsilon_y = a_7 + a_8 x \qquad (2)$$
$$\gamma_{xy} = (a_3 + a_6) + a_4 x + a_8 y$$

*** Lagrange's interpolation formula**

uses only ordinates in fitting a curve

shape function is given as

$$N_1 = \frac{(x_2 - x)(x_3 - x)\ldots(x_N - x)}{(x_2 - x_1)(x_3 - x_1)\ldots(x_N - x_1)} \qquad \text{or,} \qquad N_2 = \frac{(x_1 - x)(x_3 - x)\ldots(x_N - x)}{(x_1 - x_1)(x_3 - x_1)\ldots(x_N - x_1)}$$

And in general,

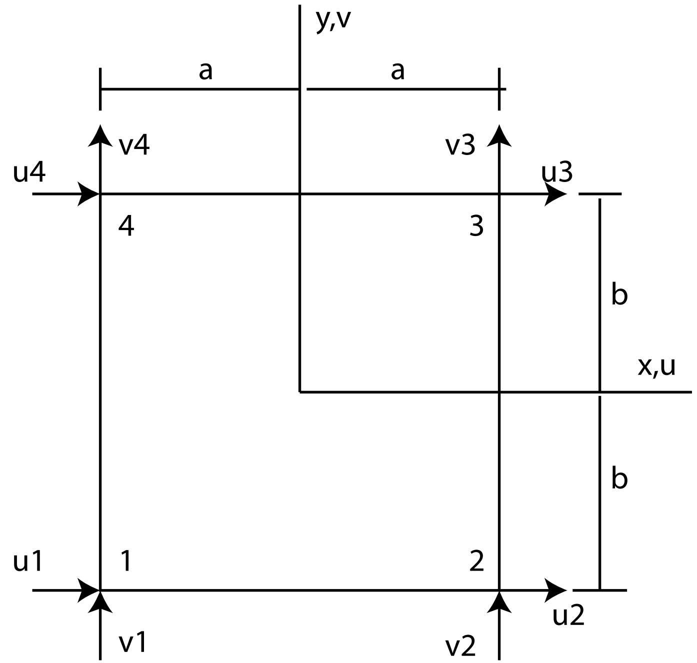
$$N_k = \frac{(x_1 - x)(x_2 - x)\ldots[x_k - x]\ldots(x_N - x)}{(x_2 - x_k)(x_3 - x_k)\ldots[x_k - x_k]\ldots(x_N - x_k)} \longrightarrow (3)$$

* terms in [ ] are omitted

Using the Lagrange interpolation formula, lets interpolate linearly along the top and bottom side along x

$$u_{12} = \frac{a-x}{2a}u_1 + \frac{a+x}{2a}u_2 \quad \longrightarrow (4)$$

$$u_{43} = \frac{a-x}{2a}u_4 + \frac{a+x}{2a}u_3 \quad \longrightarrow (5)$$



and, now in y between the above two displacements

$$u = \frac{b-y}{2b}u_{12} + \frac{b+y}{2b}u_{43} \quad \longrightarrow (6)$$

Substituting 4 and 5 into 6, we can arrive at:

$$u = \sum N_i u_i \quad \longrightarrow (7)$$

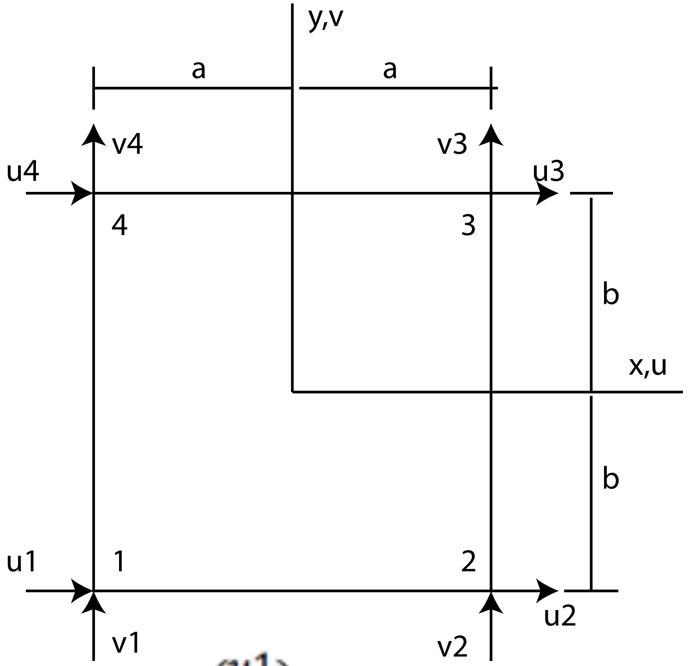
where,

$$N_1 = \frac{(a-x)(b-y)}{4ab} \; ; \qquad N_2 = \frac{(a+x)(b-y)}{4ab} \; ;$$

$$N_3 = \frac{(a+x)(b+y)}{4ab} \; ; \text{ and, } \qquad N_4 = \frac{(a-x)(b+y)}{4ab}$$

as we know, the complete element displacement field is represented as

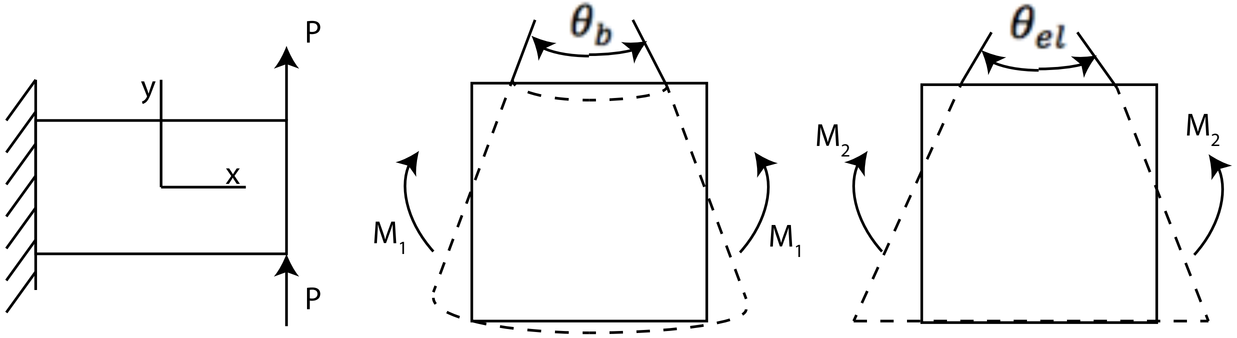
$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u1 \\ v1 \\ u2 \\ v2 \\ u3 \\ v3 \\ u4 \\ v4 \end{Bmatrix} \tag{8}$$



and, elemental strains in terms of nodal d.o.f. are:

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \underbrace{\frac{1}{4ab} \begin{bmatrix} -(b-y) & 0 & (b-y) & 0 & (b+y) & 0 & -(b+y) & 0 \\ 0 & -(a-x) & 0 & -(a+x) & 0 & (a+x) & 0 & (a-x) \\ -(a-x) & -(b-y) & -(a+x) & (b-y) & (a+x) & (b+y) & (a-x) & -(b+y) \end{bmatrix}}_{\textbf{[B]}} \begin{Bmatrix} u1 \\ v1 \\ u2 \\ v2 \\ u3 \\ v3 \\ u4 \\ v4 \end{Bmatrix} \tag{9}$$

What are the **limitations** of a bilinear quad?



In pure bending, a block of material has strains given as

$$\epsilon_x = -\frac{\theta_b y}{2a}; \quad \epsilon_y = \nu\frac{\theta_b y}{2a}; \quad \gamma_{xy} = 0 \quad \longrightarrow \quad (10)$$

A Q4 shows strains as

$$\epsilon_x = -\frac{\theta_{el} y}{2a}; \quad \epsilon_y = 0; \quad \underline{\gamma_{xy} = -\frac{\theta_{el} x}{2a}} \quad \longrightarrow \quad (11)$$

Spurious shear strain = Parasitic shear

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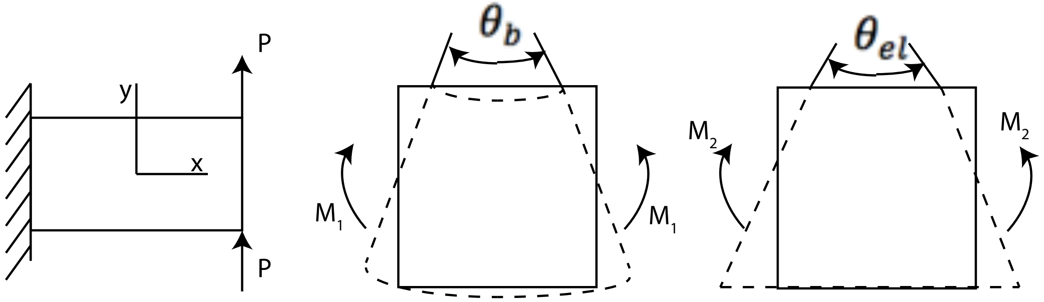
Let us assume that the block and the element include the same angle,

then, it can be shown that,

$$M_2 = \frac{1}{1+\nu}\left[\frac{1}{1-\nu} + \frac{1}{2}\left(\frac{a}{b}\right)^2\right] M_1 \longrightarrow (12)$$



If the element aspect ratio increases infinitely, so does M_2

If more moment is required to deform it, its artificially overly stiff in bending

This phenomenon is called **Shear Locking**

does a FE program “lock” the elements? —NO

- overly stiff
- lower deflections
- lower axial stresses
- higher error on transverse shear

Can we avoid it?

Yes! - meshing? element types?

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Improved bi-linear quadrilateral (Q6)

Six shape functions

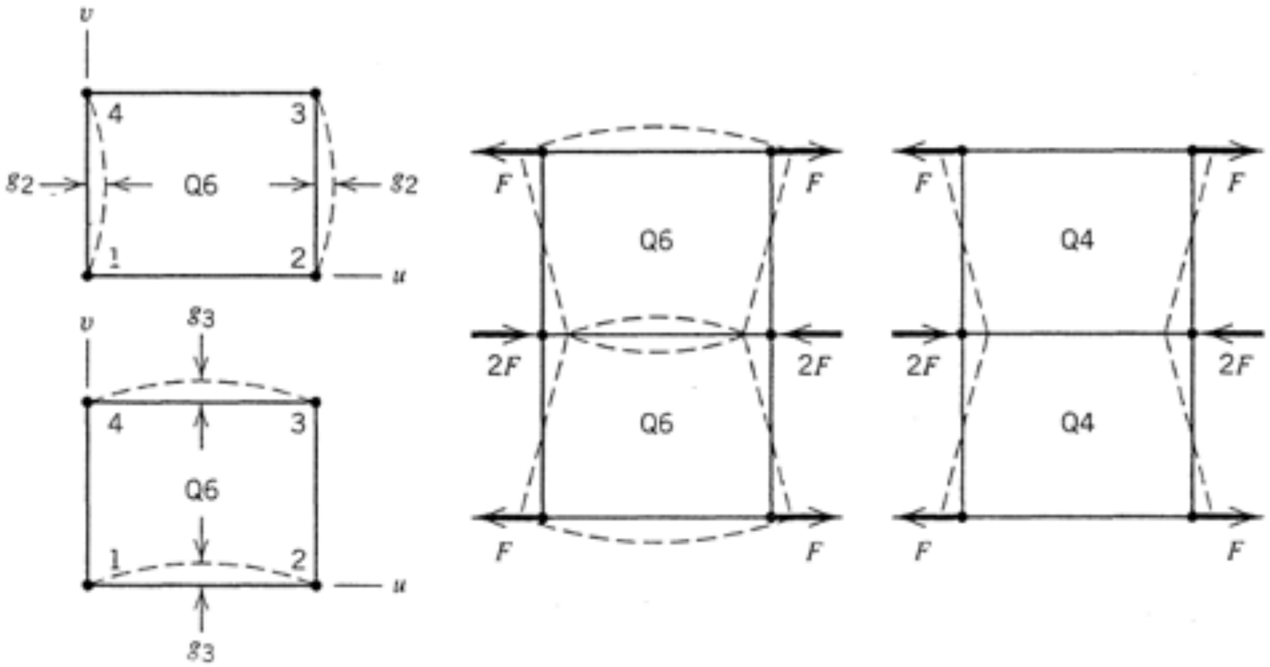
$$u = \sum_{i=1}^{4} N_i u_i + (1-\xi^2)g_1 + (1-\eta^2)g_2 \qquad \longrightarrow (13)$$

$$v = \sum_{i=1}^{4} N_i v_i + (1-\xi^2)g_3 + (1-\eta^2)g_4 \qquad \longrightarrow (14)$$

where $\xi = \dfrac{x}{a}; \quad \eta = \dfrac{y}{b}$

- can model bending
- shear strain is negligible
- element must be rectangular

Limitations?



- g_i are internal d.o.f
- internal d.o.f are incompatible with adjacent element corresponding d.o.f.
- certain loading cases cause gaps or overlaps

So why use them?

- sufficient mesh refinement leads to state of constant strain
- element edges become straight

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QUADRATIC QUAD ELEMENTS

Also called, **Q8**/Q9 element, has **eight**/nine nodes, and **sixteen**/eighteen nodal d.o.f

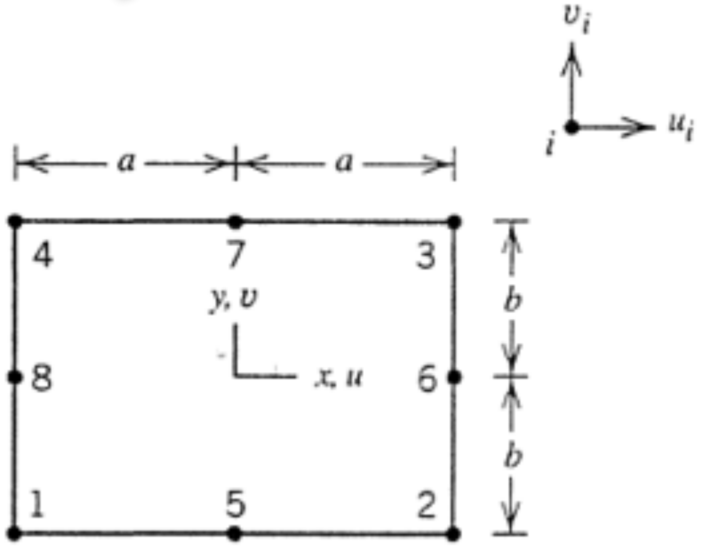
The quadratic displacement model is given as:

$$u = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^2 y + a_8 xy^2$$

$$v = a_9 + a_{10} x + a_{11} y + a_{12} x^2 + a_{13} xy + a_{14} y^2 + a_{15} x^2 y + a_{16} xy^2$$

(15)

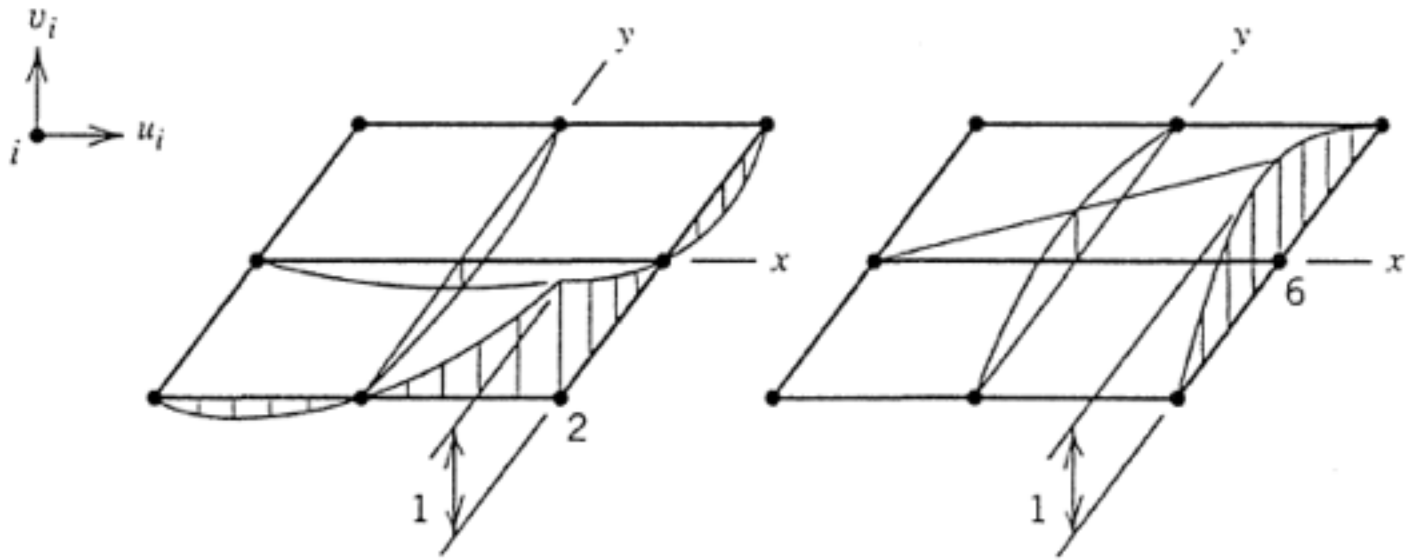
Serendipity elements!



Advantages
- no parasitic shear
- requires less mesh refinement
- faster convergence

Adding one node at x=0, y=0 location, gives a Q9 element

Displacement modes:



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Recap:

- Lagrange interpolation formula can be used to represent shape functions
- Quads (Q4,Q8,Q9) described are also called Lagrange elements
- Q9 performs better than Q8 performs better than Q4 in pure bending
- The choice is based on the problem at hand
- All elements shapes must be rectangular!

ISOPARAMETRIC FORMULATION OF QUADS

This formulation rids us of the restriction of using exclusively rectangular elements

* Shear locking behaviour is not avoided!

Some important facts and assumptions:

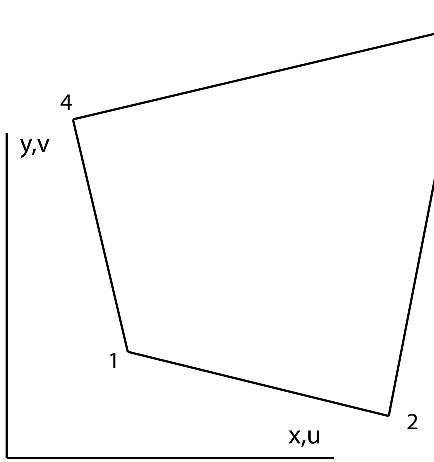
- the displacement field and the geometry definition are identical
- transformation from cartesian coordinate system to a generalised coordinate system is carried out
- the new coordinate system need not be orthogonal or parallel to the cartesian coordinates
- element sides are bisected by axes *r* and *s*
- $r = \pm 1$; $s = \pm 1$ (vertices)
- ($r = 0$, $s=0$) represents the centre of the element
- after transformation the element is always square with two units long sides
- displacement d.o.f. are parallel to cartesian coordinates and not the generalised coordinates

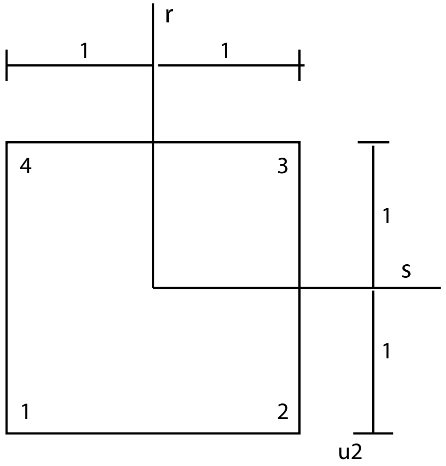
By nature of isoparametric formulation,

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} \sum N_i x_i \\ \sum N_i y_i \end{Bmatrix} = [\boldsymbol{N}]\{\boldsymbol{x}\} \quad \longrightarrow \text{(a)}$$

and,

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} \sum N_i u_i \\ \sum N_i v_i \end{Bmatrix} = [\boldsymbol{N}]\{\boldsymbol{q}\} \quad \longrightarrow \text{(b)}$$





where i = 1 to 4, and,

$$\{\boldsymbol{x}\} = [x_1 \quad y_1 \quad x_2 \quad y_2 \quad x_3 \quad y_3 \quad x_4 \quad y_4]^T$$

$$\{\boldsymbol{q}\} = [u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4]^T \quad \longrightarrow \text{(c)}$$

and, $$[\boldsymbol{N}] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \quad \longrightarrow \text{(d)}$$

Using the shape function definition from Lagrange formula,

$$N_1 = \frac{(a-x)(b-y)}{4ab} \;; \qquad N_2 = \frac{(a+x)(b-y)}{4ab} \;;$$

$$N_3 = \frac{(a+x)(b+y)}{4ab} \;; \quad \text{and,} \quad N_4 = \frac{(a-x)(b+y)}{4ab}$$

and substituting, $a = 1; b = 1; x = r; y = s$

$$N_1 = \frac{1}{4}(1-r)(1-s) \;; \qquad N_2 = \frac{1}{4}(1+r)(1-s) \;;$$

$$N_3 = \frac{1}{4}(1+r)(1+s) \;; \quad \text{and,} \quad N_4 = \frac{1}{4}(1-r)(1+s) \qquad \longrightarrow \text{(e)}$$

As always, the strain displacement relations needs to be worked out to find matrix **[B]** to evaluate the stiffness matrix

- gradients are involved
- the partial gradient w.r.t x is not related by a constant to partial gradient of r or s in this case
- a transformation matrix therefore, needs to be worked out

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Transformation matrix ***(Jacobian)***

The differentiable function, in this displacement and geometry, is a function of r and s

so, we begin with a derivative w.r.t. r and s

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \longrightarrow \text{(f)}$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \longrightarrow \text{(g)}$$

Assembling in matrix notation,

$$\begin{Bmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial s} \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix}}_{[J]} \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix} \longrightarrow \text{(h)}$$

working out the partial derivatives and substituting into (h)

$$[J] = \frac{1}{4}\begin{bmatrix} -(1-r) & (1+r) & (1+r) & -(1+r) \\ -(1-s) & -(1+s) & (1+s) & (1-s) \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \longrightarrow \text{(i)}$$

As you might remember, stiffness matrix in the isoparametric formulation is given by

$$[\boldsymbol{k}] = \int_{-1}^{1}\int_{-1}^{1} [\boldsymbol{B}]^T[\boldsymbol{E}][\boldsymbol{B}]t\,J\,dr\,ds \longrightarrow \text{(j)}$$

where, $J = \det[\boldsymbol{J}] = J_{11}J_{22} - J_{21}J_{12}$ $\longrightarrow$ (k)

* in general, jacobian determinant is a function of generalised coordinates,

* for rectangles and parallelograms, it turns out to be quarter of the area of the "physical" element

LOOSE ENDS…

- Strain energy errors
- choice of integration order

STRAIN ENERGY ERROR *FROM WEEK 6

DISCUSSION WITH EXAMPLE

- Beam problem from the practical assignment
- What’s important?
    - location of nodes
    - type of element
    - number of elements
    - directional mesh biasing
    - …

PROOF OF CONVERGENCE

- measurement of quality
  - comparison between fem and exact solution
  - discretisation error reduced to minimum

exact strain energy of the body

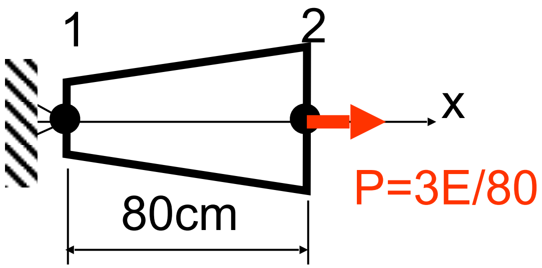
$$U = \frac{1}{2}\int_V \underline{\sigma}^T \underline{\varepsilon}\, dV$$

fe strain energy of the body (with element size h)

$$U_h = \frac{1}{2}\int_V \underline{\sigma}_h^T \underline{\varepsilon}_h\, dV$$

EXAMPLE

- Linear elastic bar



variable area

$$A(x) = \left(1 + \frac{x}{40}\right)^2 \quad cm^2$$

Boundary conditions

$$u(x=0) = 0$$

$$EA\left.\frac{du}{dx}\right|_{x=80cm} = P = \frac{3E}{80}$$

The governing differential (equilibrium) equation

$$E\frac{d}{dx}\left(A(x)\frac{du}{dx}\right) = 0 \quad for \quad x \in (0,80)$$

Analytical solution

$$u^{exact}(x) = \frac{3}{2}\left(1 - \frac{1}{1 + \frac{x}{40}}\right)$$

Exact strain energy

$$U = \frac{1}{2}\int_{x=0}^{80} \sigma\varepsilon \quad Adx = \frac{1}{2}\int_{x=0}^{80} EA\left(\frac{du^{exact}(x)}{dx}\right)^2 dx = \frac{3E}{160} = \frac{39E}{2080}$$

If we discretize the problem using a single linear finite element, the stiffness matrix is

$$\underline{K} = \frac{E\int_{x=0}^{80} A(x)dx}{80^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{13E}{240}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The strain energy of the FE system is

$$U_h = \frac{1}{2}\int_{x=0}^{80} \sigma_h \varepsilon_h Adx = \frac{1}{2}\underline{d}^T \underline{K}\underline{d} = \frac{27E}{2080}$$

where $\underline{d}^T = \begin{bmatrix} 0 & 9/13 \end{bmatrix}$

convergence in strain energy

$$U \rightarrow U_h \quad as \quad h \rightarrow 0$$

convergence in displacement

$$\left\| \underline{u} - \underline{u}_h \right\|_0 \equiv \sqrt{\int_V \left[\left(\mathrm{u}\text{-}\mathrm{u}_\mathrm{h}\right)^2 + \left(\mathrm{v}\text{-}\mathrm{v}_\mathrm{h}\right)^2 \right] dV} \rightarrow 0 \; as \; h \rightarrow 0$$

convergence rate

- measure of discretization error tending to zero
- dependent on the order of polynomial assumed as displacement model

CHOICE OF INTEGRATION ORDER

You were introduced to integration techniques in the last weeks : Gauss integration

This technique is used to generate the stiffness matrix of an element

There has to be a number of “sampling” points in the element

higher the number of sampling points in the integration rule, accuracy of integration increases

but so does the computational time

What are the obvious choices?

Low: for lower computational effort

lose some deformation modes

High: better accuracy

stiffening of higher-oder displacement modes

There is no direct answer. FE program chooses for you the best possible integration rule.
If you impose one on it, be aware of the consequences thereof!

HOMEWORK

- Continue working on Homework assignment 3
- Prepare for final practical assignment

FINAL WRAP-UP

- A wrap-up mini lecture will be posted next week
- Some additional material will also be posted for those who are curious
  - symmetry (as a consequence of your practical assignment results!)
  - dynamic fe (at request of some curious students)